Polynomial modifications of Starobinsky model

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Duality transformations

• The *duality* relation between modified $F(R)$ gravity theories and scalar-tensor gravity theories is the standard tool in modern cosmology
T.P. Sotiriou and V. Faraoni, f(R) theories of gravity, Rev. Mod. Phys. 82 (2010) 451 [arXiv:0805.1726]
A. De Felice and S. Tsujikawa, f(R) theories, Living Rev. Rel. 13 (2010) 3 [arXiv:1002.4928]
S. Capozziello and M. De Laurentis, Extended theories of gravity, Phys. Rept. 509 (2011) 167 [arXiv:1108.6266]
S.V. Ketov, Supergravity and early universe: the meeting point of cosmology and high-energy physics, Int. J. Mod. Phys. A 28 (2013) 1330021 [arXiv:1201.2239]

• The modified $F(R)$ gravity theories having the action

$$S_F[g^J_{\mu\nu}] = \int d^4x \sqrt{-g^J} F(R_J) \tag{1}$$

with a differentiable function F.

The $F(R)$ gravity action can be rewritten as

$$S_J[g^J_{\mu\nu},\sigma] = \int d^4x\sqrt{-g^J}\left[F_{,\sigma}(R_J-\sigma)+F\right]\ , \qquad (2)$$

where the new scalar field σ has been introduced, and $F_{,\sigma}(\sigma) = \frac{dF(\sigma)}{d\sigma}$.
After the Weyl transformation of the metric

$$g_{\mu\nu} = \frac{2F_{,\sigma}(\sigma)}{M_{Pl}^2} g^J_{\mu\nu} \qquad (3)$$

one gets the following action in the Einstein frame:

$$S_E[g_{\mu\nu},\sigma] = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - \frac{h(\sigma)}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - V\right], \qquad (4)$$

where we have introduced the functions

$$h(\sigma) = \frac{3M_{Pl}^2}{2F_{,\sigma}^2}F_{,\sigma\sigma}^2 \quad \text{and} \quad V(\sigma) = M_{Pl}^4\frac{F_{,\sigma}\sigma - F}{4F_{,\sigma}^2}\ . \qquad (5)$$

Introducing the canonical scalar field ϕ instead of σ as

$$\phi = \sqrt{\frac{3}{2}} M_{Pl} \ln \left[\frac{2}{M_{Pl}^2} F_{,\sigma} \right] \tag{6}$$

allows one to rewrite the action S_E to the standard (quintessence or scalar-tensor) form:

$$S_E[g_{\mu\nu}, \phi] = \int d^4x \sqrt{-g} \left[\frac{M_{Pl}^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] . \tag{7}$$

The inverse transformation reads as follows:

$$R_J = \left[\frac{\sqrt{6}}{M_{Pl}} V_{,\phi} + \frac{4V}{M_{Pl}^2} \right] \exp \left(\sqrt{\frac{2}{3}} \frac{\phi}{M_{Pl}} \right) , \tag{8}$$

$$F = \frac{M_{Pl}^2}{2} \left[\frac{\sqrt{6}}{M_{Pl}} V_{,\phi} + \frac{2V}{M_{Pl}^2} \right] \exp \left(2\sqrt{\frac{2}{3}} \frac{\phi}{M_{Pl}} \right) , \tag{9}$$

where $V_{,\phi} = \frac{dV}{d\phi}$, defining the function $F(R_J)$ in the parametric form with the parameter ϕ.

- The inflation [1] was supposed to solve problems related with the hot big-bang model[2]

[1] A. A. Starobinsky, "A New Type of Isotropic Cosmological Models Without Singularity," Phys. Lett. B **91**, 99 (1980)

