# Invariant Differential Operators : Latest Developments

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**Plan**

## 1. Introduction

Invariant differential operators play very important role in the description of physical symmetries - starting from the early occurrences in the Maxwell, d'Allembert, Dirac, equations, to the latest applications of (super-)differential operators in conformal field theory, supergravity and string theory. Thus, it is important for the applications in physics to study systematically such operators.

Some years ago we started the systematic explicit construction of invariant differential operators [1]. We gave an explicit description of the building blocks, namely, the parabolic subgroups and subalgebras from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

Since the study and description of detailed classification should be done group by group we had to decide which groups to study first. A natural choice would be non-compact groups that have discrete series of representations. By the Harish-Chandra criterion [2] these are groups where holds:

$$\mathrm{rank}\, G = \mathrm{rank}\, K, \tag{1.1}$$

where $K$ is the maximal compact subgroup of the non-compact group $G$. Another formulation is to say that the Lie algebra $\mathcal{G}$ of $G$ has a compact Cartan subalgebra.

Example: The groups $SO(p,q)$ have discrete series, except when both $p,q$ are odd numbers. $\diamondsuit$

This class is still rather big, thus, we decided to start with a subclass, namely, the class of Hermitian symmetric spaces. The practical criterion is that in these cases, the maximal compact subalgebra $\mathcal{K}$ is of the form:

$$\mathcal{K} \ = \ so(2) \oplus \mathcal{K}' \ . \tag{1.2}$$

The Lie algebras from this class are:

$$so(n,2), \quad sp(n,R), \quad su(m,n), \quad so^*(2n), \quad E_{6(-14)}\,, \quad E_{7(-25)} \tag{1.3}$$

These groups/algebras have highest/lowest weight representations, and relatedly holomorphic discrete series representations [3].

The most widely used of these algebras are the conformal algebras $so(n,2)$ in $n$-dimensional Minkowski space-time. In that case, there is a maximal Bruhat decomposition [4]:

$$\begin{aligned} so(n,2) \ &= \ \mathcal{P} \oplus \tilde{\mathcal{N}} \ = \ \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N} \oplus \tilde{\mathcal{N}}\ , \\ \mathcal{M} \ &= \ so(n-1,1)\ , \quad \dim \mathcal{A} = 1, \quad \dim \mathcal{N} = \dim \tilde{\mathcal{N}} = n \end{aligned} \tag{1.4}$$

that has direct physical meaning, namely, $so(n-1,1)$ is the Lorentz algebra of $n$-dimensional Minkowski space-time, the subalgebra $\mathcal{A} \ = \ so(1,1)$ represents the dilatations, the conjugated subalgebras $\mathcal{N}\,,\ \tilde{\mathcal{N}}$ are the algebras of translations,

[1] V.K. Dobrev, Rev. Math. Phys. **20** (2008) 407-449.

[2] Harish-Chandra, Ann. Math. **116** (1966) 1-111.

[3] A.W. Knapp, *Representation Theory of Semisimple Groups (An Overview Based on Examples)*, (Princeton Univ. Press, 1986).

[4] F. Bruhat, Bull. Soc. Math. France, **84** (1956) 97-205.

and special conformal transformations, both being isomorphic to $n$-dimensional Minkowski space-time.

The subalgebra $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ ($\cong \mathcal{M} \oplus \mathcal{A} \oplus \tilde{\mathcal{N}}$) is a maximal parabolic subalgebra.

There are other special features which are important. In particular, the complexification of the maximal compact subgroup is isomorphic to the complexification of the first two factors of the Bruhat decomposition:

$$\mathcal{K}^{\mathbb{C}} = so(n,\mathbb{C}) \oplus so(2,\mathbb{C}) \cong so(n-1,1)^{\mathbb{C}} \oplus so(1,1)^{\mathbb{C}} = \mathcal{M}^{\mathbb{C}} \oplus \mathcal{A}^{\mathbb{C}} . \tag{1.5}$$

In particular, the coincidence of the complexification of the semi-simple subalgebras:

$$\mathcal{K}'^{\mathbb{C}} = \mathcal{M}^{\mathbb{C}} \tag{1.6}$$

means that the sets of finite-dimensional (nonunitary) representations of $\mathcal{M}$ are in 1-to-1 correspondence with the finite-dimensional (unitary) representations of $\mathcal{K}'$.

It turns out that some of the hermitian-symmetric algebras share the above-mentioned special properties of $so(n,2)$. This subclass consists of:

$$so(n,2), \quad sp(n,\mathbb{R}), \quad su(n,n), \quad so^*(4n), \quad E_{7(-25)} \tag{1.7}$$

the corresponding analogs of Minkowski space-time $V$ being:

$$\mathbb{R}^{n-1,1}, \quad \mathrm{Sym}(n,\mathbb{R}), \quad \mathrm{Herm}(n,\mathbb{C}), \quad \mathrm{Herm}(n,\mathbb{Q}), \quad \mathrm{Herm}(3,\mathbb{O}) \tag{1.8}$$

where we use standard notation $\mathbb{R},\mathbb{C},\mathbb{Q},\mathbb{O}$ for the four division algebras (real, complex, quaternion, octonion).

In view of applications to physics, we proposed to call these algebras 'conformal Lie algebras', (or groups) [5].

We have started the study of the above class in the framework of the present approach in the cases: $so(n,2)$, $su(n,n)$, $sp(n,\mathbb{R})$, $E_{7(-25)}$, $SO^*(12)$. Later we have considered some algebras outside the above class: $E_{6(-14)}$, $F'_4$, $F''_4$, furthermore some cases with other parabolics, e.g., Heisenberg parabolics: $G_{2(2)}$, $SO^*(2n)$.

Later, in [6] we discovered an efficient way to extend our considerations beyond this class introducing the notion of 'parabolically related non-compact semisimple Lie algebras'.

• Definition: Let $\mathcal{G},\mathcal{G}'$ be two non-compact semisimple Lie algebras with the same complexification $\mathcal{G}^{\mathbb{C}} \cong \mathcal{G}'^{\mathbb{C}}$. We call them parabolically related if they have parabolic subalgebras $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$, $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$, such that: $\mathcal{M}^{\mathbb{C}} \cong \mathcal{M}'^{\mathbb{C}}$ ($\Rightarrow \mathcal{P}^{\mathbb{C}} \cong \mathcal{P}'^{\mathbb{C}}$). $\diamond$

Certainly, there may be more than one such parabolic relationships for any given algebra $\mathcal{G}$. Furthermore, two algebras $\mathcal{G},\mathcal{G}'$ may be parabolically related with different parabolic subalgebras.

[5] V.K. Dobrev, J. Phys. **A42** (2009) 285203.
[6] V.K. Dobrev, J. High Energy Phys. 02 (2013) 015.

## 2. Conformal algebras $so(n,2)$ and parabolically related

### *2.1. Maxwell equations hierarchy*

We start with the simplest case of conformal intertwining differential operators.

It is well known that Maxwell equations

$$\partial^\mu F_{\mu\nu} = J_\nu \qquad (2.1a)$$
$$\partial^\mu \, {}^*F_{\mu\nu} = 0 \qquad (2.1b)$$

(where ${}^*F_{\mu\nu} \equiv \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, $\epsilon_{\mu\nu\rho\sigma}$ being totally antisymmetric with $\epsilon_{0123} = 1$), or, equivalently

$$\begin{aligned} \partial_k E_k &= J_0 \ (= 4\pi\rho), \quad \partial_0 E_k - \varepsilon_{k\ell m}\partial_\ell H_m = J_k \ (= -4\pi j_k), \\ \partial_k H_k &= 0\,, \quad \partial_0 H_k + \varepsilon_{k\ell m}\partial_\ell E_m = 0\,, \end{aligned} \qquad (2.2)$$

where $E_k \equiv F_{k0}$, $H_k \equiv (1/2)\varepsilon_{k\ell m}F_{\ell m}$, may be rewritten in the following manner:

$$\partial_k F_k^\pm = J_0\,, \quad \partial_0 F_k^\pm \pm i\varepsilon_{k\ell m}\partial_\ell F_m^\pm = J_k\,, \qquad (2.3)$$

where

$$F_k^\pm \equiv E_k \pm iH_k\,. \qquad (2.4)$$

Not so well known is the fact that the eight equations in (??) may be rewritten as two conjugate scalar equations in the following way:

$$I^+\ F^+(z) = J(z,\bar z)\,, \qquad (2.5a)$$
$$I^-\ F^-(\bar z) = J(z,\bar z)\,, \qquad (2.5b)$$

where

$$I^+ = \bar z\partial_+ + \partial_v - \tfrac{1}{2}\Big(\bar z z\partial_+ + z\partial_v + \bar z\partial_{\bar v} + \partial_-\Big)\partial_z\,, \qquad (2.6a)$$
$$I^- = z\partial_+ + \partial_{\bar v} - \tfrac{1}{2}\Big(\bar z z\partial_+ + z\partial_v + \bar z\partial_{\bar v} + \partial_-\Big)\partial_{\bar z}\,, \qquad (2.6b)$$

$$x_\pm \equiv x_0 \pm x_3, \quad v \equiv x_1 - ix_2, \quad \bar v \equiv x_1 + ix_2, \qquad (2.7a)$$
$$\partial_\pm \equiv \partial/\partial x_\pm, \quad \partial_v \equiv \partial/\partial v, \quad \partial_{\bar v} \equiv \partial/\partial \bar v, \qquad (2.7b)$$

$$F^+(z) \equiv z^2(F_1^+ + iF_2^+) - 2zF_3^+ - (F_1^+ - iF_2^+)\,, \qquad (2.8a)$$
$$F^-(\bar z) \equiv \bar z^2(F_1^- - iF_2^-) - 2\bar zF_3^- - (F_1^- + iF_2^-)\,, \qquad (2.8b)$$
$$\begin{aligned} J(z,\bar z) &\equiv \bar z z(J_0+J_3) + z(J_1+iJ_2) + \bar z(J_1-iJ_2) + (J_0-J_3) = \\ &\equiv \bar z zJ_+ + zJ_v + \bar zJ_{\bar v} + J_- \end{aligned} \qquad (2.8c)$$

It is easy to recover (??) from (??) - just note that both sides of each equation are first order polynomials in each of the two variables $z$ and $\bar z$, then comparing the independent terms in (??) one gets at once (??).

Writing the Maxwell equations in the simple form (??) has also important conceptual meaning. The point is that each of the two scalar operators $I^+, I^-$ is indeed a single object, namely it is an intertwiner of the conformal group, or conformally invariant differential operator, while the individual components in (??) - (??) do not have this interpretation. This is also the simplest way to see that the Maxwell equations are conformally invariant, since this is equivalent to the intertwining property.

Let us be more explicit. The physically relevant representations $T^\chi$ of the 4-dimensional conformal algebra $so(4,2) = su(2,2)$ may be labelled by $\chi = [n_1, n_2; d]$, where $n_1, n_2$ are non-negative integers fixing finite-dimensional irreducible representations of the Lorentz subalgebra, (the dimension being $(n_1 + 1)(n_2 + 1)$), and $d$ is the conformal dimension (or energy). (In the literature these Lorentz representations are labelled also by $(j_1, j_2) = (n_1/2, n_2/2)$.) Then the intertwining properties of the operators in (??) are given by:

$$I^+ \ :\ C^+ \longrightarrow C^0\,, \quad I^+ \circ T^+ \ =\ T^0 \circ I^+\,, \tag{2.9a}$$
$$I^- \ :\ C^- \longrightarrow C^0\,, \quad I^- \circ T^- \ =\ T^0 \circ I^-\,, \tag{2.9b}$$

where $T^a = T^{\chi^a}$, $a = 0, +, -$, $C^a = C^{\chi^a}$ are the representation spaces, and the signatures are given explicitly by:

$$\chi^+ = [2,0;2]\,, \quad \chi^- = [0,2;2]\,, \quad \chi^0 = [1,1;3]\,, \tag{2.10}$$

as anticipated. Indeed, $(n_1, n_2) = (1,1)$ is the four-dimensional Lorentz representation, (carried by $J_\mu$ above), and $(n_1, n_2) = (2,0), (0,2)$ are the two conjugate three-dimensional Lorentz representations, (carried by $F_k^\pm$ above), while the conformal dimensions are the canonical dimensions of a current $(d = 3)$, and of the Maxwell field $(d = 2)$. We see that the variables $z, \bar{z}$ are related to the spin properties and we shall call them 'spin variables'.

It is also important that the variables $x_\pm, v, \bar{v}, z, \bar{z}$ have definite group-theoretical meaning, namely, they are six local coordinates on the coset $\mathcal{Y} = SL(4)/B$, where $B$ is the Borel subgroup of $SL(4)$ consisting of all upper diagonal matrices. (Equally well one may take the coset $SL(4)/B^-$, where $B^-$ is the Borel subgroup of lower diagonal matrices.) Under the natural conjugation (cf. also below) this is also a coset of the conformal group $SU(2,2)$.

Now we recollect that closely related to the above fields is the potential $A_\mu$ with signature

$$\tilde{\chi}^0 = [1,1;1] \tag{2.11}$$

so that the analog of (??a) is

$$\partial_\mu A_\nu = F_{\mu\nu} \tag{2.12}$$

(not forgetting that the RHS is only a subspace). We also recall that there are two more conformal operators involving two scalar fields with signatures:

$$\phi = [0,0;0], \quad \Phi = [0,0;4] \tag{2.13}$$

so that

$$\partial\mu\phi = A_\mu, \quad \partial^\mu J_\mu = \Phi \tag{2.14}$$

(again the RHSs are subspaces).

Altogether we have the following picture:

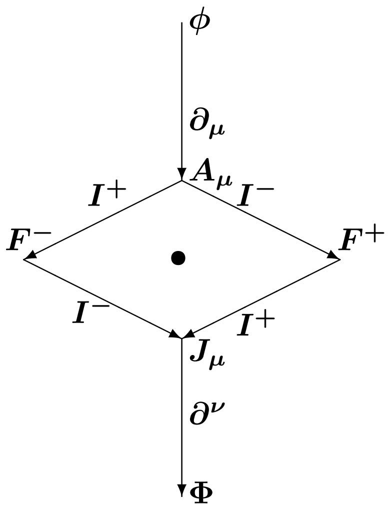



**Fig. 1. Simplest occurrence of conformal invariant differential operators**

**Remark:** Note that the $\pm$ pairs (that are symmetrical w.r.t. the bullet in the figure) are related by integral operators $G_{KS}$, so-called Knapp-Stein operators [7], with kernels which are conformal two-point functions. Their action on the signatures is:

$$G_{KS} \ : \ [n_1, n_2; d] \ \longrightarrow \ [n_2, n_1; 4-d] \ . \qquad \diamond \tag{2.15}$$

The above picture is the simplest occurrence of 4D conformally invariant differential operators. The general case is given by a 3-parameter generalization given as follows:

$$\begin{aligned}
\chi^-_{p\nu n} &= [p-1, n-1;\ 2-\nu-\tfrac{1}{2}(p+n)] \quad (\phi) \\
\chi^+_{p\nu n} &= [n-1, p-1;\ 2+\nu+\tfrac{1}{2}(p+n)] \quad (\Phi) \\
\chi'^-_{p\nu n} &= [p+\nu-1, n+\nu-1;\ 2-\tfrac{1}{2}(p+n)] \quad (A_\mu) \\
\chi'^+_{p\nu n} &= [n+\nu-1, p+\nu-1;\ 2+\tfrac{1}{2}(p+n)] \quad (J_\mu) \\
\chi''^-_{p\nu n} &= [\nu-1, p+n+\nu-1;\ 2+\tfrac{1}{2}(p-n)] \quad (F^-) \\
\chi''^+_{p\nu n} &= [p+n+\nu-1, \nu-1;\ 2+\tfrac{1}{2}(n-p)] \quad (F^+)
\end{aligned} \tag{2.16}$$

where $p, \nu, n$ are positive integers which are exactly the Dynkin labels $m_1, m_2, m_3$ of $sl(4)$ for $\chi^-_{p\nu n}$.

We call "multiplets" such collection of representations related by intertwining differential operators.

The simplest example we considered first is obtained for $p = \nu = n = 1$.

The multiplets (sextets here) are given now in the following figure:

[7] A.W. Knapp and E.M. Stein, Ann. Math. **93** (1971) 489-578; II : Inv. Math. **60** (1980) 9-84.

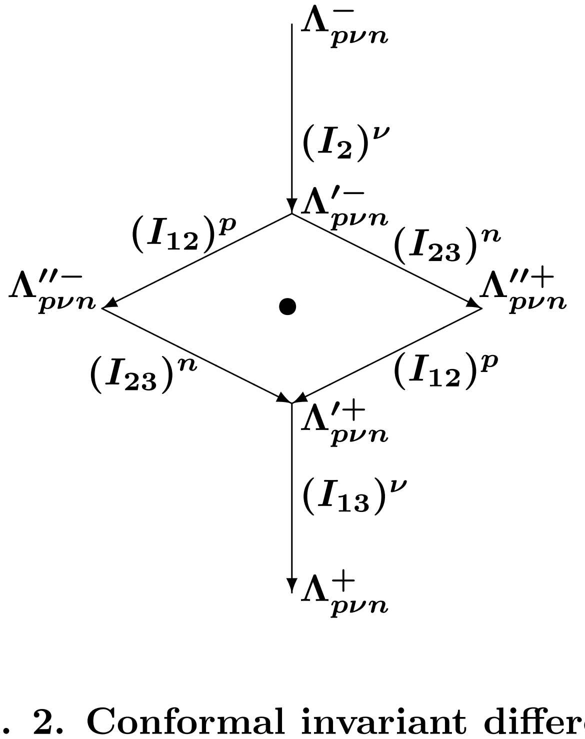



**Fig. 2. Conformal invariant differential operators in 4D, general case**

where the differential operators are given explicitly by:

$$\begin{aligned}(I_2)^m &= (\bar{z}_1 z_1 \partial_+ + z_1 \bar{z}_2 \partial_v + \bar{z}_1 z_2 \partial_{\bar{v}} + \bar{z}_2 z_2 \partial_-)^m = \\ &= \left( (\bar{z}_1, \bar{z}_2)\ \sigma^\mu \partial_\mu \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right)^m , \qquad (2.17a)\\ (I_{12})^m &= \left( (\bar{z}_1, \bar{z}_2)\ \sigma^\mu \partial_\mu\, \varepsilon \begin{pmatrix} \partial_{z_1} \\ \partial_{z_2} \end{pmatrix} \right)^m , \qquad (2.17b)\\ (I_{23})^m &= \left( (\partial_{\bar{z}_1}, \partial_{\bar{z}_2})\ \varepsilon \sigma^\mu \partial_\mu \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right)^m , \qquad (2.17c)\\ (I_{13})^m &= \left( (\partial_{\bar{z}_1}, \partial_{\bar{z}_2})\ \sigma^\mu \partial_\mu \begin{pmatrix} \partial_{z_1} \\ \partial_{z_2} \end{pmatrix} \right)^m , \qquad (2.17d)\end{aligned}$$

where $\sigma_\mu$ are the Pauli matrices, $\varepsilon = i\sigma_2$. Note that here for the finite-dimensional irreps of the Lorentz subalgebra we have passed from polynomials in $z, \bar{z}$ of degrees $n_1, n_2$, to homogeneous polynomials in $z_1, z_2$ of degree $n_1$ and in $\bar{z}_1, \bar{z}_2$ of degree $n_2$. The two realizations are easily related via $z = z_1/z_2$, $\bar{z} = \bar{z}_1/\bar{z}_2$.

The above picture is valid also for the 4-dimensional Euclidean conformal algebra $so(5,1)$, [8], and also for the Lie algebra $so(3,3)$.

### 2.2. *Generalization : $so(n,2)$ and $so(p,q)$*

Next we recall that the conformal algebra of n-dimensional Minkowski space-time is the algebra $so(2n,2)$. Actually we shall consider a more general picture, namely, the Lie algebras $\mathcal{G} = so(p,q)$.

The analogue of the Lorentz subalgebra is:

$$\mathcal{M} = so(p-1, q-1) . \qquad (2.18)$$

The analogue of Minkowski space-time is $\mathcal{N}$ with:

$$\dim \mathcal{N} = p + q - 2 . \qquad (2.19)$$

We label the signature of the representations of $\mathcal{G}$ as follows:

$$\begin{aligned}\chi &= \{ n_1 , \dots , n_h ; c \} , \qquad (2.20)\\ &n_j \in \mathbb{Z}/2 , \quad c = d - \tfrac{p+q-2}{2} , \quad h \equiv [\tfrac{p+q-2}{2}],\\ &|n_1| < n_2 < \cdots < n_h , \quad p+q \ \text{ even} ,\\ &0 < n_1 < n_2 < \cdots < n_h , \quad p+q \ \text{ odd} ,\end{aligned}$$

where the last entry of $\chi$ labels the characters of $\mathcal{A}$, and the first $h$ entries are labels of the finite-dimensional nonunitary irreps of $\mathcal{M} = so(p-1, q-1)$.

[8] V.K. Dobrev and V.B. Petkova, Rept. Math. Phys. **13** (1978) 233-277.

The reason to use the parameter $c$ instead of $d$ is that the parametrization of the ERs in the multiplets is given in a simple intuitive way:

$$\begin{aligned}
\chi_1^{\pm} &= \{\epsilon n_1\,, \dots,\, n_h\,;\, \pm n_{h+1}\}\ , \quad n_h < n_{h+1}\ , \\
\chi_2^{\pm} &= \{\epsilon n_1\,, \dots,\, n_{h-1}\,,\, n_{h+1}\,;\, \pm n_h\} \\
\chi_3^{\pm} &= \{\epsilon n_1, \dots, n_{h-2}, n_h, n_{h+1}\,;\, \pm n_{h-1}\} \\
&\dots \\
\chi_h^{\pm} &= \{\epsilon n_1\,, n_3\,, \dots,\, n_h\,,\, n_{h+1}\,;\, \pm n_2\} \\
\chi_{h+1}^{\pm} &= \{\epsilon n_2\,, \dots,\, n_h\,,\, n_{h+1}\,;\, \pm n_1\} \\
\epsilon &= \begin{cases} \pm\,, & n\ even \\ 1, & n\ odd \end{cases}
\end{aligned} \tag{2.21}$$

($\epsilon = \pm$ is correlated with $\chi^{\pm}$).

Further, we denote by $\tilde{\mathcal{C}}_i^{\pm}$ the representation space with signature $\chi_i^{\pm}$.

The number of ERs in a multiplet is:

$$|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})|\,/\,|W(\mathcal{M}^{\mathbb{C}}, \mathcal{H}_m^{\mathbb{C}})| \tag{2.22}$$

where $\mathcal{H}^{\mathbb{C}}$, $\mathcal{H}_m^{\mathbb{C}}$ are Cartan subalgebras of $\mathcal{G}^{\mathbb{C}}$, $\mathcal{M}^{\mathbb{C}}$, resp.

The above in our case gives:

$$|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})|\,/\,|W(\mathcal{M}^{\mathbb{C}}, \mathcal{H}_m^{\mathbb{C}})| \ = \ 2(1+h) \tag{2.23}$$

Below we give the multiplets pictorially first for $p+q$ even, then for $p+q$ odd.

$$
\begin{tikzcd}
 & C_1^- \arrow[d, "d_1"] & \\
 & C_2^- & \\
 & \vdots & \\
 & C_{h-1}^- \arrow[d, "d_{h-1}"] & \\
 & C_h^- \arrow[dl, "d_h"'] \arrow[dr, "d'_h"] \arrow[dd, phantom, "\bullet"] & \\
C_{h+1}^- \arrow[dr, "d'_h"'] & & C_{h+1}^+ \arrow[dl, "d_h"] \\
 & C_h^+ \arrow[d, "d'_{h-1}"] & \\
 & C_{h-1}^+ & \\
 & \vdots & \\
 & C_2^+ \arrow[d, "d'_1"] & \\
 & C_1^+ &
\end{tikzcd}
$$

**Fig. 3. Invariant differential operators for $so(p,q)$ for $p+q$ even, $h = \frac{1}{2}(p+q-2)$**

$$
\begin{tikzcd}
C_1^- \arrow[d, "d_1"] \\
C_2^- \\
\vdots \\
C_{h-1}^- \arrow[d, "d_{h-1}"] \\
C_h^- \arrow[d, "d_h"] \\
C_{h+1}^- \arrow[d, "d_{h+1}"] \\
C_{h+1}^+ \arrow[d, "d'_h"] \\
C_h^+ \arrow[d, "d'_{h-1}"] \\
C_{h-1}^+ \\
\vdots \\
C_2^+ \arrow[d, "d'_1"] \\
C_1^+
\end{tikzcd}
$$

**Fig. 4. Invariant differential operators for $so(p,q)$ for $p+q$ odd, $h = \frac{1}{2}(p+q-1)$**

The degrees of the operators in the two pictures are:

$$\begin{aligned} &\deg d_i = \deg d'_i = n_{h+2-i} - n_{h+1-i}, \qquad i = 1, \ldots, h, \\ &\deg d'_h = n_2 + n_1, \quad (p+q) - \text{even} \ , \\ &\deg d_{h+1} = 2n_1, \quad (p+q) - \text{odd} \end{aligned} \tag{2.24}$$

where $d'_h$ is omitted from the first line for $(p+q)$ even.

Again the pairs $\tilde{\mathcal{C}}_i^{\pm}$ are related by Knapp-Stein operators that correspond to elements of the restricted Weyl group of $\mathcal{G}$, namely, we have:

$$G_i^{\pm} \ : \ \tilde{\mathcal{C}}_i^{\mp} \longrightarrow \tilde{\mathcal{C}}_i^{\pm} \ , \quad i \ = \ 1, \ldots, 1+h \tag{2.25}$$

There is a peculiarity, namely, that for $p+q$ odd, for the pair $C_{h+1}^{\pm}$ the KS operator acting from $C_{h+1}^{-}$ to $C_{h+1}^{+}$ has degenerated (due to regularization of the kernel) to the differential operator $d_{h+1}$.

Matters are arranged so that in every multiplet only the ER with signature $\chi_1^-$ contains a finite-dimensional nonunitary subrepresentation in a subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional unitary irrep of $so(n+2)$ with signature $\{n_1, \ldots, n_h, n_{h+1}\}$. The subspace $\mathcal{E}$ is annihilated by the operator $G_1^+$, and is the image of the operator $G_1^-$.

Although the diagrams are valid for arbitrary $so(p,q)$ $(p+q \geq 5)$ the contents is very different. We comment only on the ER with signature $\chi_1^+$. In all cases it contains an UIR of $so(p,q)$ realized on an invariant subspace $\mathcal{D}$ of the ER $\chi_1^+$. That subspace is annihilated by the operator $G_1^-$, and is the image of the operator $G_1^+$. (Other ERs contain more UIRs.)

If $pq \in 2\mathbb{N}$ the mentioned UIR is a discrete series representation. (Other ERs contain more discrete series UIRs.)

And if $q = 2$ the invariant subspace $\mathcal{D}$ is the direct sum of two subspaces $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$, in which are realized a holomorphic discrete series representation and its conjugate anti-holomorphic discrete series representation, resp. Note that the corresponding lowest weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series.

Above for $so(n,2)$ we restricted to $n > 2$. The case $n = 2$ is reduced to $n = 1$ since $so(2,2) \cong so(1,2) \oplus so(1,2)$. The case $so(1,2)$ is special and must be treated separately. But in fact, it is contained in what we presented already. In that case the multiplets contain only two ERs which may be depicted by the top pair $\chi_1^{\pm}$ in all pictures that we presented. And they have the properties that we described. That case was the first given already in 1946-7 independently by Gel'fand et al [9] and Bargmann [10].

[9] I.M. Gelfand and M.A. Naimark, Acad. Sci. USSR. J. Phys. **10** (1946) 93-94.
[10] V. Bargmann, Annals Math. **48**, (1947) 568-640.

## 3. The Lie algebra $su(n,n)$ and parabolically related

Let $\mathcal{G} = su(n,n)$, $n \geq 2$. The maximal compact subgroup is $\mathcal{K} \cong u(1) \oplus su(n) \oplus su(n)$, while $\mathcal{M} = sl(n,\mathbb{C})_{\mathbb{R}}$.

The signature of the ERs of $\mathcal{G}$ is:

$$\chi = \{ n_1 , \ldots , n_{n-1} , n_{n+1} \ldots , n_{2n-1} ; c \} , \qquad n_j \in \mathbb{N} , \quad c = d - \tfrac{1}{2} n^2 \tag{3.1}$$

The Knapp–Stein restricted Weyl reflection is given by:

$$\begin{aligned} &G_{KS} \ : \ \mathcal{C}_\chi \longrightarrow \mathcal{C}_{\chi'} , \\ &\chi' = \{(n_1,\ldots,n_{n-1},n_{n+1},\ldots,n_{2n-1})^*; -c\} , \\ &(n_1,\ldots,n_{n-1},n_{n+1},\ldots,n_{2n-1})^* \doteq (n_{n+1},\ldots,n_{2n-1},n_1,\ldots,n_{n-1}) \end{aligned} \tag{3.2}$$

Further, we use the root system of the complex algebra $sl(2n,\mathbb{C})$. The positive roots $\alpha_{ij}$ in terms of the simple roots $\alpha_i$ are:

$$\begin{aligned} &\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j , \qquad 1 \leq i < j \leq 2n-1 , \\ &\alpha_{ii} \equiv \alpha_i , \quad 1 \leq i \leq 2n-1 \end{aligned} \tag{3.3}$$

from which the non-compact are:

$$\alpha_{ij} , \qquad 1 \leq i \leq n , \qquad n \leq j \leq 2n-1 \tag{3.4}$$

The correspondence between the signatures $\chi$ and the highest weight $\Lambda$ is through the Dynkin labels:

$$\begin{aligned} &n_i = m_i \equiv (\Lambda+\rho,\alpha_i^\vee) = (\Lambda+\rho,\alpha_i) , \qquad i = 1,\ldots,2n-1, \\ &c = -\tfrac{1}{2}(m_{\tilde\alpha} + m_n) = -\tfrac{1}{2}(m_1 + \cdots + m_{n-1} + 2m_n + m_{n+1} + \cdots + m_{2n-1}) , \end{aligned} \tag{3.5}$$

$\Lambda = \Lambda(\chi)$, $\tilde\alpha = \alpha_1 + \cdots + \alpha_{2n-1}$ is the highest root.

The number of ERs in the corresponding multiplets by (??) is equal to:

$$|W(\mathcal{G}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}})| \, / \, |W(\mathcal{M}^{\mathbb{C}},\mathcal{H}_m^{\mathbb{C}})| = \binom{2n}{n} \tag{3.6}$$

In our diagrams we need also the Harish-Chandra parameters for the non-compact roots using the following notation:

$$m_{ij} \equiv m_{\alpha_{ij}} = m_i + \cdots + m_j , \quad i < j \tag{3.7}$$

We use the following conventions. Each intertwining differential operator is represented by an arrow accompanied by a symbol $i_{j\ldots k}$ encoding the root $\beta_{j\ldots k}$ and the number $m_{\beta_{j\ldots k}}$ which is involved in the BGG criterion. This notation is used to save space, but it can be used due to the fact that only intertwining differential operators which are non-composite are displayed, and that the data $\beta, m_\beta$, which is involved in the embedding $V^\Lambda \longrightarrow V^{\Lambda - m_\beta,\beta}$ turns out to involve only the $m_i$ corresponding to simple roots, i.e., for each $\beta, m_\beta$ there exists $i = i(\beta, m_\beta, \Lambda) \in \{1,\ldots,r\}$, ($r = \mathrm{rank}\,\mathcal{G}$), such that $m_\beta = m_i$. Hence the data $\beta_{j\ldots k}, m_{\beta_{j\ldots k}}$ is represented by $i_{j\ldots k}$ on the arrows.

**Below we give the diagrams for $su(n,n)$ for $n=3,4$. (The case $n=$ was already considered since $su(2,2)=so(4,2)$.**

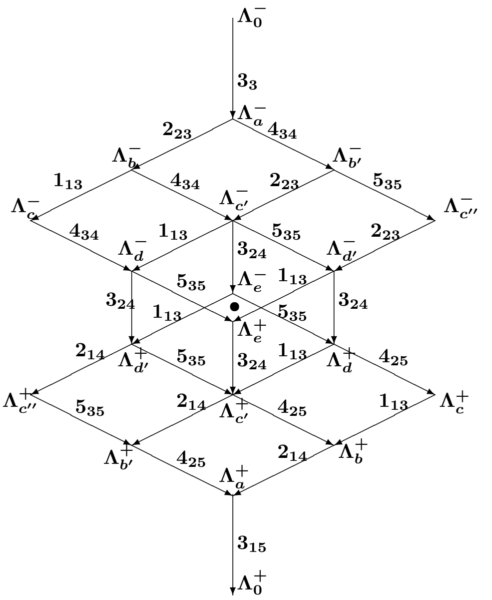



**Fig. 5. Main multiplets for $su(3,3)$ and $sl(6,\mathbb{R})$**
**with parabolic $\mathcal{M}$-factors $sl(3,\mathbb{C})_{\mathbb{R}}$, $sl(3,\mathbb{R}) \oplus sl(3,\mathbb{R})$, resp.**