[2] R. Brout, F. Englert, E. Gunzig, The Creation of the Universe as a Quantum Phenomenon, Annals Phys., **115**, 78 (1978).
D. Kazanas, "Dynamics of the Universe and Spontaneous Symmetry Breaking," Astrophys. J., **241**, L59 (1980);
K. Sato, "First-order phase transition of a vacuum and the expansion of the Universe," MNRAS, **195**, 467 (1981);
A. H. Guth, "The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems," Phys. Rev. D **23**, 347 (1981);
A. D. Linde, " A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems," Phys. Lett. B **108**, 389 (1982);
A. Albrecht, P. J. Steinhardt, "Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking," Phys. Rev. Lett. **48**, 1220 (1982);
A. D. Linde, "Chaotic Inflation," Phys. Lett. B **129**, 177 (1983);
V. F. Mukhanov and G. V. Chibisov, "Quantum Fluctuation and Nonsingular Universe. (In Russian)," JETP Lett. **33**, 532 (1981) [Pisma Zh. Eksp. Teor. Fiz. **33**, 549 (1981)];

The Starobinsky R^2 inflationary model

- The main cosmological parameters of inflation are given by the spectral index n_s and the tensor-to-scalar ratio r, whose values are constrained by the combined Planck, WMAP and BICEP/Keck observations of CMB as

$$n_s = 0.9649 \pm 0.0042 \quad (68\%\mathrm{CL}) \qquad \text{and} \qquad r < 0.036 \quad (95\%\mathrm{CL})\,.$$

- The Starobinsky model[3], whose action is given by

$$S_{\mathrm{Star.}}[g^J_{\mu\nu}] = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-g_J} \left(R_J + \frac{1}{6m^2} R_J^2 \right) \,, \tag{10}$$

is known as the excellent model of large-field slow-roll cosmological inflation with very good agreement to the observation data.

[3] A.A. Starobinsky, *Phys. Lett. B* **91** (1980) 99,
A.A. Starobinsky, *Phys. Lett. B* **117** (1982) 175.

Dual presentation of R^2 inflationary model

The action (10) is dual to the quintessence (or scalar-tensor gravity) action

$$S_{\rm quint.}[g_{\mu\nu},\phi] = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V_{\rm Star.}(\phi)\right] \quad (11)$$

in terms of the canonical scalar ϕ and another metric $g_{\mu\nu}$ in the Einstein frame, related to $g^J_{\mu\nu}$ (in the Jordan frame) by a Weyl transformation. The induced scalar potential is given by

$$V_{\rm Star.}(\phi) = \frac{3}{4}M_{Pl}^2 m^2\left[1-\exp\left(-\sqrt{\frac{2}{3}}\frac{\phi}{M_{Pl}}\right)\right]^2 . \quad (12)$$

Equations of motion

In the spatially flat FLRW universe with the metric

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right) ,$$

the action (7) leads to the standard system of evolution equations:

$$6M_{Pl}^2 H^2 = \dot{\phi}^2 + 2V, \quad (13)$$

$$2M_{Pl}^2 \dot{H} = -\dot{\phi}^2, \quad (14)$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0, \quad (15)$$

where $H = \dot{a}/a$ is the Hubble parameter,
$a(t)$ is the scale factor,
and the dots denote the derivatives with respect to the cosmic time t.

Slow-roll approximation and inflation observables

The inflationary stage is slow-roll on the quasi de Sitter solution, $|\dot{H}| \ll H^2$

In the Einstein frame, the slow-roll parameters are

$$\epsilon = \frac{M_{Pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 ,$$

$$\eta = M_{Pl}^2\left(\frac{V_{,\phi\phi}}{V}\right)$$

The scalar spectral index n_s and the tensor-to-scalar ratio r in terms of the slow-roll parameters are given by

$$n_s = 1 - 6\epsilon + 2\eta, \qquad r = 16\epsilon \ . \tag{16}$$

The amplitude of scalar perturbations is given by

$$A_s = \frac{2V}{3\pi^2 M_{Pl}^4 r} \ , \tag{17}$$

while its observed value (Planck) is $A_s = 2.1 \times 10^{-9}$. Therefore, Eq. (17) relates the height of the inflationary potential to the tensor-to-scalar ratio r.

The analyze of slow-roll inflation in terms of the e-folding number representation with $A' = dA/dN$ is the most convenient:

$$\frac{d}{dt} = -H\frac{d}{dN}$$

The inflationary interval in the e-folding formulation is $0 < N < 65$ positive [4]

In the inflationary model building, the e-foldings number

$$N_e = \ln\left(\frac{a_{\mathrm{end}}}{a}\right) \ , \tag{18}$$

where a_{end} is the value of a at the end of inflation, is considered instead of the time variable.