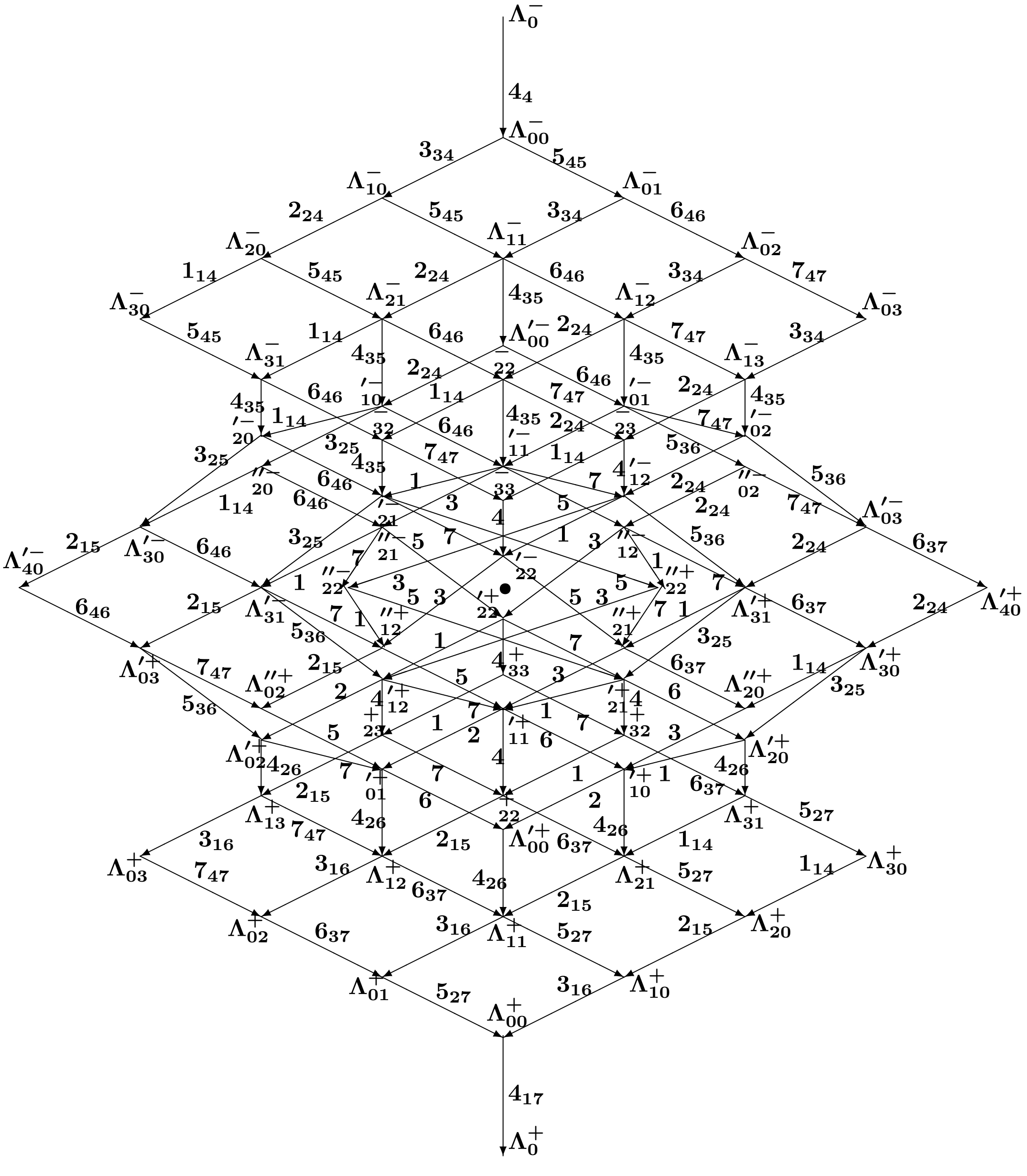


**Fig. 6. Main multiplets for** $su(4,4)$, $sl(8,\mathbb{R})$, $su^*(8)$
**with parabolic** $\mathcal{M}$**-factors** $sl(4,\mathbb{C})_{\mathbb{R}}$, $sl(4,\mathbb{R})\oplus sl(4,\mathbb{R})$, $su^*(4)\oplus su^*(4)$, **resp.**

## 4. The Lie algebras $sp(n,\mathbb{R})$ and $sp(\frac{n}{2},\frac{n}{2})$ ($n$–even)

Let $n \geq 2$. Let $\mathcal{G} = sp(n,\mathbb{R})$, the rank $n$ split real form of $sp(n,\mathbb{C}) = \mathcal{G}^{\mathbb{C}}$. The maximal compact subgroup is $\mathcal{K} \cong u(1) \oplus su(n)$, while $\mathcal{M} = sl(n,\mathbb{R})$.

The signature of the ERs of $\mathcal{G}$ is:

$$\chi = \{\, n_1\,, \ldots, \, n_{n-1}\,;\, c\,\}\,, \qquad n_j \in \mathbb{N}\,. \tag{4.1}$$

The Knapp-Stein restricted Weyl reflection acts as follows:

$$\begin{aligned} &G_{KS} \ :\ \mathcal{C}_\chi \ \longrightarrow\ \mathcal{C}_{\chi'}\,, \\ &\chi' \ =\ \{\,(n_1,\ldots,n_{n-1})^*\,;\, -c\,\}\,, \\ &(n_1,\ldots,n_{n-1})^* \ \doteq\ (n_{n-1},\ldots,n_1) \end{aligned} \tag{4.2}$$

In terms of an orthonormal basis $\epsilon_i$, $i = 1,\ldots,n$, the positive roots are:

$$\Delta^+ \ =\ \{\, \epsilon_i \pm \epsilon_j, \ \ 1 \leq i < j \leq n; \qquad 2\epsilon_i, \ \ 1 \leq i \leq n\,\} \tag{4.3}$$

the simple roots are:

$$\pi \ =\ \{\, \alpha_i = \epsilon_i - \epsilon_{i+1}, \ 1 \leq i \leq n-1, \qquad \alpha_n = 2\epsilon_n\,\} \tag{4.4}$$

the non-compact roots:

$$\beta_{ij} \ \equiv\ \epsilon_i + \epsilon_j, \ , \ \ 1 \leq i \leq j \leq n\ , \tag{4.5}$$

the Harish-Chandra parameters: $m_\beta \equiv (\Lambda+\rho,\beta)$ for the noncompact roots are:

$$\begin{aligned} m_{\beta_{ij}} \ &=\ \Big(\sum_{s=i}^{n} + \sum_{s=j}^{n}\Big) m_s\,, \qquad i<j\,, \\ m_{\beta_{ii}} \ &=\ \sum_{s=i}^{n} m_s \end{aligned} \tag{4.6}$$

The correspondence between the signatures $\chi$ and the highest weight $\Lambda$ is:

$$n_i = m_i\,, \quad c \ =\ -\tfrac{1}{2}(m_{\tilde\alpha} + m_n) \ =\ -\ \tfrac{1}{2}(m_1 + \cdots + m_{n-1} + 2m_n) \tag{4.7}$$

where $\tilde\alpha = \beta_{11}$ is the highest root.

The number of ERs in the corresponding multiplets by (??) is:

$$|W(\mathcal{G}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}})| \,/\, |W(\mathcal{M}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}}_m)| \ =\ 2^n \tag{4.8}$$

Below we give pictorially the multiplets for $sp(n,\mathbb{R})$ for $n = 3,4,5,6$. For $n = 2r$ these are also multiplets for $sp(r,r)$, $r = 1,2,3$ with parabolic $\mathcal{M}$-factor $su^*(2r)$.

Note that the cases $n = 1,2$ were already considered recalling that $sp(1,\mathbb{R}) \cong sl(2,\mathbb{R})$, $sp(2,\mathbb{R}) \cong so(3,2)$). Also the case $sp(1,1)$ was considered recalling that $sp(1,1) \cong so(4,1)$.

**Also note that the diagram for $sp(3, \mathbb{R})$ looks similar to the one for $so(6, 2)$, however the parametrizations are obviously different as the ranks are different.**

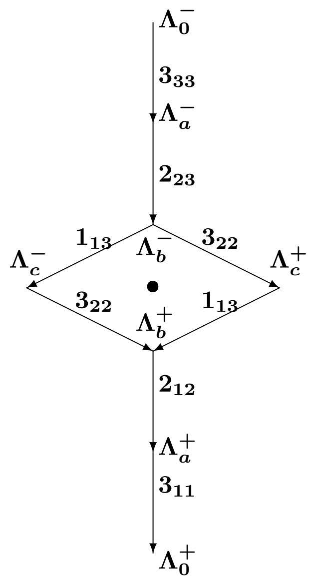



**Fig. 7. Main multiplets for** $sp(3, I\!R)$

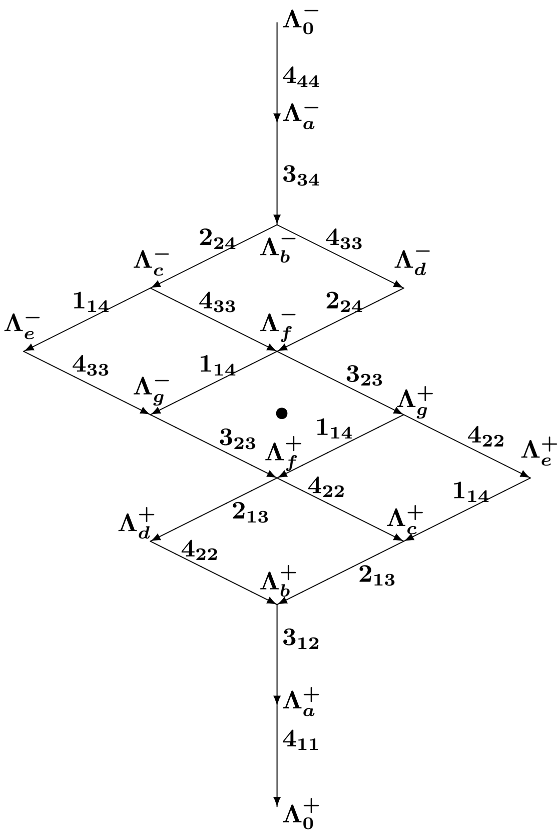



**Fig. 8. Main multiplets for $sp(4,\mathbb{R})$ and $sp(2,2)$ with parabolic $\mathcal{M}$-factors $sl(4,\mathbb{R})$, $su^*(4)$, resp.**

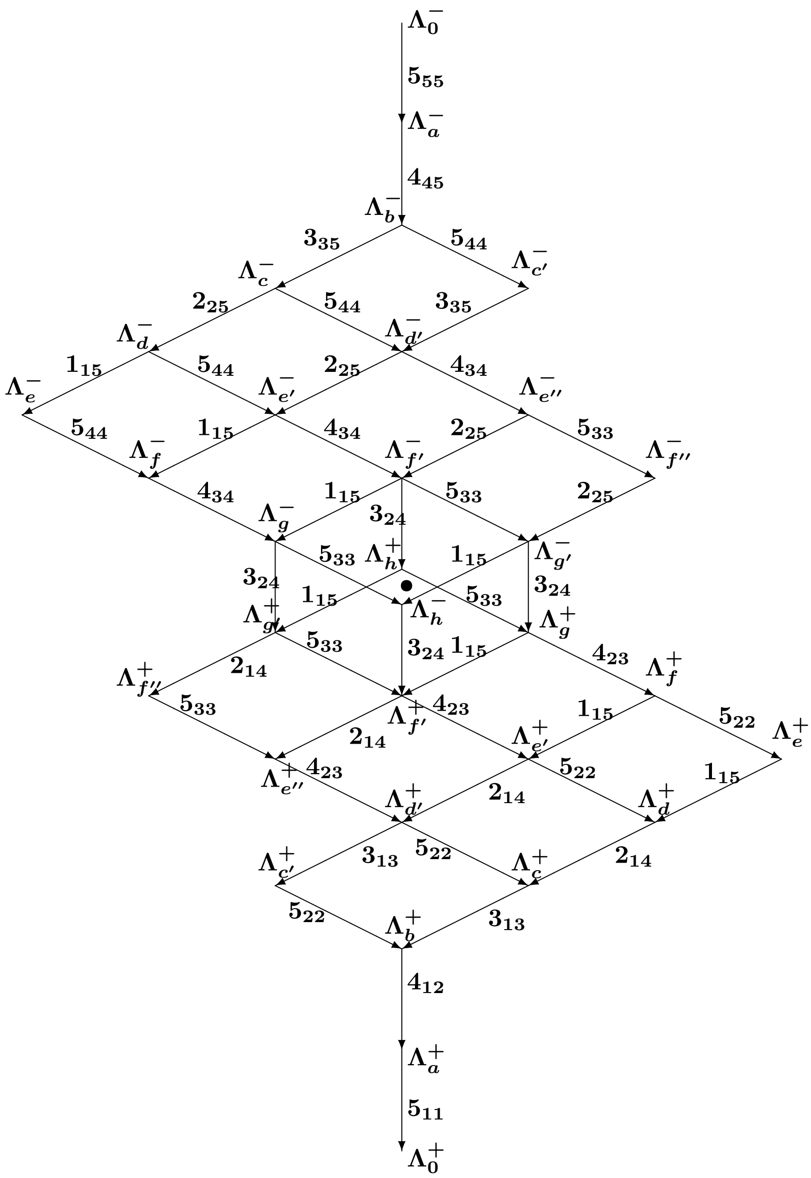



**Fig. 9. Main multiplets for $sp(5,\mathbb{R})$**

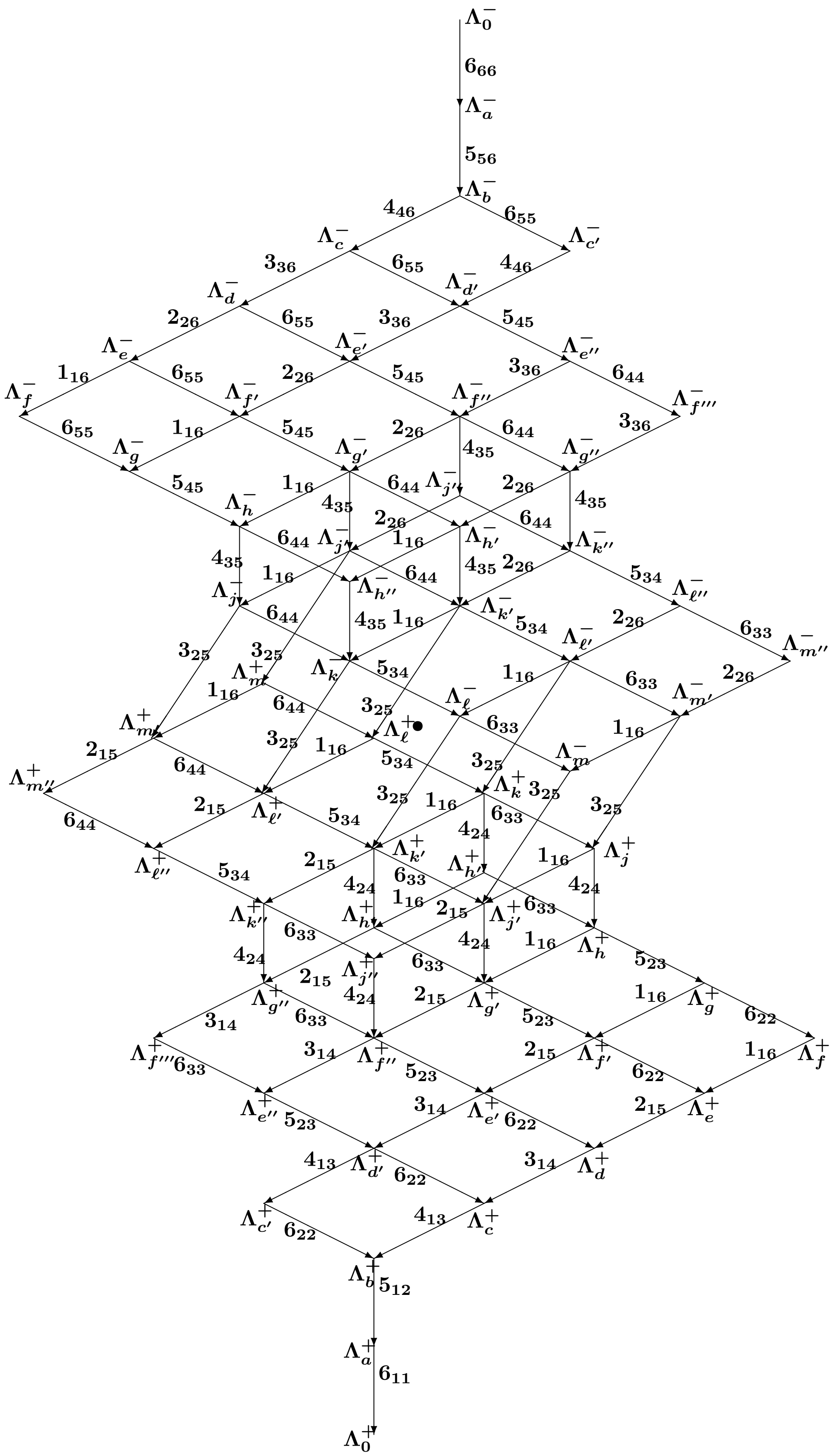


**Fig. 10.** **Main multiplets for $Sp(6,\mathbb{R})$ and $Sp(3,3)$ with parabolic $\mathcal{M}$-factors $sl(6,\mathbb{R})$, $su^*(6)$, resp.**

## 5. SO*(4n) case

Let $\mathcal{G} = so^*(4n)$. We choose a maximal parabolic $\mathcal{P} = \mathcal{MAN}$ such that $\mathcal{A} \cong so(1,1)$, $\mathcal{M} = su^*(2n)$. Since the algebras $so^*(4n)$ belong to the class called 'conformal Lie algebras' we have:

$$\mathcal{K}^{\mathbb{C}} \cong u(1)^{\mathbb{C}} \oplus sl(2n, \mathbb{C}) \cong \mathcal{A}^{\mathbb{C}} \oplus \mathcal{M}^{\mathbb{C}} \tag{5.1}$$

Here we have the series of algebras: $so^*(4)$, $so^*(8)$, $so^*(12)$, ...
However the first two cases are reduced to well known conformal algebras due to the coincidences: $so^*(4) \cong so(3) \oplus so(2,1)$, $so^*(8) \cong so(6,2)$.

Thus, we shall study the algebra $\mathcal{G} \equiv so^*(12)$.

We label the signature of the ERs of $\mathcal{G}_6$ as follows:

$$\chi = \{ n_1 , n_2 , n_3 , n_4 , n_5 ; c \} , \qquad n_j \in \mathbb{Z}_+ , \quad c = d - \tfrac{15}{2} \tag{5.2}$$

where the last entry of $\chi$ labels the characters of $\mathcal{A}$, and the first five entries are labels of the finite-dimensional nonunitary irreps of $su^*(6)$.

Finally, we remind that the above considerations are applicable also for the parabolically related algebra $so(6,6)$ with parabolic $\mathcal{M}$-factor $sl(6,\mathbb{R})$. It has discrete series representations but no highest/lowest weight representations.

The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of $so^*(12)$, i.e., they are labelled by the six positive Dynkin labels $m_i \in \mathbb{N}$. The number of ERs/GVMs in the main multiplets is:

$$|W(\mathcal{G}_6^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| \,/\, |W(\mathcal{M}_6^{\mathbb{C}}, \mathcal{H}_m^{\mathbb{C}})| = |W(so(12,\mathbb{C}))| \,/\, |W(sl(6,\mathbb{C}))| = 32 \tag{5.3}$$

where $\mathcal{H}^{\mathbb{C}}, \mathcal{H}_m^{\mathbb{C}}$ are Cartan subalgebras of $\mathcal{G}_6^{\mathbb{C}}, \mathcal{M}_6^{\mathbb{C}}$, resp.