[4] M. Galante, R. Kallosh, A. Linde and D. Roest, "Unity of Cosmological Inflation Attractors," Phys. Rev. Lett. **114** (2015) no.14, 141302 [arXiv:1412.3797 [hep-th]]. R. Kallosh and A. Linde, "Universality Class in Conformal Inflation," JCAP **1307**, 002 (2013) [arXiv:1306.5220 [hep-th]].

We equations of motion can be presented such as:

$$Q = \frac{2V}{6M_{Pl}^2 - \chi^2} \ , \tag{19}$$

$$Q' = \frac{1}{M_{Pl}^2} Q\chi^2 \ , \quad \chi' = 3\chi - \frac{1}{2M_{Pl}^2}\chi^3 - \frac{1}{Q}\frac{dV}{d\phi} \ . \tag{20}$$

where $Q \equiv H^2$ and $\chi = \phi' = -\dot{\phi}/H$, and the primes denote the derivatives with respect to N_e.
We rewrite the last equation as

$$\chi' = 3\chi - \frac{1}{2M_{Pl}^2}\chi^3 - \frac{6M_{Pl}^2 - \chi^2}{2V}\frac{dV}{d\phi} \ . \tag{21}$$

Being motivated by the potential (12), we find useful to introduce the non-canonical dimensionless field

$$y \equiv \exp\left(-\sqrt{\frac{2}{3}}\frac{\phi}{M_{Pl}}\right) = \frac{M_{Pl}^2}{2F_{,\sigma}} > 0 \tag{22}$$

because it is (physically) *small* during slow-roll inflation. Defining $\tilde{V}(y) = V(\phi)$ and using

$$\frac{dV}{d\phi} = -\sqrt{\frac{2}{3}}\frac{y}{M_{Pl}}\frac{d\tilde{V}}{dy} \ ,$$

we simplify Eqs. (8) and (9) as follows:

$$R_J = \frac{2}{M_{Pl}^2}\left(2\frac{\tilde{V}}{y} - \tilde{V}_{,y}\right), \tag{23}$$

$$F = \frac{\tilde{V}}{y^2} - \frac{\tilde{V}_{,y}}{y} \ , \tag{24}$$

respectively.
In the Starobinsky model, we have

$$\tilde{V}_{\rm Star.}(y) = V_0(1-y)^2 \ , \quad \text{where} \quad V_0 = \frac{3}{4}m^2 M_{Pl}^2 \quad . \tag{25}$$

In the Einstein frame, the slow-roll parameters are

$$\epsilon = \frac{y^2}{3}\left(\frac{\tilde{V}_{,y}}{\tilde{V}}\right)^2 ,$$

$$\eta = \frac{2y}{3\tilde{V}}\left(\tilde{V}_{,y} + y\tilde{V}_{,yy}\right) .$$

In the slow-roll approximation, the function $\phi(N_e)$ can be found as a solution of

$$\chi \equiv \phi' \simeq \frac{M_{Pl}^2}{V} V_{,\phi} \tag{26}$$

this equation is equivalent to

$$y' = \frac{2y^2\tilde{V}_{,y}}{3\tilde{V}} . \tag{27}$$

$\epsilon = 1$ corresponds to the end of inflation with $a = a_{\mathrm{end}}$.

The main cosmological parameters of inflation are given by the scalar tilt n_s and the tensor-to-scalar ratio r, whose values are constrained by the combined Planck, WMAP and BICEP/Keck observations of CMB as

$$n_s = 0.9649 \pm 0.0042 \quad (68\%\mathrm{CL}) \qquad \text{and} \qquad r < 0.036 \quad (95\%\mathrm{CL})\ .$$

The Starobinsky model is known as the excellent model of large-field slow-roll cosmological inflation with very good agreement to the observation data.

ϕ_i/M_{Pl}	5.2262	5.4971
n_s	0.961	0.969
r	0.0043	0.0027
N_e	49.258	62.335

The values of the inflationary parameters are sensitive to the duration of inflation and the initial value of the inflaton field, ϕ_i. In the Starobinsky model, we have

$$A_s = \frac{N_e^2 m^2}{24\pi^2 M_{Pl}^2} \tag{28}$$

that determines the value of $m/M_{Pl} \sim \mathcal{O}(10^{-5})$.