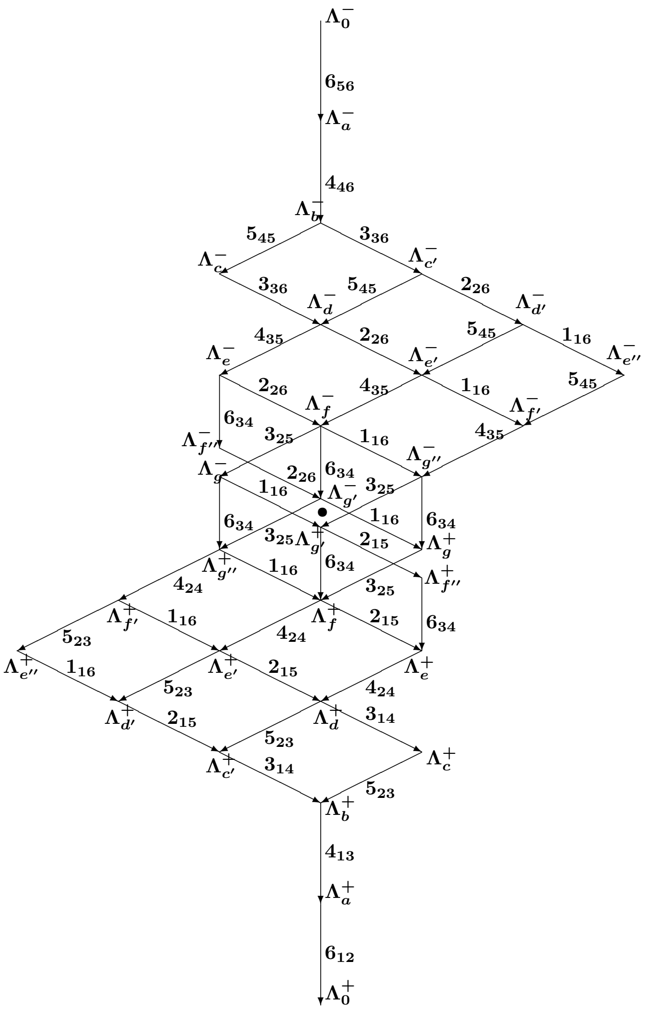



**Fig. 11. Main multiplets for $so^*(12)$ and $so(6,6)$ with parabolic $\mathcal{M}$ factors $su^*(6)$, $sl(6,\mathbb{R})$, resp.**

## 6. The Lie algebras $E_{7(-25)}$ and $E_{7(7)}$

Let $\mathcal{G} = E_{7(-25)}$. The maximal compact subgroup is $\mathcal{K} \cong e_6 \oplus so(2)$. We work with maximal parabolic $\mathcal{P} = \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{N}$ with $\mathcal{M} \cong E_{6(-26)}$.

The signatures of the ERs of $\mathcal{G}$ are:

$$\chi = \{ n_1 , \ldots , n_6 ; c \} , \qquad n_j \in \mathbb{N} , \tag{6.1}$$

expressed through the Dynkin labels:

$$n_i = m_i , \quad c = -\tfrac{1}{2}(m_{\tilde{\alpha}} + m_7) = -\tfrac{1}{2}(2m_1 + 2m_2 + 3m_3 + 4m_4 + 3m_5 + 2m_6 + 2m_7)$$

The same holds for the parabolically related exceptional Lie algebra $E_{7(7)}$ (with $\mathcal{M}$-factor $E_{6(6)}$).

The noncompact roots of the complex algebra $E_7$ are:

$$\begin{aligned} &\alpha_7 , \ \ \alpha_{17} , \ \ \ldots , \ \alpha_{67} , \\ &\alpha_{1,37} , \ \alpha_{2,47} , \ \alpha_{17,4} , \ \alpha_{27,4} , \\ &\alpha_{17,34} , \ \alpha_{17,35} , \ \alpha_{17,36} , \ \alpha_{17,45} , \ \alpha_{17,46} , \\ &\alpha_{27,45} , \ \alpha_{27,46} , \\ &\alpha_{17,25,4} , \ \alpha_{17,26,4} , \ \alpha_{17,35,4} , \ \alpha_{17,36,4} , \\ &\alpha_{17,26,45} , \ \alpha_{17,36,45} , \\ &\alpha_{17,26,35,4} , \ \alpha_{17,26,45,4} , \\ &\alpha_{17,16,35,4} = \tilde{\alpha} , \end{aligned} \tag{6.2}$$

using compact notation:

$$\begin{aligned} \alpha_{ij} &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_j , \ i < j , \\ \alpha_{ij,k} = \alpha_{k,ij} &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_j + \alpha_k , \qquad i < j , \qquad \text{etc.} \end{aligned} \tag{6.3}$$

The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of $E_7$, i.e., they will be labelled by the seven positive Dynkin labels $m_i \in \mathbb{N}$. The number of ERs in the corresponding multiplets by (??) is equal to:

$$|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| \,/\, |W(\mathcal{M}^{\mathbb{C}}, \mathcal{H}_m^{\mathbb{C}})| = 56 \tag{6.4}$$

The Knapp-Stein operators $G^{\pm}_{\chi}$ act pictorially as reflection w.r.t. the bullet intertwining each $\mathcal{T}^-_{\chi}$ member with the corresponding $\mathcal{T}^+_{\chi}$ member.

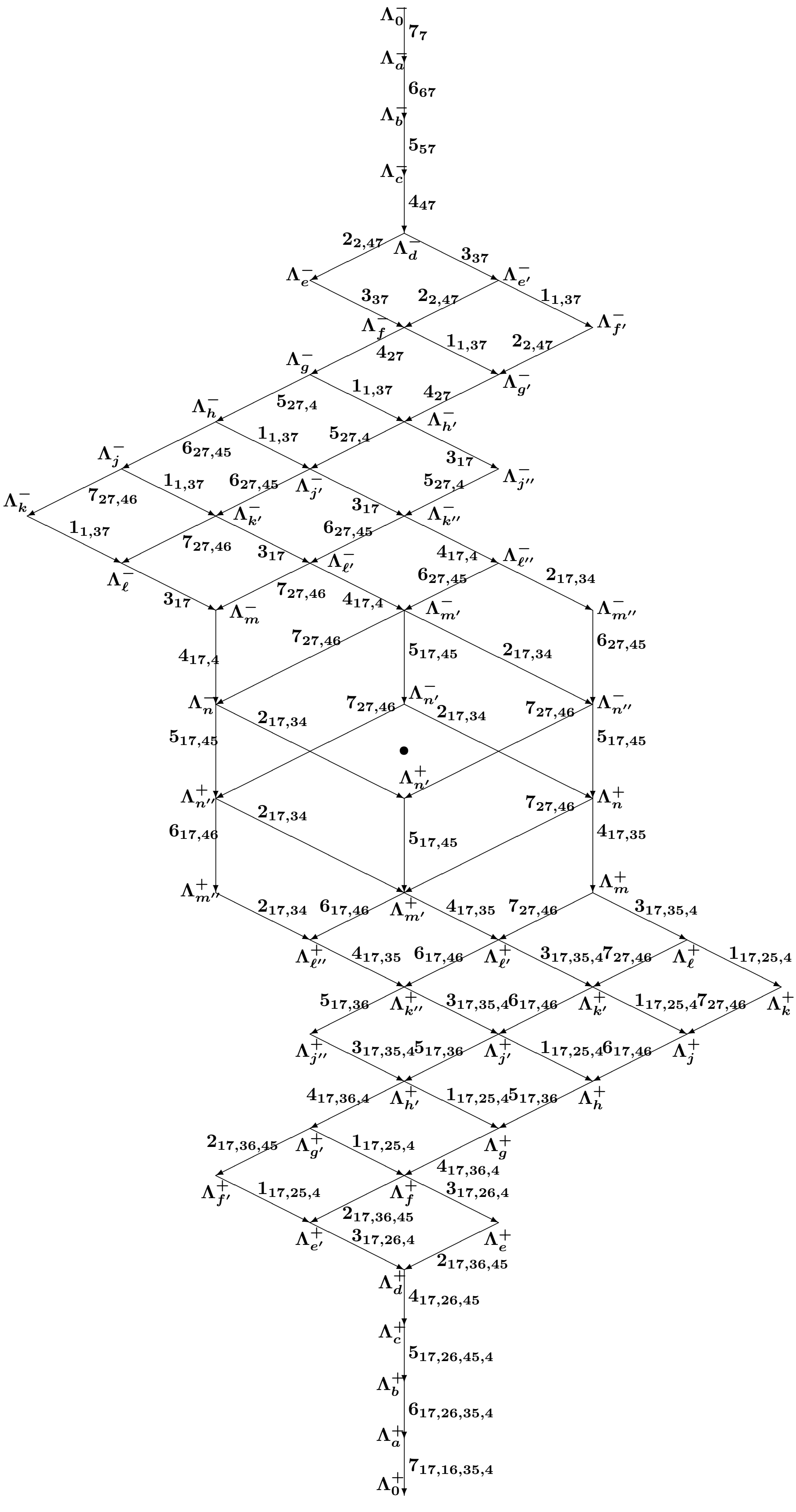


Fig. 12. Main Type for $E_{7(-25)}$ and $E_{7(7)}$

## 7. The Lie algebras $E_{6(-14)}$, $E_{6(6)}$ and $E_{6(2)}$

Let $\mathcal{G} = E_{6(-14)}$ . The maximal compact subalgebra is $\mathcal{K} \cong so(10) \oplus so(2)$, while $\mathcal{M} \cong su(5,1)$.

The signature of the ERs of $\mathcal{G}$ is:

$$\chi = \{ n_1 , n_3 , n_4 , n_5 , n_6 ; c \} , \quad c = d - \tfrac{11}{2} , \tag{7.1}$$

expressed through the Dynkin labels as:

$$n_i = m_i , \quad -c = \tfrac{1}{2} m_{\tilde{\alpha}} = \tfrac{1}{2}(m_1 + 2m_2 + 2m_3 + 3m_4 + 2m_5 + m_6) \tag{7.2}$$

The same holds for the parabolically related exceptional Lie algebras $E_{6(6)}$ and $E_{6(2)}$ with $\mathcal{M}$–factors $sl(6,\mathbb{R})$ and $su(3,3)$, resp.

Further, we need the noncompact roots of the complex algebra $E_6$ :

$$\begin{gathered} \alpha_2 , \ \alpha_{14} , \ \alpha_{15} , \ \alpha_{16} , \ \alpha_{24} , \ \alpha_{25} , \ \alpha_{26} \\ \alpha_{2,4} , \ \alpha_{2,45} , \ \alpha_{2,46} , \ \alpha_{25,4} , \ \alpha_{15,4} , \ \alpha_{26,4} \\ \alpha_{16,4} , \ \alpha_{15,34} , \ \alpha_{26,45} , \ \alpha_{16,34} , \ \alpha_{16,45} \\ \alpha_{16,35} , \ \alpha_{16,35,4} , \ \alpha_{16,25,4} = \tilde{\alpha} \end{gathered} \tag{7.3}$$

The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of $\mathcal{G}$, i.e., they will be labelled by the six positive Dynkin labels $m_i \in \mathbb{N}$. It turns out that each such multiplet contains 70 ERs/GVMs - see the figure below.

The Knapp-Stein operators $G^\pm_\chi$ act pictorially as reflection w.r.t. the dotted line separating the $\mathcal{T}^-_\chi$ members from the $\mathcal{T}^+_\chi$ members.

Note that there are five cases when the embeddings correspond to the highest root $\tilde{\alpha}$ : $V^{\Lambda^-} \longrightarrow V^{\Lambda^+}$, $\Lambda^+ = \Lambda^- - m_{\tilde{\alpha}}\,\tilde{\alpha}$ . In these five cases the weights are denoted as: $\Lambda^\pm_{k''}$ , $\Lambda^\pm_{k'}$ , $\Lambda^\pm_{\tilde{k}}$ , $\Lambda^\pm_{k}$ , $\Lambda^\pm_{k^o}$ , then: $m_{\tilde{\alpha}} = m_1, m_3, m_4, m_5, m_6$ , resp. Thus, their action coincides with the action of the Knapp-Stein operators $G^+_\chi$ which in the above five cases degenerate to differential operators as we discussed for $so(3,2)$.

Note that the figure has the standard $E_6$ symmetry, namely, conjugation exchanging indices $1 \longleftrightarrow 6$, $3 \longleftrightarrow 5$.

Full details are given in [11].

[11] V.K. Dobrev, in: Proceedings, 5th Mathematical Physics Meeting: Summer School and Conference on Modern Mathematical Physics, Belgrade, 6-17.07.2008, Eds. B. Dragovich, Z. Rakic, (Institute of Physics, Belgrade, 2009) pp. 95-124.

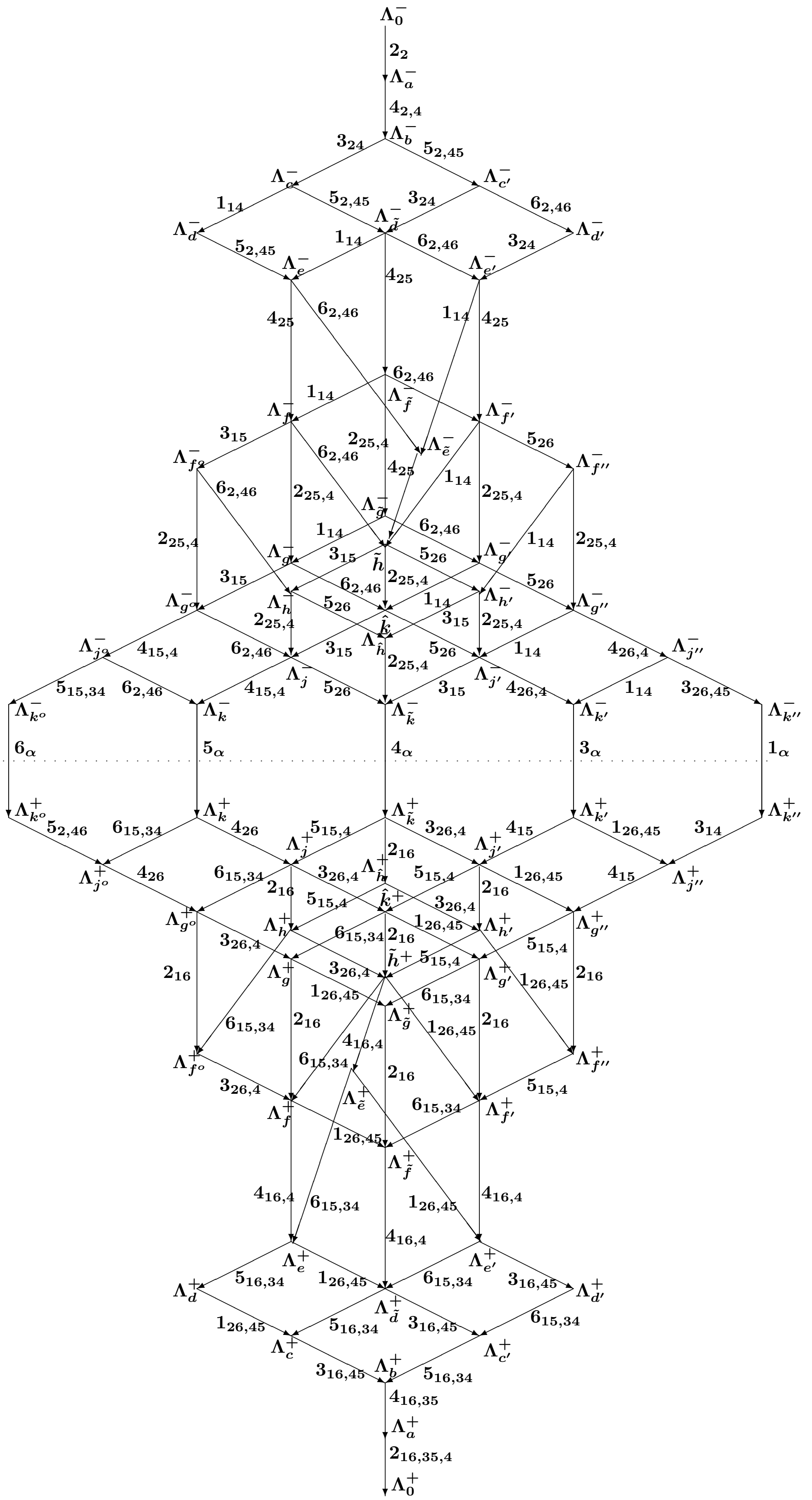


**Fig. 13. Main Type for $E_{6(-14)}$, $E_{6(6)}$ and $E_{6(2)}$ with parabolic $\mathcal{M}$-factors $su(5,1)$, $sl(6,\mathbb{R})$, $su(3,3)$, resp.**

## 8. Heisenberg Parabolic Subgroups of Exceptional Noncompact $G_{2(2)}$

Let $\mathcal{G}^{\mathbb{C}} = G_2$, with Cartan matrix: $(a_{ij}) = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$, simple roots $\alpha_1, \alpha_2$ with products: $(\alpha_2,\alpha_2) = 3(\alpha_1,\alpha_1) = -2(\alpha_2,\alpha_1)$. We choose $(\alpha_1,\alpha_1) = 2$, then $(\alpha_2,\alpha_2) = 6$, $(\alpha_2,\alpha_1) = -3$. As we know $G_2$ is 14–dimensional. The positive roots may be chosen as:

$$\Delta^+ = \{\alpha_1,\ \ \alpha_2,\ \ \alpha_3 = \alpha_1 + \alpha_2,\ \ \alpha_4 = \alpha_2 + 2\alpha_1,\ \ \alpha_5 = \alpha_2 + 3\alpha_1,\ \ \alpha_6 = 2\alpha_2 + 3\alpha_1\} \tag{8.1}$$

The Weyl group $W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})$ of $G_2$ is the dihedral group of order 12.

The complex Lie algebra $G_2$ has one non-compact real form: $\mathcal{G} = G_{2(2)}$ which is naturally split. Its maximal compact subalgebra is $\mathcal{K} = su(2) \oplus su(2)$, also written as $\mathcal{K} = su(2)_S \oplus su(2)_L$ to emphasize the relation to the root system (after complexification the first factor contains a short root, the second - a long root). We remind that $\mathcal{G} = G_{2(2)}$ has discrete series representations. Actually, it is quaternionic discrete series since $\mathcal{K}$ contains as direct summand (at least one) $su(2)$ subalgebra. The number of discrete series is equal to the ratio $|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})|/|W(\mathcal{K}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})|$, where $\mathcal{H}$ is a compact Cartan subalgebra of both $\mathcal{G}$ and $\mathcal{K}$, $W$ are the relevant Weyl groups. Thus, the number of discrete series in our setting is three. One case will be explicitly identified below.

The compact Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$ will be chosen to coincide with the Cartan subalgebra of $\mathcal{K}$ and we may write: $\mathcal{H} = u(1)_S \oplus u(1)_L$ .

The minimal parabolic of $\mathcal{G}$ is:

$$\mathcal{P}_0 \ = \ \mathcal{M}_0 \oplus \mathcal{A}_0 \oplus \mathcal{N}_0 \ = \ \mathcal{A}_0 \oplus \mathcal{N}_0 \tag{8.2}$$

There are two isomorphic maximal cuspidal parabolic subalgebras of $\mathcal{G}$ which are of Heisenberg type:

$$\begin{aligned} \mathcal{P}_k \ &= \ \mathcal{M}_k \oplus \mathcal{A}_k \oplus \mathcal{N}_k, \quad k = 1,2; \\ \mathcal{M}_k &= sl(2,\mathbb{R})_k, \quad \dim \mathcal{A}_k = 1, \quad \dim \mathcal{N}_k = 5 \end{aligned} \tag{8.3}$$

Let us denote by $\mathcal{T}_k$ the compact Cartan subalgebra of $\mathcal{M}_k$. Then $\mathcal{H}_k = \mathcal{T}_k \oplus \mathcal{A}_k$ is a non-compact Cartan subalgebra of $\mathcal{G}$. We choose $\mathcal{T}_1$ to be generated by the short $\mathcal{K}$-compact root $\alpha_1 + \alpha_2$ and $\mathcal{A}_1$ to be generated by the long root $\alpha_2$, while $\mathcal{T}_2$ to be generated by the long $\mathcal{K}$-compact root $\alpha_2 + 3\alpha_1$ and $\mathcal{A}_2$ to be generated by the short root $\alpha_1$.

Equivalently, the $\mathcal{M}_1$-compact root of $\mathcal{G}^{\mathbb{C}}$ is $\alpha_1 + \alpha_2$, while the $\mathcal{M}_2$-compact root is $\alpha_2 + 3\alpha_1$. In each case the remaining five positive roots of $\mathcal{G}^{\mathbb{C}}$ are $\mathcal{M}_k$-noncompact.

To characterize the Verma modules we shall use first the Dynkin labels:

$$m_i \ \equiv \ (\Lambda + \rho, \alpha_i^\vee), \quad i = 1,2, \tag{8.4}$$

where $\rho$ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$. Thus, we shall use :

$$\chi_\Lambda \ = \ \{m_1, m_2\} \tag{8.5}$$

Note that when both $m_i \in \mathbb{N}$ then $\chi_\Lambda$ characterizes the finite-dimensional irreps of $\mathcal{G}^{\mathbb{C}}$ and its real forms, in particular, $\mathcal{G}$. Furthermore, $m_k \in \mathbb{N}$ characterizes the finite-dimensional irreps of the $\mathcal{M}_k$ subalgebra.

We shall use also the Harish-Chandra parameters:

$$m_\beta \ = \ (\Lambda + \rho, \beta^\vee) \ , \tag{8.6}$$

for any positive root $\beta$, and explicitly in terms of the Dynkin labels:

$$\begin{aligned} \chi_{HC} \quad = \quad \{ \ m_1, \quad m_3 = 3m_2 + m_1, \quad m_4 = 3m_2 + 2m_1 \qquad & (8.7a) \\ m_2, \quad m_5 = m_2 + m_1, \quad m_6 = 2m_2 + m_1, \} \qquad & (8.7b) \end{aligned}$$

*8.1. Induction from minimal parabolic*

The main multiplets are in 1-to-1 correspondence with the finite-dimensional irreps of $G_2$, i.e., they are labelled by the two positive Dynkin labels $m_i \in \mathbb{N}$.

Using this labelling the signatures may be given in the following pair-wise manner:

$$\begin{aligned} \chi_0^\pm \ &= \ \{\mp m_1, \mp m_2; \ \pm \tfrac{1}{2}(2m_2 + m_1)\} \qquad (8.8) \\ \chi_2^\pm \ &= \ \{\mp(3m_2 + m_1), \pm m_2; \ \pm \tfrac{1}{2}(m_2 + m_1)\}, \\ \chi_1^\pm \ &= \ \{\pm m_1, \mp(m_2 + m_1); \ \pm \tfrac{1}{2}(2m_2 + m_1)\}, \\ \chi_{12}^\pm \ &= \ \{\mp(3m_2 + 2m_1), \pm(m_2 + m_1); \ \pm \tfrac{1}{2}m_2\} \\ \chi_{21}^\pm \ &= \ \{\pm(3m_2 + m_1), \mp(2m_2 + m_1); \ \pm \tfrac{1}{2}(m_2 + m_1)\} \\ \chi_{121}^\pm \ &= \ \{\mp(3m_2 + 2m_1), \pm(2m_2 + m_1); \ \mp \tfrac{1}{2}m_2\}, \end{aligned}$$

We have included as third entry also the parameter $c \ = \ -\frac{1}{2}(2m_2 + m_1)$, related to the Harish-Chandra parameter of the highest root (recalling that $m_{\alpha_6} = 2m_2 + m_1$). It is also related to the conformal weight $d = \frac{3}{2} + c$.

The ERs in the multiplet are related also by intertwining integral Knapp-Stein operators. These operators are defined for any ER, the general action in our situation being:

$$\begin{aligned} & G_{KS} \ : \ \mathcal{C}_\chi \ \longrightarrow \ \mathcal{C}_{\chi'} \ , \\ & \chi \ = \ [\, n_1, n_2 \, ; \, c \,] \ , \qquad \chi' \ = \ [\, -n_1, -n_2 \, ; \, -c \,]. \end{aligned} \tag{8.9}$$

The main multiplets are given explicitly in the next figure:

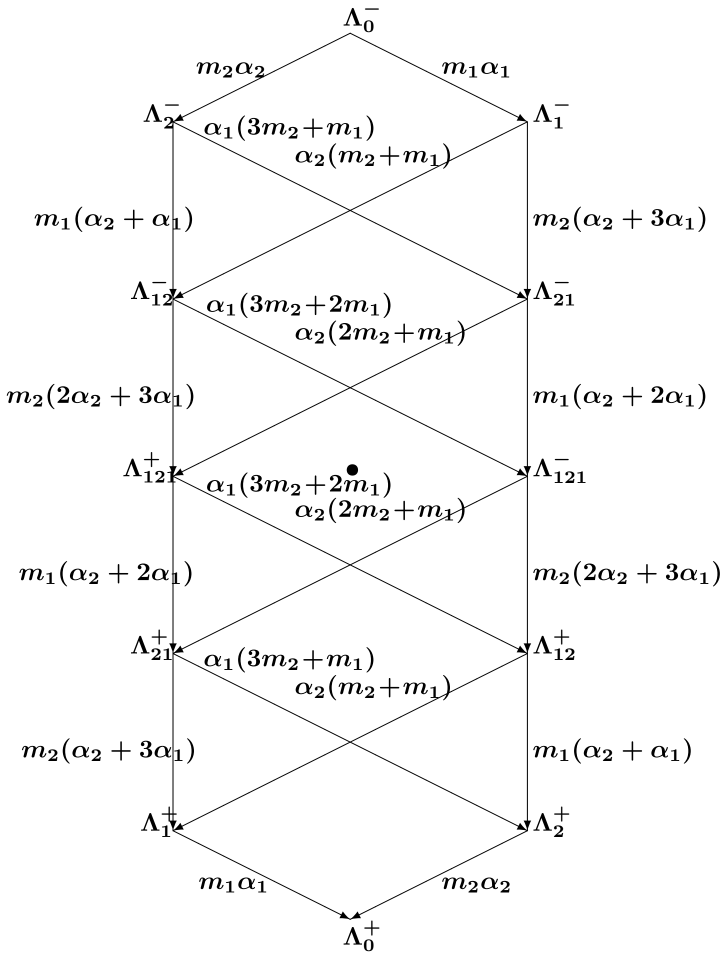



**Fig. 14. Main multiplets for $G_{2(2)}$ using induction from the minimal parabolic**

The pairs $\chi^\pm$ are symmetric w.r.t. the bullet in the middle of the picture - this symbolizes the Weyl symmetry realized by the Knapp-Stein operators (??): $G^\pm\ :\ \mathcal{C}_{\chi^\mp} \longrightarrow \mathcal{C}_{\chi^\pm}$ .

Some comments are in order.

Matters are arranged so that in every multiplet only the ER with signature $\chi_0^-$ contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional irrep of $G_{2(2)}$ with signature $[m_1, m_2]$. The subspace $\mathcal{E}$ is annihilated by the operators $G^+$ , $\mathcal{D}^{m_1}_{\alpha_1}$, $\mathcal{D}^{m_2}_{\alpha_2}$ and is the image of the operator $G^-$ .
When both $m_i = 1$ then dim $\mathcal{E} = 1$, and in that case $\mathcal{E}$ is also the trivial one-dimensional UIR of the whole algebra $\mathcal{G}$. Furthermore in that case the conformal weight is zero: $d = \frac{3}{2} + c = \frac{3}{2} - \frac{1}{2}(2m_2 + m_1)_{|m_i=1} = 0$.

In the conjugate ER $\chi_0^+$ there is a unitary discrete series representation (according to the Harish-Chandra criterion) in an infinite-dimensional subspace $\tilde{\mathcal{D}}_0$ with conformal weight $d = \frac{3}{2} + c = \frac{3}{2} + \frac{1}{2}(2m_2 + m_1)\ =\ 3, \frac{7}{2}, 4, ....$ It is annihilated by the operator $G^-$, and is in the intersection of the images of the operators $G^+$ (acting from $\chi_0^-$), $\mathcal{D}^{m_1}_{\alpha_1}$ (acting from $\chi_1^+$), $\mathcal{D}^{m_2}_{\alpha_2}$ (acting from $\chi_2^+$).

Full details are given in [12].

*8.2. Induction from maximal parabolics*

When inducing from the maximal parabolic $\mathcal{P}_1 = \mathcal{M}_1 \oplus \mathcal{A}_1 \oplus \mathcal{N}_1$ there is one $\mathcal{M}_1$-compact root, namely, $\alpha_1$. We take again the Verma module with $\Lambda_{HC} = \Lambda_0^{1-}$. We take $\chi_0^{1-} = \chi_{HC}$. Altogether, the main multiplet in this case includes the same number of ERs/GVMs as in (??), so we may use the same notation only adding super index 1, but in order to avoid coincidence with (??) we must impose the conditions: $m_1 \notin \mathbb{N}$, $m_1 \notin \mathbb{N}/2$.

What is peculiar is that the ERs/GVMs of the main multiplet here actually

[12] V.K. Dobrev, Symmetry **14** (4) 660 (2022).

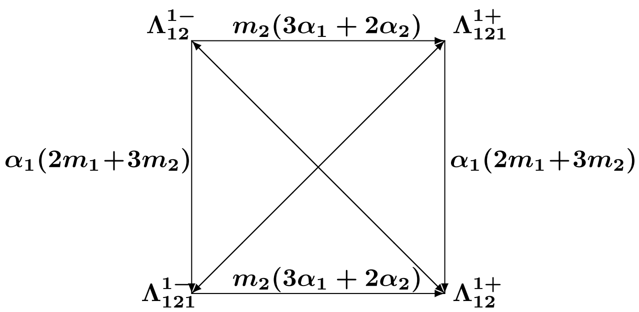
consists of three submultiplets with intertwining diagrams as follows:

$$\begin{array}{ccc} \Lambda_0^{1-} & \stackrel{\mathcal{D}_{\alpha_2}^{m_2}}{\longrightarrow} & \Lambda_2^{1-} \\ \updownarrow & & \updownarrow \\ \Lambda_0^{1+} & \stackrel{\mathcal{D}_{\alpha_2}^{m_2}}{\longleftarrow} & \Lambda_2^{1+} \end{array} \quad \text{subtype } (\mathbf{A_1}) \tag{8.10a}$$

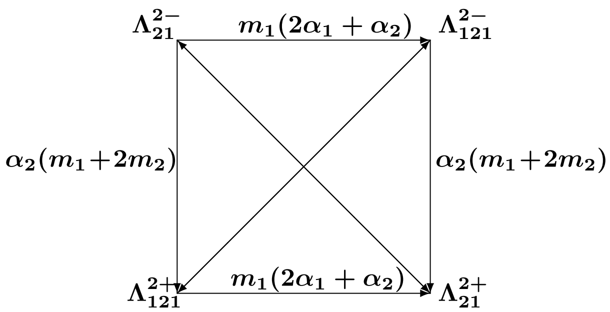
$$\begin{array}{ccc} \Lambda_1^{1-} & \stackrel{\mathcal{D}_{\alpha_5}^{m_2}}{\longrightarrow} & \Lambda_{21}^{1-} \\ \updownarrow & & \updownarrow \\ \Lambda_1^{1+} & \stackrel{\mathcal{D}_{\alpha_5}^{m_2}}{\longleftarrow} & \Lambda_{21}^{1+} \end{array} \quad \text{subtype } (\mathbf{B_1}) \tag{8.10b}$$

$$\begin{array}{ccc} \Lambda_{12}^{1-} & \stackrel{\mathcal{D}_{\alpha_6}^{m_2}}{\longrightarrow} & \Lambda_{121}^{1+} \\ \updownarrow & & \updownarrow \\ \Lambda_{12}^{1+} & \stackrel{\mathcal{D}_{\alpha_6}^{m_2}}{\longleftarrow} & \Lambda_{121}^{1-} \end{array} \quad \text{subtype } (\mathbf{C_1}) \tag{8.10c}$$

**Next we relax one of the conditions, namely, we allow $m_1 \in \mathbb{N}/2$, still keeping $m_2 \in \mathbb{N}$, $m_1 \notin \mathbb{N}$. This changes the diagram of subtype $(C_1)$, (??c), as given in Fig. 15. :**




$C_1$




$C_2$

**Fig. 15. Subdiagrams $C_1$ and $C_2$ of $G_{2(2)}$ multiplets using induction from maximal parabolics $\mathcal{P}_1$, $\mathcal{P}_2$, resp.**

**Inducing from the other maximal parabolic $\mathcal{P}_2$ is partly dual to the previous one. The main multiplet is given as (??) only adding superscript 2 but in order to avoid coincidence with (??) we must impose the conditions: $m_2 \notin \mathbb{N}$, $m_2 \notin \mathbb{N}/2$, $m_2 \notin \mathbb{N}/3$.**

**Similarly to the $\mathcal{P}_1$ case the ERs/GVMs of the main miltiplet here actually consists of three submultiplets with intertwining diagrams as follows:**

$$\begin{array}{ccc} \Lambda_0^{2-} & \stackrel{\mathcal{D}_{\alpha_1}^{m_1}}{\longrightarrow} & \Lambda_1^{2-} \\ \updownarrow & & \updownarrow \\ \Lambda_0^{2+} & \stackrel{\mathcal{D}_{\alpha_1}^{m_1}}{\longleftarrow} & \Lambda_1^{2+} \end{array} \quad \textbf{subtype } (\mathbf{A_2}) \tag{8.11a}$$

$$\begin{array}{ccc} \Lambda_2^{2-} & \stackrel{\mathcal{D}_{\alpha_3}^{m_1}}{\longrightarrow} & \Lambda_{12}^{2-} \\ \updownarrow & & \updownarrow \\ \Lambda_2^{2+} & \stackrel{\mathcal{D}_{\alpha_3}^{m_1}}{\longleftarrow} & \Lambda_{12}^{2+} \end{array} \quad \textbf{subtype } (\mathbf{B_2}) \tag{8.11b}$$

$$\begin{array}{ccc} \Lambda_{21}^{2-} & \stackrel{\mathcal{D}_{\alpha_4}^{m_1}}{\longrightarrow} & \Lambda_{121}^{2-} \\ \updownarrow & & \updownarrow \\ \Lambda_{21}^{2+} & \stackrel{\mathcal{D}_{\alpha_4}^{m_1}}{\longleftarrow} & \Lambda_{121}^{2+} \end{array} \quad \textbf{subtype } (\mathbf{C_2}) \tag{8.11c}$$

**Next we relax one of the conditions, namely, we allow $m_2 \in \mathbb{N}/2$, still keeping $m_2 \notin \mathbb{N}$, $m_2 \notin \mathbb{N}/3$. This changes the diagram of subtype $(C_2)$, (??c), as given in Fig. 15.**