The $(R+R^2+R^3)$ gravity models of inflation

To the best of our knowledge, adding the higher-order terms in R was first proposed in *J.D. Barrow and S. Cotsakis, Phys. Lett. B* **214** *(1988) 515.*
A generic $(R+R^2+R^3)$ gravity action is given by

$$S_{3-\text{gen.}} = \frac{M_{Pl}^2}{2}\int d^4x\sqrt{-g_J}\left[(1+\delta_1)R_J + \frac{(1+\delta_2)}{6m^2}R_J^2 + \frac{\delta_3}{36m^4}R_J^3\right] ,$$

where we have introduced the three dimensionless parameters δ_i.
The corresponding inflaton scalar potential (5) is given by

$$V(\sigma) = \frac{16V_0\tilde{\sigma}^2\left[3(1+\delta_2)+\delta_3\tilde{\sigma}\right]}{3\left[12(1+\delta_1)+4(1+\delta_2)\tilde{\sigma}+\delta_3\tilde{\sigma}^2\right]^2} ,$$

where the dimensionless variable $\tilde{\sigma} = \sigma/m^2$ has been introduced.
$V(0)=0$, $V(\tilde{\sigma})>0$ at $\tilde{\sigma}>0$, and V tends to zero at $\tilde{\sigma}\to+\infty$, while the potential has a maximum at some positive value of $\tilde{\sigma}$. The equation $V'=0$ has only one positive root given by $\tilde{\sigma}_{\text{max.}} = 6\sqrt{\frac{1+\delta_1}{\delta_3}}$.

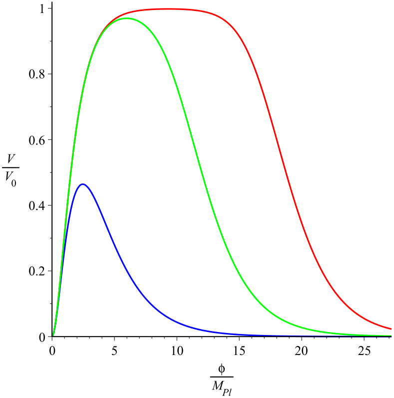


Figure: The normalized potential $V(\phi)/V_0$ with $\delta_1 = \delta_2 = 0$ for $\delta_3 = 0.000001$ (red), $\delta_3 = 0.000247$ (blue), and $\delta_3 = 1/3$ (green).

To study the impact of the R^3-term on inflation in more detail, let us consider the simplest non-trivial case with $\delta_1 = \delta_2 = 0$, in which Eq. (22) implies

$$\frac{1}{y} = 1 + \frac{1}{3}\tilde{\sigma} + \frac{\delta_3}{12}\tilde{\sigma}^2\,. \tag{29}$$

Equation (29) is a quadratic equation on $\tilde{\sigma}$ as a function of y. The only positive root of this equation is given by

$$\tilde{\sigma} = \frac{2}{\delta_3}\left[\sqrt{1+3\delta_3\left(y^{-1}-1\right)}-1\right] = \frac{2}{\delta_3}\left[\sqrt{1+3\delta_3\left(\mathrm{e}^{\sqrt{\frac{2}{3}}\phi/M_{Pl}}-1\right)}-1\right].$$

Using Eqs. (5) and (22), we find the scalar potential in terms of y or the inflaton field ϕ as follows:

$$\tilde{V}(y) = \frac{4V_0}{27\delta_3^2 y}\left[y+2\sqrt{y(y+3\delta_3(1-y))}\right]\left(y-\sqrt{y(y+3\delta_3(1-y))}\right)^2.$$

It is worth noticing that $\tilde{V}_{\mathrm{Star.}}(y)$ is reproduced in the limit $\delta_3 \to 0$.

The condition $\phi_i < \phi_{\rm max.}$ yields the additional restriction on the possible initial values of ϕ, being represented by the blue curve on the left-hand-side of Fig. 2.