**Next we relax another condition, namely, we allow $m_2 \in \mathbb{N}/3$, still keeping $m_2 \notin \mathbb{N}$, $m_2 \notin \mathbb{N}/2$. This changes the diagrams of subtypes $(B_2)$ and $(C_2)$ combining them as given in the next figure:**

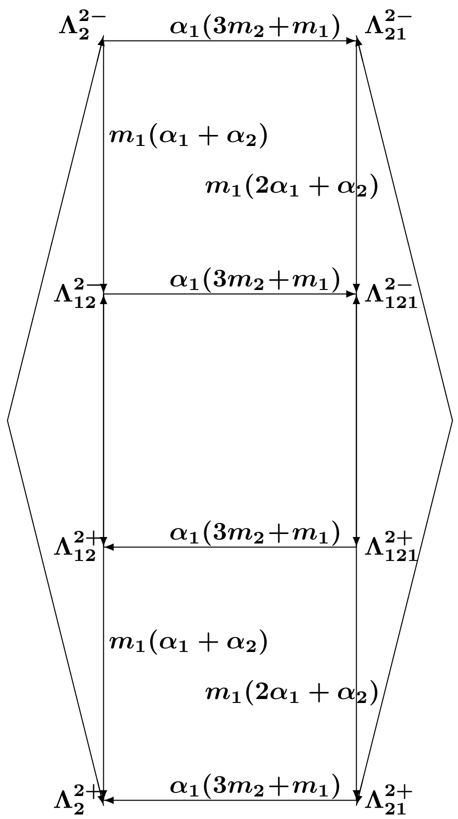



**Fig. 16. Combined subdiagrams $B_2$ and $C_2$ of $G_{2(2)}$ multiplets using induction from maximal parabolic $\mathcal{P}_2$ for $m_1 \in \mathbb{N}$, $m_2 \in \mathbb{N}/3$, $m_2 \notin \mathbb{N}$, $m_2 \notin \mathbb{N}/2$.**

## 9. Exceptional Lie Algebra $F_4'$

We start with the complex exceptional Lie algebra $\mathcal{G}^{\mathbb{C}} = F_4$. We use the standard definition of $\mathcal{G}^{\mathbb{C}}$ given in terms of the Chevalley generators $X_i^{\pm}$, $H_i$, $i = 1,2,3,4(=\mathrm{rank}\, F_4)$, by the relations :

$$[H_j\,,\,H_k] = 0\,,\ [H_j\,,\,X_k^{\pm}] = \pm a_{jk} X_k^{\pm}\,,\ [X_j^{+}\,,\,X_k^{-}] = \delta_{jk}\, H_j\,, \tag{9.1}$$

$$\textstyle\sum_{m=0}^{n} (-1)^m \binom{n}{m} \left(X_j^{\pm}\right)^m X_k^{\pm} \left(X_j^{\pm}\right)^{n-m} = 0\,,\ j \neq k\,,\ n = 1 - a_{jk}\,,$$

where

$$(a_{ij}) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} ; \tag{9.2}$$

is the Cartan matrix of $\mathcal{G}^{\mathbb{C}}$, $\alpha_j^{\vee} \equiv \frac{2\alpha_j}{(\alpha_j,\alpha_j)}$ is the co-root of $\alpha_j$, $(\cdot,\cdot)$ is the scalar product of the roots, so that the nonzero products between the simple roots are: $(\alpha_1,\alpha_1) = (\alpha_2,\alpha_2) = 2(\alpha_3,\alpha_3) = 2(\alpha_4,\alpha_4) = 2$, $(\alpha_1,\alpha_2) = -1$, $(\alpha_2,\alpha_3) = -1$, $(\alpha_3,\alpha_4) = -1/2$. The elements $H_i$ span the Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}^{\mathbb{C}}$, while the elements $X_i^{\pm}$ generate the subalgebras $\mathcal{G}^{\pm}$. We shall use the standard triangular decomposition

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}_+ \oplus \mathcal{H} \oplus \mathcal{G}_-\,, \qquad \mathcal{G}_{\pm} \equiv \mathop{\oplus}_{\alpha\in\Delta^{\pm}} \mathcal{G}_{\alpha}\,, \tag{9.3}$$

where $\Delta^+$, $\Delta^-$, are the sets of positive, negative, roots, resp. Explicitly we have that there are roots of two lengths with length ratio 2 : 1.

The long roots are: $\alpha_1$, $\alpha_2$, $\alpha_1+\alpha_2$, $\alpha_2+2\alpha_3$, $\alpha_1+\alpha_2+2\alpha_3$, $\alpha_1+2\alpha_2+2\alpha_3$, $\alpha_2+2\alpha_3+2\alpha_4$, $\alpha_1+\alpha_2+2\alpha_3+2\alpha_4$, $\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4$, $\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4$, $\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$, $2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$. With the chosen normalization they have length 2.

The short roots are: $\alpha_3$, $\alpha_4$, $\alpha_2+\alpha_3$, $\alpha_3+\alpha_4$, $\alpha_1+\alpha_2+\alpha_3$, $\alpha_2+\alpha_3+\alpha_4$, $\alpha_1+\alpha_2+\alpha_3+\alpha_4$, $\alpha_2+2\alpha_3+\alpha_4$, $\alpha_1+2\alpha_2+2\alpha_3+\alpha_4$, $\alpha_1+\alpha_2+2\alpha_3+\alpha_4$, $\alpha_1+2\alpha_2+3\alpha_3+\alpha_4$, $\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$, and they have length 1.

(Note that the short roots are exactly those which contain $\alpha_3$ and/or $\alpha_4$ with coefficient 1, while the long roots contain $\alpha_3$ and $\alpha_4$ with even coefficients.)

Thus, $F_4$ is 52–dimensional ($52 = |\Delta| +$ rank $F_4$).

In terms of the normalized basis $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$ we have:

$$\begin{aligned}\Delta^+ = \{\varepsilon_i,\ 1\le i\le 4;\ \varepsilon_j\pm\varepsilon_k,\ 1\le j<k\le 4;\\ \frac{1}{2}(\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4),\ \text{all signs}\}\ .\end{aligned} \tag{9.4}$$

The simple roots are:

$$\pi = \{\alpha_1 = \varepsilon_2-\varepsilon_3,\ \alpha_2 = \varepsilon_3-\varepsilon_4, \alpha_3 = \varepsilon_4,\ \alpha_4 = \frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)\}\ . \tag{9.5}$$

The maximal compact subalgebra is $\mathcal{K} = sp(3)\oplus su(2)$. Its complexification $\mathcal{K}^{\mathbb{C}}$ may be embedded most easily in $F_4$ as the Lie algebra generated by the subalgebras with simple roots $\{\alpha_2,\alpha_3,\alpha_4\}$ and $\{\alpha_1\}$. The long roots of $sp(3,\mathbb{C})$ in this embedding are: $\alpha_2$, $\alpha_2+2\alpha_3$, $\alpha_2+2\alpha_3+2\alpha_4$. The short roots are: $\alpha_3$, $\alpha_4$, $\alpha_2+\alpha_3$, $\alpha_3+\alpha_4$, $\alpha_2+\alpha_3+\alpha_4$, $\alpha_2+2\alpha_3+\alpha_4$.

Note that the 16 roots on the first line of (??) form the positive root system of $B_4$ with simple roots $\varepsilon_i - \varepsilon_{i+1}$ , $i = 1, 2, 3$, $\varepsilon_4$ .

The Weyl group of $F_4$ is the semidirect product of $S_3$ with a group which itself is the semidirect product of $S_4$ with $(\mathbb{Z}/2\mathbb{Z})^3$, thus, $|W| = 2^7\, 3^2 = 1152$.

*9.1. Structure theory of the real split form*

The real split form of $F_4$ is denoted as $F_4'$, sometimes as $F_{2(2)}$ . This real form has discrete series representations since $\mathrm{rank} F_4' = \mathrm{rank}\, \mathcal{K}$. We can use the same basis (but over $\mathbb{R}$) and the same root system.

The Iwasawa decomposition of the real split form $\mathcal{G} \equiv F_4'$, is:

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{A}_0 \oplus \mathcal{N}_0 \ , \tag{9.6}$$

the Cartan decomposition is:

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{Q}, \tag{9.7}$$

where we use: the maximal compact subalgebra $\mathcal{K} \cong sp(3) \oplus su(2)$, $\dim_{\mathbb{R}} \mathcal{Q} = 28$, $\dim_{\mathbb{R}} \mathcal{A}_0 = 4$, $\mathcal{N}_0 = \mathcal{N}_0^+$ , or $\mathcal{N}_0 = \mathcal{N}_0^- \cong \mathcal{N}_0^+$ , $\dim_{\mathbb{R}} \mathcal{N}_0^\pm = 24$.

Since $\mathcal{G}$ is maximally split, then the centralizer $\mathcal{M}_0$ of $\mathcal{A}_0$ in $\mathcal{K}$ is zero, thus, the minimal parabolic $\mathcal{P}_0$ and the corresponding Bruhat decomposition are:

$$\mathcal{P}_0 = \mathcal{A}_0 \oplus \mathcal{N}_0 \ , \qquad \mathcal{G} = \mathcal{A}_0 \oplus \mathcal{N}_0^+ \oplus \mathcal{N}_0^- \tag{9.8}$$

*9.2. Intertwining differential operators for $F_4'$*

The real form $F_4'$ has several parabolic subalgebras. We shall consider the maximal parabolic subalgebra:

$$\begin{aligned} \mathcal{P} &= \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N} \ , \\ \mathcal{M} &= sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}) \ , \\ \dim \mathcal{A} &= 1, \qquad \dim \mathcal{N} = 20 \end{aligned} \tag{9.9}$$

such that the embedding of $\mathcal{M}$ and $\mathcal{M}^{\mathbb{C}}$ in $\mathcal{G}^{\mathbb{C}}$ is given by:

$$sl(3,\mathbb{R})^{\mathbb{C}} : \{\alpha_1, \alpha_2, \alpha_{12} = \alpha_1 + \alpha_2\}, \qquad sl(2,\mathbb{R})^{\mathbb{C}} : \{\alpha_4\} \tag{9.10}$$

Remark: Note that $F_4'$ has a another maximal parabolic subalgebra that is also written as (??) but the embedding of $\mathcal{M}$ and $\mathcal{M}^{\mathbb{C}}$ flips the short and long roots:

$$sl(3,\mathbb{R})^{\mathbb{C}} : \{\alpha_3, \alpha_4, \alpha_{34} = \alpha_3 + \alpha_4\}, \qquad sl(2,\mathbb{R})^{\mathbb{C}} : \{\alpha_1\} \tag{9.11}$$

That case is also very interesting and was considered in [13]. $\diamondsuit$

The result of our classification is a follows. The multiplets of GVMs (and ERs) induced from (??) are parametrized by four positive integers $\chi = [m_1, m_2, m_3, m_4]$. Each multiplet contains 96 GVMs (ERs). They are presented in the next figure.

[13] V.K. Dobrev, in: Springer Proceedings in Mathematics and Statistics, Vol. 335 (Springer, Heidelberg-Tokyo, 2020) pp. 383-398.

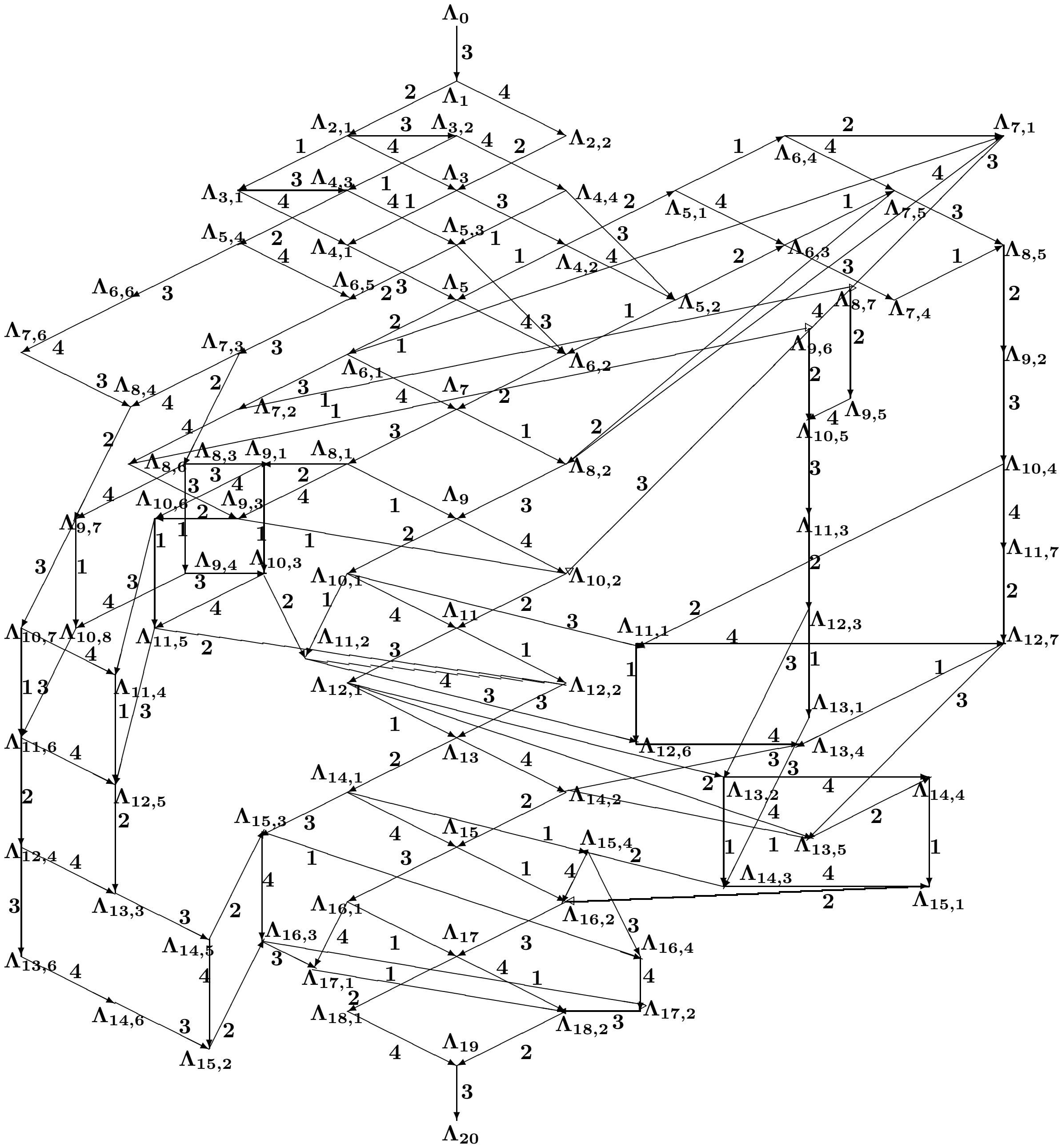



**Fig. 17.** Multiplets for the real split form $F'_4$ using maximal parabolic with $\mathcal{M} = sl(3,\mathbb{R})_{\text{long roots}} \oplus sl(2,\mathbb{R})_{\text{short roots}}$

On the figure each arrow represents an embedding between two Verma modules, $V^\Lambda$ and $V^{\Lambda'}$, the arrow pointing to the embedded module $V^{\Lambda'}$. Each arrow carries a number $n$, $n = 1,2,3,4$, which indicates the level of the embedding, $\Lambda' = \Lambda - m_n\,\beta$. Another feature is indicated by the enumeration of the GVMs (ERs). Namely, if $\Lambda$ corresponds to signature $\chi_{k,\ell}$, then $\Lambda'$ corresponds to signature $\chi_{k+1,\ell'}$ (where $\ell, \ell'$ are secondary enumerations that are absent in some cases).

Further, we mention the additional symmetry w.r.t. to the central point of the diagram (marked by a bullet) which indicates the integral intertwining Knapp-Stein (KS) operators acting between the ERs. Due to this symmetry in the actual parametrization we shall use the conformal weight $d = 7/2 + c$, more precisely, the parameter $c$, instead of the non-compact Dynkin label $m_3$. The parameter $c$ is more convenient since the KS operators just flip its sign. The KS operators also involve $sl(3)$ flip of the Dynkin labels $m_1, m_2$ (see below). Thus, the entries are:

$$\chi \;=\; \{n_1, n_2, c, n_4\} \tag{9.12}$$

so that for the top ER (GVM) on the figure $\Lambda_0^-$ we have:

$$\chi_0^- \;=\; \{n_1 = m_1, n_2 = m_2, c = -(m_1 + m_2 + m_3 + m_4/2), n_4 = m_4\} \tag{9.13}$$

Furthermore the $sl(3)$ flip $(n_1, n_2)^\pm$ will be given below by:

$$(n_1, n_2)^+ \;=\; (n_1, n_2), \qquad (n_1, n_2)^- \;=\; (n_2, n_1) \tag{9.14}$$

The explicit parametrization of the multiplets is given in [14].

*9.3. Concluding remark*

We expect that the discrete series are contained in the representation $\chi_0^+$ since it is dual to $\chi_0^-$ where are located the finite-dimensional (non-unitary) irreps. Following the Harish-Chandra criterion we must check which $\mathcal{M}$-non-compact entries are negative. We recall that the $\mathcal{M}$-compact entries are $m'_1, m'_2, m'_{12}, m'_4$ , all other are non-compact. It is easy to see that all the $\mathcal{M}$-non-compact entries are negative. The discrete series irrep with lowest possible conformal weight $d = 7$ happens naturally when $m_1 = m_2 = m_3 = m_4 = 1$. It corresponds to the one-dimensional irrep contained in $\chi_0^-$.