The upper bound on the parameter δ_3 can be estimated by assuming the observable value of n_s to be calculated at the maximum of the potential. Then we find

$$n_s(\phi_{\rm max.}) = 1 - \frac{8\sqrt{\delta_3}\left(1+4\sqrt{\delta_3}+4\delta_3\right)}{3\left(3\sqrt{\delta_3}+1\right)\left(2\sqrt{\delta_3}+1\right)^2} \quad . \tag{30}$$

Since observations require $n_s > 0.960$, we get $\delta_3 < 0.0002467$. The dependence of n_s upon δ_3 is given on the right-hand-side of Fig. 2. Therefore, the domain of allowed values of δ_3 and ϕ is highly restricted.

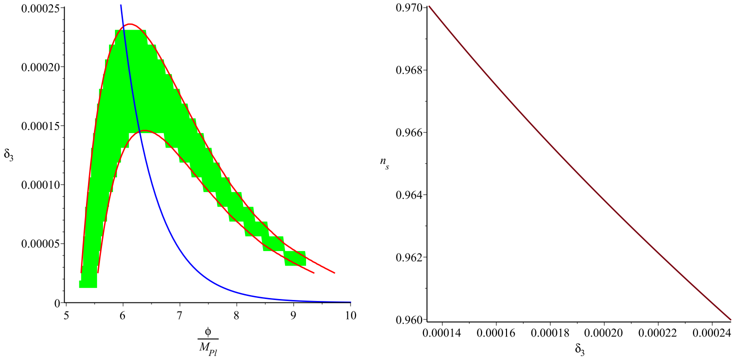


Figure: The allowed range of δ_3 and ϕ from the observational constraints (Planck): $0.961 < n_s < 0.969$ (left), and the dependence of n_s upon δ_3 (right), under the assumption that inflation started at the maximum of the potential.

Since viable inflation requires $\delta_3 \ll 1$, it allows us to consider δ_3 as a truly small parameter and expand the potential in power series of δ_3 as follows:

$$\tilde{V}(y) = V_0(y-1)^2\left[1 + \frac{y-1}{2y}\delta_3 + \frac{9(y-1)^2}{16y^2}\delta_3^2 + \mathcal{O}\left(\delta_3^3\right)\right] . \qquad (31)$$

In this approximation we get

$$\epsilon \simeq \frac{4y^2}{3(y-1)^2} + \frac{2}{3(y-1)}\delta_3 + \frac{14y+1}{12y^2}\delta_3^2 \quad ,$$

$$\eta \simeq \frac{4y(2y-1)}{3(y-1)^2} + \frac{3y+1}{3y(y-1)}\delta_3 + \frac{21y+16}{12y^2}\delta_3^2 \quad ,$$

and, hence,

$$n_s \simeq 1 - \frac{8y(y+1)}{3(y-1)^2} - \frac{2(3y-1)}{3y(y-1)}\delta_3 - \frac{21y-13}{6y^2}\delta_3^2 \quad . \qquad (32)$$

The conditions $\epsilon(y_{end}) = 1$ and $y_{end} = 2\sqrt{3} - 3$ at $\delta_3 = 0$ give

$$y_{end} = 2\sqrt{3} - 3 + \frac{45 - 26\sqrt{3}}{12\sqrt{3} - 21}\delta_3 + \frac{2146\sqrt{3} - 3717}{9\left(4\sqrt{3} - 7\right)^2}\delta_3^2 \,, \qquad (33)$$

We get e-folding number for small δ_3:

$$N_e = \frac{3(y\ln(y) + 1)}{4y} + \frac{3y^2 - 3y + 1}{16y^3}\delta_3 + \frac{3(70y^4 - 110y^3 + 80y^2 - 25y + 2)}{640y^5}\delta_3^2 \qquad (34)$$

$-N_0$, where N_0 can be obtained by the condition $N_e(y_{end}) = 0$:

$$N_0 = \frac{(1896102\sqrt{3} - 3284145)\ln(2\sqrt{3} - 3) - 293328\sqrt{3} + 508059}{4(2\sqrt{3} - 3)^5(4\sqrt{3} - 7)^2} + \frac{2578194\sqrt{3} - 4465563}{16(2\sqrt{3} - 3)^5(4\sqrt{3} - 7)^2}\delta_3 + \frac{187776774\sqrt{3} - 325238913}{640(2\sqrt{3} - 3)^5(4\sqrt{3} - 7)^2}\delta_3^2 . \qquad (35)$$

With a value of δ_3 within $0 \leqslant \delta_3 \leqslant 0.00025$ we obtain $N_0 \simeq 1.040$.