[14] V.K. Dobrev, in: Proceedings, of Workshop on Quantum Geometry, Field Theory and Gravity, Corfu, 18-25.9.2019; Volume 376, PoS (CORFU2019) (Published 2020) 233.

## 10. Exceptional Lie Algebra $F_4''$

The split real form of $F_4$ is denoted as $F_4''$, sometimes as $F_{4(-20)}$. It has rank four. Its maximal compact subalgebra is $\mathcal{K} \cong so(9)$, also of rank four. This real form has discrete series representations since $\mathrm{rank} F_4'' = \mathrm{rank}\,\mathcal{K}$. The number of discrete series is equal to the ratio $|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})|/|W(\mathcal{K}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})|$, where $\mathcal{H}$ is a compact Cartan subalgebra of both $\mathcal{G}$ and $\mathcal{K}$, $W$ are the relevant Weyl groups. Thus, the number of discrete series in our setting is three. They will be identified below. Here there is only one nontrivial parabolic: $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$, where $\mathcal{M} = so(7)$, $\dim_{\mathbb{R}} \mathcal{N} = 15$.

Note that the root system of $\mathcal{M}^{\mathbb{C}} = so(7,\mathbb{C}) = B_3$ consists of the roots

$$\Delta_3^+ = \{\varepsilon_i,\ 2 \le i \le 4;\ \varepsilon_j \pm \varepsilon_k,\ 2 \le j < k \le 4\} \qquad (10.1)$$

which are part of (??), while the simple roots are part of (??)

$$\pi_3 \ = \ \{\alpha_1 = \varepsilon_2 - \varepsilon_3, \quad \alpha_2 = \varepsilon_3 - \varepsilon_4, \quad \alpha_3 = \varepsilon_4\} \qquad (10.2)$$

The roots of $\mathcal{M}^{\mathbb{C}}$ are called $\mathcal{M}$-compact roots of the $F_4$ root system (??), the rest are called $\mathcal{M}$-noncompact roots. The latter give rise to intertwining differential operators, as explained below.

More explicitly, the $\mathcal{M}$-compact roots are:

$$\begin{aligned} &\alpha_1,\ \alpha_2,\ \alpha_1+\alpha_2 \equiv \alpha_{12},\ \alpha_2+2\alpha_3 \equiv \alpha_{23,3},\ \alpha_1+\alpha_2+2\alpha_3 \equiv \alpha_{13,3},\\ &\ \alpha_1+2\alpha_2+2\alpha_3 \equiv \alpha_{13,23}, && (10.3a)\\ &\alpha_3,\ \alpha_2+\alpha_3 \equiv \alpha_{23},\ \alpha_1+\alpha_2+\alpha_3 \equiv \alpha_{13}, && (10.3b)\end{aligned}$$

(??a) are long roots, (??b) - short.
The $\mathcal{M}$-noncompact roots are:

$$\begin{aligned} &\alpha_2+2\alpha_3+2\alpha_4 \equiv \alpha_{24,23},\ \alpha_1+\alpha_2+2\alpha_3+2\alpha_4 \equiv \alpha_{14,34},\\ &\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4 \equiv \alpha_{14,24},\\ &\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4 \equiv \alpha_{14,24,3,3},\ \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4 \equiv \alpha_{14,24,23,3},\\ &2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4 \equiv \alpha_{14,14,23,3} && (10.4a)\\ &\alpha_4,\ \alpha_3+\alpha_4 \equiv \alpha_{34},\ \alpha_2+\alpha_3+\alpha_4 \equiv \alpha_{24},\ \alpha_1+\alpha_2+\alpha_3+\alpha_4 \equiv \alpha_{14},\\ &\alpha_2+2\alpha_3+\alpha_4 \equiv \alpha_{24,3},\ \alpha_1+2\alpha_2+2\alpha_3+\alpha_4 \equiv \alpha_{14,23},\\ &\ \alpha_1+\alpha_2+2\alpha_3+\alpha_4 \equiv \alpha_{14,3},\ \alpha_1+2\alpha_2+3\alpha_3+\alpha_4 \equiv \alpha_{14,23,3},\\ &\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4 \equiv \alpha_{14,24,3} && (10.4b)\end{aligned}$$

(??a) are long roots, (??b) - short.

Correspondingly, the Dynkin labels $m_1, m_2, m_3$ are called $\mathcal{M}$-compact, while $m_4$ is called $\mathcal{M}$-noncompact.

The result of our classification is a follows. The multiplets of GVMs (and ERs) induced from $\mathcal{P}$ are parametrized by four positive integers - the Dynkin labels. Each multiplet contains 24 GVMs (ERs). These multiplets are presented in the figure below. On the figure each arrow represents an embedding between two Verma modules, $V^\Lambda$ and $V^{\Lambda'}$, the arrow pointing to the embedded module $V^{\Lambda'}$. Each arrow carries a number $n$, $n = 1,2,3,4$, which indicates the level of the embedding, $\Lambda' = \Lambda - m_n\,\beta$. By our construction it also represents the invariant differential operator $\mathcal{D}_{n,\beta}$.

Further, we use the additional symmetry w.r.t. to the dashed line in the figure which indicates the integral intertwining Knapp-Stein (KS) operators acting between the spaces $\mathcal{C}_{\chi^\mp}$ in opposite directions:

$$G^+_{KS} \ : \ \mathcal{C}_{\chi^-} \longrightarrow \mathcal{C}_{\chi^+} \ , \qquad G^-_{KS} \ : \ \mathcal{C}_{\chi^+} \longrightarrow \mathcal{C}_{\chi^-} \tag{10.5}$$

Note that the KS opposites are induced from the same irreps of $\mathcal{M}$.

This symmetry may be more explicit if we change the parametrization:

$$\{m_1, m_2, m_3, m_4\} \longrightarrow [m_1, m_2, m_3; c] \tag{10.6}$$

so that the action of the KS operators on this signature is:

$$G^\pm_{KS} \ : \ [m_1, m_2, m_3; c] \longrightarrow [m_1, m_2, m_3; -c] \tag{10.7}$$

This enables us to write the multiplet in a more compact way:

$$\begin{aligned}
\chi^\pm_0 &= [m_1, m_2, m_3; \pm(m_{14,2,4} + m_3/2)] \\
\chi^\pm_a &= [m_1, m_2, m_{34}; \pm\tfrac{1}{2}m_{14,13,23,2}] \\
\chi^\pm_b &= [m_1, m_{23}, m_4; \pm\tfrac{1}{2}m_{14,13,2,2}] \\
\chi^\pm_c &= [m_{12}, m_{23}, m_4; \pm\tfrac{1}{2}m_{14,13}] \\
\chi^\pm_d &= [m_2, m_{13}, m_4; \pm\tfrac{1}{2}m_{24,23}] \\
\chi^\pm_e &= [m_{13}, m_2, m_{34}; \pm\tfrac{1}{2}m_{14,12}] \\
\chi^\pm_f &= [m_{23}, m_{12}, m_{34}; \pm\tfrac{1}{2}m_{24,2}] \\
\chi^\pm_g &= [m_{14}, m_2, m_3; \pm\tfrac{1}{2}m_{13,12}] \\
\chi^\pm_h &= [m_{23}, m_1, m_{24,2}; \pm\tfrac{1}{2}m_{34}] \\
\chi^\pm_i &= [m_{24}, m_{12}, m_3; \pm\tfrac{1}{2}m_{23,2}] \\
\chi^\pm_j &= [m_2, m_1, m_{24,23}; \pm\tfrac{1}{2}m_4] \\
\chi^\pm_k &= [m_{24}, m_1, m_{23,2}; \pm\tfrac{1}{2}m_3]
\end{aligned} \tag{10.8}$$

Note that if in (??) we denote generically

$$\chi^\pm = \{m_1, m_2, m_3, m_4^\pm\} = [m_1, m_2, m_3; c^\pm] \tag{10.9}$$

then there is the relation

$$|c^+| + |c^-| = |m_4^+| + |m_4^-| \ . \tag{10.10}$$

Remark: Note that the pairs $\chi^\pm_j$ and $\chi^\pm_k$ are related by KS operators, but in each case the operator $G^+_{KS}$ is degenerated into a differential operator, namely, we have

$$\Lambda^-_j \quad \overset{m_4\alpha_{14,24,3}}{\longrightarrow} \quad \Lambda^+_j \tag{10.11a}$$

$$\Lambda^-_k \quad \overset{m_3\alpha_{14,24,3}}{\longrightarrow} \quad \Lambda^+_k \tag{10.11b}$$

*10.1. Concluding remarks*

**Matters are arranged so that in every main multiplet only the ER with signature $\chi_0^-$ contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional irrep of $F_4''$ with signature $\{m_1\,, m_2\,, m_3\,, m_4\}$. Thus, the main multiplets are in 1-to-1 correspondence with the finite-dimensional representations of $F_4''$.**

**The subspace $\mathcal{E}$ is annihilated by the operator $G^+$, and is the image of the operator $G^-$. The subspace $\mathcal{E}$ is annihilated also by the intertwining differential operator $\mathcal{D}_{m_4\alpha_4}$ acting from $\chi_0^-$ to $\chi_a^-$. When all $m_i = 1$ then $\dim \mathcal{E} = 1$, and in that case $\mathcal{E}$ is also the trivial one-dimensional UIR of the whole algebra $\mathcal{G}$. Furthermore in that case the conformal weight is zero: $d = \frac{7}{2} + c_{|_{m_i=1}} = 0$.**

**In the conjugate ER $\chi_0^+$ there is a unitary discrete series subrepresentation in an infinite-dimensional subspace $\mathcal{D}_0$. It is annihilated by the operator $G^-$, and is in the image of the operator $G^+$ acting from $\chi_0^-$ and in the image of the intertwining differential operator $\mathcal{D}^{m_4}_{\alpha_{14,23,3}}$ acting from $\chi_a^+$.**

**Two more occurrences of discrete series are in the infinite-dimensional subspaces $\mathcal{D}_a, \mathcal{D}_b$ of the ERs $\chi_a^+,\ \chi_b^+$, resp. As above they are annihilated by the operator $G^-$, and are in the images of the operator $G^+$ acting from $\chi_a^-,\ \chi_b^-$, resp. Furthermore the subspace $\mathcal{D}_a$ is in the image of the operator $\mathcal{D}^{m_3}_{\alpha_{14,23}}$ acting from $\chi_b^+$ and is annihilated by the intertwining differential operator $\mathcal{D}^{m_4}_{\alpha_{14,23,3}}$. Furthermore the subspace $\mathcal{D}_b$ is in the image of the operator $\mathcal{D}^{m_2}_{\alpha_{14,14,23,3}}$ acting from $\chi_c^+$ and is annihilated by the intertwining differential operator $\mathcal{D}^{m_3}_{\alpha_{14,23}}$.**

**Full details are given in [15].**

[15] V.K. Dobrev, Contribution to Peter Suranyi 87th Birthday Festschrift: "A Life in Quantum Field Theory", https://doi.org/10.1142/13025 (World Scientific, November 2022), Edited by: P. Argyres, G. Dunne, G. Semenoff, R. Wijewardhana.

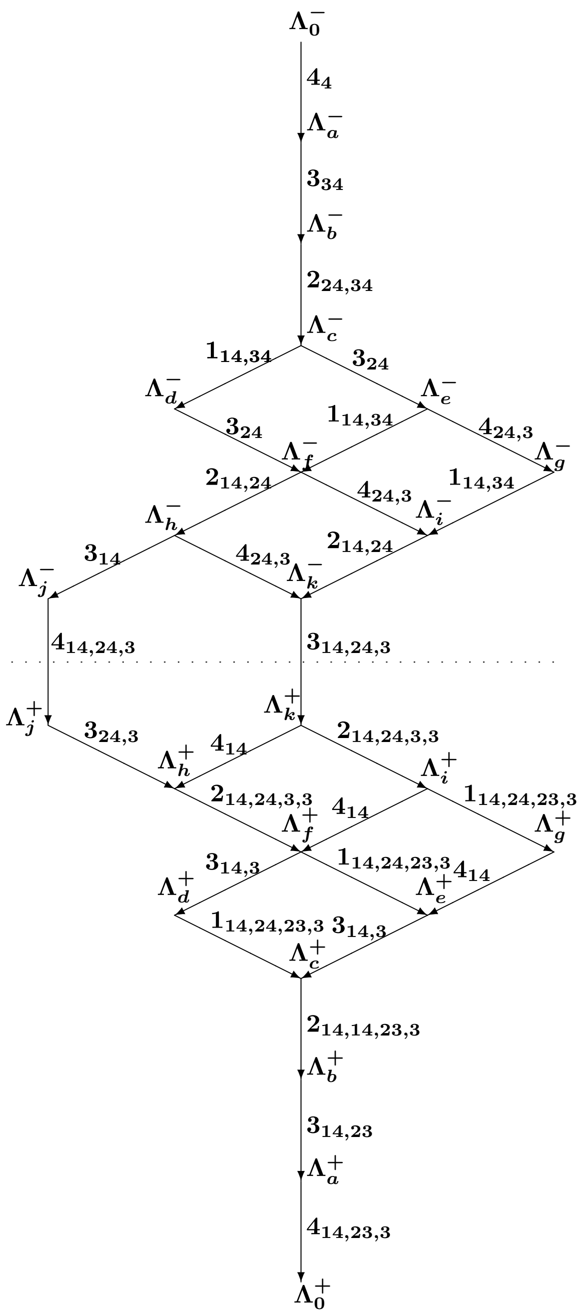



**Fig. 18. Main multiplets for $F_4''$**

## 11. Heisenberg Parabolic Subgroups of $SO^*(2n)$

Here we focus on the algebras $SO^*(2n)$. In Section ?? we considered already part of this family, namely, $SO^*(4n)$, with maximal parabolic factor $\mathcal{M}^{\mathbb{C}}$ equal to the semisimple part of the maximal compact subalgebra. Here the maximal parabolic factor belongs to a different case, namely, Heisenberg parabolics.

The compact roots w.r.t. the real form $SO^*(2n)$ are $\alpha_{ij}$ - they form (by restriction) the root system of the semisimple part of $\mathcal{K}^{\mathbb{C}}$, namely, $\mathcal{K}_s^{\mathbb{C}} \cong su(n)^{\mathbb{C}} \cong sl(n,\mathbb{C})$, while the roots $\beta_{ij}$ are $\mathcal{K}$-noncompact.

The minimal parabolics of $SO^*(2n)$ depend on whether $n$ is even or odd and are:

$$\begin{aligned} \mathcal{M}_0 &= so(3) \oplus \cdots \oplus so(3), \quad r \text{ factors, for } n = 2r && (11.1a)\\ &= so(2) \oplus so(3) \oplus \cdots \oplus so(3), \quad r \text{ factors, for } n = 2r+1 && (11.1b)\end{aligned}$$

The subalgebras $\mathcal{N}_0^{\pm}$ which form the root spaces of the root system $(\mathcal{G},\mathcal{A}_0)$ are of real dimension $n(n-1) - [n/2]$.

The maximal parabolic subalgebras have $\mathcal{M}$-factors as follows:

$$\mathcal{M}_j^{\max} = so^*(2n-4j) \oplus su^*(2j) \ , \quad j = 1,\ldots,r \ . \tag{11.2}$$

The $\mathcal{N}^{\pm}$ factors in the maximal parabolic subalgebras have dimensions: $\dim(\mathcal{N}_j^{\pm})^{\max} = j(4n-6j-1)$.

The case $j=1$ is special.[16] In this case we have a maximal Heisenberg parabolic with $\mathcal{M}$-factor:

$$\begin{aligned} \mathcal{M}^{\max}_{\text{Heisenberg}} &= so^*(2n-4) \oplus su(2) && (11.3a)\\ \operatorname{rank} \mathcal{M}^{\max}_{\text{Heisenberg}} &= n-1 && (11.3b)\end{aligned}$$

which we use in this paper.

### *11.1. The case $SO(p,q)$*

The Lie algebras $\mathcal{G}_{p,q} = so(p,q)$ $(p \geq q \geq 2)$ in general belong to the class that have maximal Heisenberg parabolic subalgebras. The latter have the factor $\mathcal{M}_{p,q} = sl(2,\mathbb{R}) \oplus so(p-2,q-2)$ which has $\operatorname{rank}\mathcal{M}_{p,q} = [(p+q)/2] - 1 = \operatorname{rank}\mathcal{G}_{p,q} - 1$.

For us it is important that when $p+q=2n$ is even, then $\mathcal{G}_{p,q}$ is parabolically related to $so^*(2n)$. For this are needed the following facts:

$$\begin{aligned} &\mathcal{G}^{\mathbb{C}}_{p,q} = (so^*(2n))^{\mathbb{C}}, \quad p+q = 2n \\ &\mathcal{M}^{\mathbb{C}}_{p,q} = (\mathcal{M}^{\max}_{\text{Heisenberg}})^{\mathbb{C}} \end{aligned} \tag{11.4}$$

Let us consider the data for this relation.