We come to the conclusion that the model under investigation gives the inflationary parameters that do not contradict observations only if $\delta_3 < 0.00012$ and the inflation started in the narrow domain of the scalar field ϕ values (a part of the marked green domain in the left picture of Fig. 2), which implies that this inflationary scenario is rather unrealistic. The same model was also studied in detail in G. *Rodrigues-da Silva, J. Bezerra-Sobrinho and L.G. Medeiros, 2110.15502*.

The $(R + R^{3/2} + R^2)$ gravity model of inflation

Let

$$F(R_J) = \frac{M_{Pl}^2}{2}\left[R_J + \frac{1}{6m^2}R_J^2 + \frac{\delta}{m}R_J^{3/2}\right] \ , \qquad (36)$$

where we have introduced the dimensionless parameter δ.
The $R^{3/2}$ term appears in the (chiral) modified supergravity
S.V. Ketov and A.A. Starobinsky, Phys. Rev. D 83 (2011) 063512 [1011.0240].
S.V. Ketov and S. Tsujikawa, Phys. Rev. D 86 (2012) 023529 [1205.2918].
The $R^{3/2}$-term in $F(R)$ gravity arises in an approximate description of the Higgs field with a small cubic term in its scalar potential and a large non-minimal coupling to R
J.S. Martins, O.F. Piattella, I.L. Shapiro and A.A. Starobinsky, 2010.14639.

Given $\tilde{\sigma} > 0$, we find

$$F_{,R_J} = \frac{M_{Pl}^2}{6}\left(\sqrt{R_J} + \frac{9\delta}{4}\right)^2 - \frac{3}{16}\left(27\delta^2 - 16\right) > 0 \qquad (37)$$

when $\delta > -4\sqrt{3}/9$, and

$$F_{,R_J R_J} = \frac{M_{Pl}^2}{24m^2}\left(4 + \frac{9\delta}{\sqrt{R_J}}\right) > 0 \qquad (38)$$

only when $\delta > 0$.
Hence, the condition $\delta > 0$ is necessary to get a stable $F(R)$ gravity model for all $R_J > 0$.

The corresponding scalar potential (5) is given by

$$V = \frac{4V_0\tilde{\sigma}(3\delta\sqrt{\tilde{\sigma}} + \tilde{\sigma})}{(6 + 9\delta\sqrt{\tilde{\sigma}} + 2\tilde{\sigma})^2} \quad . \tag{39}$$

Equation (22) in this case is a quadratic equation on $\sqrt{\tilde{\sigma}}$, and its only real solution is

$$\tilde{\sigma} = \frac{3(1-y)}{y} + \frac{9\delta}{8y}\left[9\delta\, y - \sqrt{3y(27\delta^2 y - 16y + 16)}\right] \quad . \tag{40}$$

The potential can be rewritten as

$$\begin{aligned} \tilde{V} &= \frac{V_0}{2304y^2}\,(s + 3\delta\, y)\,(s - 9\delta\, y)^3 \\ &= \frac{243V_0\delta^4 y^2}{256}\left(3\sqrt{1 + \frac{16(1-y)}{27\delta^2 y}} + 1\right)\left(\sqrt{1 + \frac{16(1-y)}{27\delta^2 y}} - 1\right)^3 \quad , \end{aligned}$$

where we have introduced $s = \sqrt{3y(27\delta^2 y - 16y + 16)}$.