We need the root system of the complexification: $so(p+q,\mathbb{C})$ for $p+q=2n$. The positive roots are given standardly as:

$$\begin{aligned} \alpha_{ij} &= \epsilon_i - \epsilon_j \ , \quad 1 \leq i < j \leq n \ , && (11.5a)\\ \beta_{ij} &= \epsilon_i + \epsilon_j \ , \quad 1 \leq i < j \leq n && (11.5b)\end{aligned}$$

[16] In Section ?? we considered the case $j = r$, $n = 2r$.

where $\epsilon_i$ are standard orthonormal basis: $\langle\epsilon_i, \epsilon_j\rangle = \delta_{ij}$ . The simple roots are:

$$\pi_n \ = \ \{\gamma_i = \alpha_{i,i+1},\ 1 \leq i \leq n-1, \quad \gamma_n = \beta_{n-1,n}\} \tag{11.6}$$

Thus, the root system of $\mathcal{M}^{\mathbb{C}}_{p,q} = sl(2,\mathbb{C}) \oplus so(2n-4,\mathbb{C})$ is given by:

$$\alpha_{12},\quad \alpha_{ij},\quad \beta_{ij} \quad 3 \leq i < j \leq n\ , \quad n \geq 4 \tag{11.7a}$$
$$\alpha_{23},\quad n = 3 \tag{11.7b}$$

The simple roots of $\mathcal{M}^{\mathbb{C}}_{p,q}$ are:

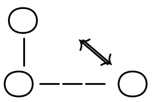
$$\pi^{\mathcal{M}}_{p,q} \ = \ \{\gamma_1,\ \gamma_i = \alpha_{i,i+1},\ 3 \leq i \leq n-1,\ \gamma_n = \beta_{n-1,n}\},\ \frac{p+q}{2} = n \geq 4 \tag{11.8a}$$
$$\pi^{\mathcal{M}}_{3} \ = \ \{\ \gamma_2\ \},\quad \frac{p+q}{2} = 3 \tag{11.8b}$$

We see that the cases $p+q=6$ are not representative in relation to the Satake-Dynkin diagrams. Namely, the Satake-Dynkin diagram of $so(3,3)$ is:

$$\bigcirc - - - \bigcirc - - - \bigcirc \tag{11.9}$$

since the algebra is split and $\mathcal{M}_0 = 0$.

The Satake-Dynkin diagram of $so(4,2)$ is:



(11.10)

where by standard convention the left-right arrow represents the $so(2)$ subalgebra (actually equal to $\mathcal{M}_0$).

We recall that the Satake-Dynkin diagram of $so(2n)^{\mathbb{C}}$ for $n \geq 4$ contains a node related to three nodes unrelated to each other (see also next subsection).

We mention also that for $p+q = 6$ the parabolically related Lie algebra $so^*(6) \cong su(3,1)$ is not included in the list of algebras with maximal Heisenberg parabolic subalgebra, since being of split rank 1 it has one non-trivial parabolic (which is both minimal and maximal) with $so(2) \oplus so(3)$.

Finally, we mention that one representative case of $p+q=6$, namely, $so(4,2)$ with Heisenberg parabolic $\mathcal{M}_{4,2} = sl(2,\mathbb{R}) \oplus so(2)$ was considered in detail in [17].

Thus, it is clear that we can safely consider $so^*(2n)$ only for $n \geq 4$. Below we consider the nontrivial case $so^*(10)$.

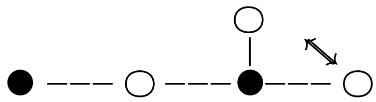
### *11.2.* $SO^*(10)$

Further we restrict to our case of study $\mathcal{G} = so^*(10)$ with minimal parabolic:

$$\mathcal{M}_0 \ = \ so(2) \oplus so(3) \oplus so(3) \tag{11.11}$$

[17] V.K. Dobrev, Physics of Atomic Nuclei, **80**, No. 2 (2017) 347–352.

**The Satake-Dynkin diagram of $\mathcal{G}$ is:**



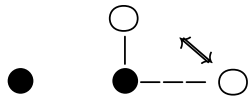
(11.12)

**where by standard convention the black dots represent the $so(3)$ subalgebras of $\mathcal{M}_0$ and the left-right arrow represents the $so(2)$ subalgebra of $\mathcal{M}_0$.**

**We shall use the Heisenberg maximal parabolic (??) with $\mathcal{M}$-subalgebra:**

$$\mathcal{M} \;=\; so^*(6) \oplus so(3) \cong su(3,1) \oplus su(2) \tag{11.13}$$

**The Satake-Dynkin diagram of $\mathcal{M}$ is a subdiagram of (??):**



(11.14)

**where the single black dot represents the $so(3)$ subalgebra, while the connected part of the diagram represents the $su(3,1)$ subalgebra.**

From the above follows that the $\mathcal{M}$-compact roots of $\mathcal{G}^{\mathbb{C}}$ are (given in terms of the simple roots):

$$\begin{aligned}
&\alpha_{12} = \gamma_1, && (11.15a)\\
&\alpha_{34} = \gamma_3,\ \alpha_{45} = \gamma_4,\ \beta_{45} = \gamma_5, && (11.15b)\\
&\alpha_{35} = \gamma_3 + \gamma_4,\ \beta_{34} = \gamma_3 + \gamma_4 + \gamma_5,\ \beta_{35} = \gamma_3 + \gamma_5 &&
\end{aligned}$$

By definition the above are the positive roots of $\mathcal{M}^{\mathbb{C}}$, namely: $su(2)^{\mathbb{C}}$ (??a), and $su(3,1)^{\mathbb{C}} = sl(4,\mathbb{C})$ (??b).

The positive $\mathcal{M}$-noncompact roots of $\mathcal{G}^{\mathbb{C}}$ in terms of the simple roots are:

$$\begin{aligned}
&\gamma_{12} = \gamma_1 + \gamma_2,\ \gamma_{13} = \gamma_1 + \gamma_2 + \gamma_3,\ \ \gamma_{14} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4, &&\\
&\gamma_2,\ \gamma_{23} = \gamma_2 + \gamma_3,\ \gamma_{24} = \gamma_2 + \gamma_3 + \gamma_4, && (11.16a)\\
&\beta_{12} = \gamma_1 + 2\gamma_2 + 2\gamma_3 + \gamma_4 + \gamma_5,\ \beta_{13} = \gamma_1 + \gamma_2 + 2\gamma_3 + \gamma_4 + \gamma_5, &&\\
&\beta_{14} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5,\ \beta_{15} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_5, &&\\
&\beta_{23} = \gamma_2 + 2\gamma_3 + \gamma_4 + \gamma_5,\ \ \beta_{24} = \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5, &&\\
&\beta_{25} = \gamma_2 + \gamma_3 + \gamma_5 && (11.16b)
\end{aligned}$$

where for convenience we use the notation $\gamma_{ij} \equiv \alpha_{i,j+1}$

To characterize the Verma modules we shall use first the Dynkin labels:

$$m_i \ \equiv\ (\Lambda + \rho, \gamma_i^\vee) \ =\ (\Lambda + \rho, \gamma_i)\ , \quad i = 1, \ldots, 5, \tag{11.17}$$

where $\rho$ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$. Thus, we shall use :

$$\chi_\Lambda \ =\ \{m_1, m_2, m_3, m_4, m_5\} \tag{11.18}$$

Note that when all $m_i \in \mathbb{N}$ then $\chi_\Lambda$ characterizes the finite-dimensional irreps of $\mathcal{G}^{\mathbb{C}}$ and its real forms, in particular, $so^*(10)$. Furthermore, $m_1 \in \mathbb{N}$ characterizes the finite-dimensional irreps of the $su(2)$ subalgebra, while the set of positive integers $\{m_3, m_4, m_5\}$ characterizes the finite-dimensional irreps of $su(3,1)$.

For the $\mathcal{M}$-noncompact roots of $\mathcal{G}^{\mathbb{C}}$ we shall use also the Harish-Chandra parameters:

$$\begin{aligned}
m_{ij} \ &=\ (\Lambda + \rho, \gamma_{ij}^\vee)\ , && (11.19a)\\
\hat{m}_{ij} \ &=\ (\Lambda + \rho, \beta_{ij}^\vee) && (11.19b)
\end{aligned}$$

and explicitly in terms of the Dynkin labels (compare (??)):

$$\begin{aligned}
\chi_{HC} \ =\ \ &\{m_{12} = m_1 + m_2,\ m_{13} = m_1 + m_2 + m_3, &&\\
&m_{14} = m_1 + m_2 + m_3 + m_4, \quad m_2, &&\\
&m_{23} = m_2 + m_3,\ m_{24} = m_2 + m_3 + m_4, && (11.20a)\\
&\hat{m}_{12} = m_1 + 2m_2 + 2m_3 + m_4 + m_5, &&\\
&\hat{m}_{13} = m_1 + m_2 + 2m_3 + m_4 + m_5, &&\\
&\hat{m}_{14} = m_1 + m_2 + m_3 + m_4 + m_5, &&\\
&\hat{m}_{15} = m_1 + m_2 + m_3 + m_5, &&\\
&\hat{m}_{23} = m_2 + 2m_3 + m_4 + m_5, &&\\
&\hat{m}_{24} = m_2 + m_3 + m_4 + m_5, &&\\
&\hat{m}_{25} = m_2 + m_3 + m_5\} && (11.20b)
\end{aligned}$$

The main multiplets are in 1-to-1 correspondence with the finite-dimensional irreps of $so^*(10)$, i.e., they are labelled by the five positive Dynkin labels $m_i \in \mathbb{N}$.

We take $\chi_0 = \chi_{HC}$. It has one embedded Verma module with HW $\Lambda_a = \Lambda_0 - m_2\gamma_2$. The number of ERs/GVMs in a main multiplet is 40.

We shall label the signature of the ERs of $\mathcal{G}$ also as follows:

$$\chi = [n;\ c;\ n_1, n_2, n_3]\ , \qquad n \in \mathbb{N}, \quad c = -\tfrac{1}{2}m_{15,23}, \quad n_j = m_{j+2} \in \mathbb{Z}_+\ , \tag{11.21}$$

where the first entry $n = m_1$ labels the finite-dimensional irreps of $su(2)$, the second entry labels the characters of $\mathcal{A}$, the last three entries of $\chi$ are labels of the finite-dimensional (nonunitary) irreps of $\mathcal{M} = su(3,1)$ when all $n_j > 0$ or limits of the latter when some $n_j = 0$. Note that $m_{15,23} = m_1 + 2m_2 + 2m_3 + m_4 + m_5$ is the Harish-Chandra parameter for the highest root $\beta_{12}$.

Using this labelling signatures may be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= [m_1; m_3, m_4, m_5\ ;\ \pm\tfrac{1}{2}m_{15,23}] \\
\chi_a^\pm &= [m_{12}; m_{23}, m_4, m_5\ ;\ \pm\tfrac{1}{2}m_{15,3}] \\
\chi_b^\pm &= [m_2; m_{13}, m_4, m_5\ ;\ \pm\tfrac{1}{2}m_{25,3}] \\
\chi_c^\pm &= [m_{13}; m_2, m_{34}, m_{3,5}\ ;\ \pm\tfrac{1}{2}m_{15}] \\
\chi_d^\pm &= [m_{23}; m_{12}, m_{34}, m_{3,5}\ ;\ \pm\tfrac{1}{2}m_{25}] \\
\chi_e^\pm &= [m_{14}; m_2, m_3, m_{35}\ ;\ \pm\tfrac{1}{2}m_{13,5}] \\
\chi_f^\pm &= [m_{13,5}; m_2, m_{35}, m_3\ ;\ \pm\tfrac{1}{2}m_{14}] \\
\chi_g^\pm &= [m_3; m_1, m_{24}, m_{23,5}\ ;\ \pm\tfrac{1}{2}m_{35}] \\
\chi_h^\pm &= [m_{24}; m_{12}, m_3, m_{35}\ ;\ \pm\tfrac{1}{2}m_{23,5}] \\
\chi_i^\pm &= [m_{23,5}; m_{12}, m_{35}, m_3\ ;\ \pm\tfrac{1}{2}m_{24}] \\
\chi_j^\pm &= [m_{15}; m_2, m_{3,5}, m_{34}\ ;\ \pm\tfrac{1}{2}m_{13}] \\
\chi_k^\pm &= [m_{34}; m_1, m_{23}, m_{25}\ ;\ \pm\tfrac{1}{2}m_{3,5}] \\
\chi_l^\pm &= [m_{3,5}; m_1, m_{25}, m_{23}\ ;\ \pm\tfrac{1}{2}m_{34}] \\
\chi_m^\pm &= [m_{25}; m_{12}, m_{3,5}, m_{34}\ ;\ \pm\tfrac{1}{2}m_{23}] \\
\chi_n^\pm &= [m_{15,3}; m_{23}, m_5, m_4\ ;\ \pm\tfrac{1}{2}m_{12}] \\
\chi_p^\pm &= [m_4; m_1, m_2, m_{25,3}\ ;\ \pm\tfrac{1}{2}m_5] \\
\chi_q^\pm &= [m_{35}; m_1, m_{23,5}, m_{24}\ ;\ \pm\tfrac{1}{2}m_3] \\
\chi_r^\pm &= [m_5; m_1, m_{25,3}, m_2\ ;\ \pm\tfrac{1}{2}m_4] \\
\chi_s^\pm &= [m_{25,3}; m_{13}, m_5, m_4\ ;\ \pm\tfrac{1}{2}m_2] \\
\chi_t^\pm &= [m_{15,23}; m_3, m_5, m_4\ ;\ \pm\tfrac{1}{2}m_1]
\end{aligned}$$

The ERs in the multiplet are related also by the Knapp-Stein intertwining integral operators. These operators are defined for any ER, the general action here is: being:

$$\begin{aligned}
&G_{KS}\ :\ \mathcal{C}_\chi \longrightarrow \mathcal{C}_{\chi'}\ , \\
&\chi = \{\, n\,;\, n_1, n_2, n_3\,;\, c\,\}\ , \qquad \chi' = \{\, n\,;\, n_1, n_2, n_3\,;\, -c\,\}.
\end{aligned} \tag{11.22}$$

The main multiplets are given explicitly in the figure below. The pairs $\chi^{\pm}$ are symmetric w.r.t. to the dashed line in the middle the figure - this represents the Weyl symmetry realized by the Knapp-Stein operators (??): $G_{KS} : \mathcal{C}_{\chi^{\mp}} \longrightarrow \mathcal{C}_{\chi^{\pm}}$ .

Some comments are in order.

Matters are arranged so that in every multiplet only the ER with signature $\chi_0^-$ contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional irrep of $so^*(10)$ with signature $\{m_1, \ldots, m_5\}$. The subspace $\mathcal{E}$ is annihilated by the operator $G^+$, and is the image of the operator $G^-$. The subspace $\mathcal{E}$ is annihilated also by the intertwining differential operator acting from $\chi_0^-$ to $\chi_a^-$. When all $m_i = 1$ then $\dim \mathcal{E} = 1$, and in that case $\mathcal{E}$ is also the trivial one-dimensional UIR of the whole algebra $\mathcal{G}$. Furthermore in that case the conformal weight is zero: $d = \frac{7}{2} + c = \frac{7}{2} - \frac{1}{2}(m_1 + 2m_2 + 2m_3 + m_4 + m_5)_{|m_i=1} = 0$.

In the conjugate ER $\chi_0^+$ there is a unitary discrete series subrepresentation in an infinite-dimensional subspace $\mathcal{D}$. It is annihilated by the operator $G^-$, and is the image of the operator $G^+$.

Thus, for $so^*(10)$ the ER with signature $\chi_0^+$ contains both a holomorphic discrete series representation and a conjugate anti-holomorphic discrete series representation. The direct sum of the holomorphic and the antiholomorphic representations spaces form the invariant subspace $\mathcal{D}$ mentioned above. Note that the corresponding lowest weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series.

Finally, we remind that the above considerations for the intertwining differential operators are applicable also for the algebras $so(p, q)$ (with $p + q = 10$, $p \geq q \geq 2$) with maximal Heisenberg parabolic subalgebras: $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$, $\mathcal{M}' = so(p-2, q-2) \oplus sl(2, \mathbb{R})$.

Full details are given in [18].

[18] V.K. Dobrev, Symmetry 2022, 14 (8), 1592

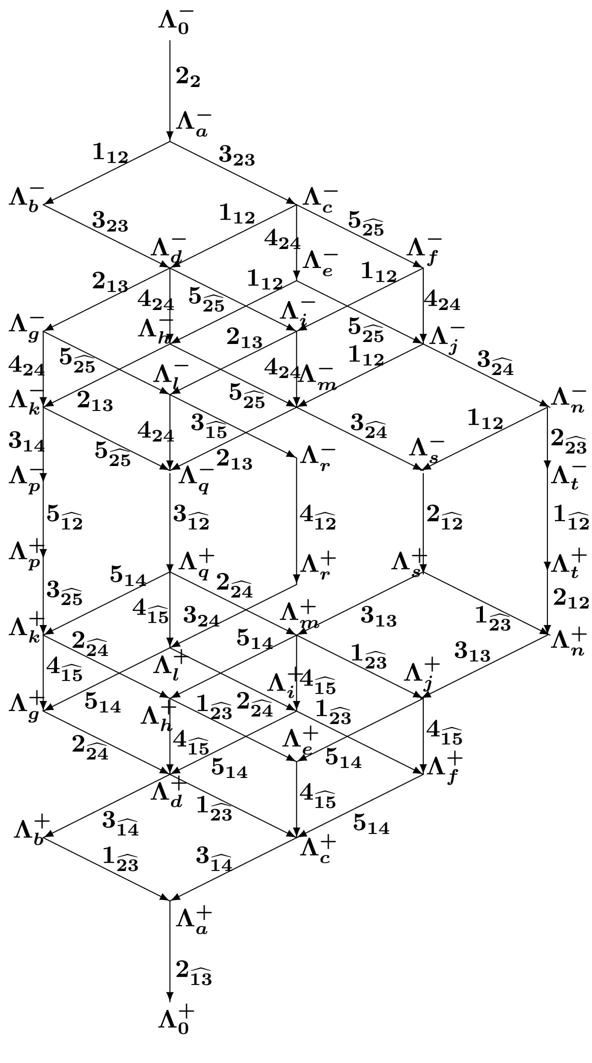



**Fig. 19. Main multiplets for $SO^*(10)$ using induction from maximal Heisenberg parabolic**

## Thank you for your attention!