When $\delta = 4\sqrt{3}/9$, the $F_{,\sigma}$ function is a perfect square, and the potential simplifies as

$$\tilde{V}_{\rm special}(y) = \frac{V_0}{3}\,(3+\sqrt{y})\,(1-\sqrt{y})^3 \ , \tag{41}$$

or

$$V_{\rm special}(\phi) = \frac{V_0}{3}\left({\rm e}^{\phi/(\sqrt{6}M_{Pl})} - 1\right)^3\left(1 + 3{\rm e}^{\phi/(\sqrt{6}M_{Pl})}\right){\rm e}^{-2\sqrt{2/3}\,\phi/M_{Pl}} \ .$$

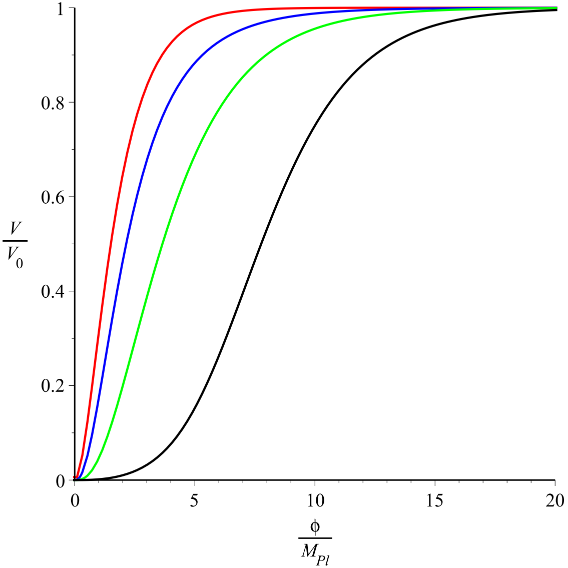


Figure: The potential $V(\phi)$ for $\delta = 0$ (red), $\delta = 1/5$ (blue), $\delta = 4\sqrt{3}/9$ (green), and $\delta = 5$ (black).

The inflationary parameters

The inflationary parameters are given by

$$n_s = 1 + \frac{8y\left(3s(3\delta(9\delta^2-16)s+720\delta^2-256)y^2 - s^3\delta(39\delta y+s)\right)}{(-9\,\delta\,y+s)^2\,(3\,\delta\,y+s)^2\,s} -$$
$$\frac{8y\left[72\delta(4-9\delta^2)(27\delta^2-16)y^4 + [(768-1215\delta^4-432\delta^2)s - 144\delta(45\delta^2-16)]y^3\right]}{(-9\,\delta\,y+s)^2\,(3\,\delta\,y+s)^2\,s} \tag{42}$$

and

$$r = \frac{768\,y^2\left(-9\,\delta^2 y+s\delta+8\,y\right)^2}{(-9\,\delta\,y+s)^2\,(3\,\delta\,y+s)^2} \quad . \tag{43}$$

The amplitude of scalar perturbations is given by

$$A_s = \frac{(-9\,\delta\,y+s)^5\,(3\,\delta\,y+s)^3\,m^2}{3538944y^4\pi^2\left(-9\,\delta^2 y+s\delta+8\,y\right)^2} \quad . \tag{44}$$

The observed value of A_s determines the value of the parameter m.

The slow-roll evolution equation allows us to relate N_e with y at the end of inflation,

$$N_e = \left(\frac{9}{8} - \delta^{-2}\right) \ln \left[9\delta^3 \left(9\delta\, y - s\right) + 24 \left(1 - 4y\right) \delta^2 + 8 \left(\delta s + 4y\right)\right] + \left(\delta^{-2} - \frac{3}{8}\right) \ln y + \frac{s}{4\delta\, y} - N_0 \ ,$$

where the integration constant N_0 is fixed by the condition $N_e(y_{end}) = 0$. The analytic formula for $N_0(\delta)$ is obtained by substituting $N_e = 0$ and $y = y_{end}$.
The condition $\epsilon = 1$ gives

$$y_{end} = \frac{3(4 - 3\delta^2 + \sqrt{3}\delta^2) - \sqrt{9(4 - 3\delta^2 + \sqrt{3}\delta^2)^2 - 72(2 - 3\delta^2)}}{2(2 - 3\delta^2)(3 + 2\sqrt{3})} \ .$$

It is worth noticing that this solution has no singularity at $\delta = \sqrt{2/3}$, while $y_{end}(\delta)$ is a smooth monotonically decreasing function.

The slow-roll parameters ϵ and η remain finite in the limit $\delta \to +\infty$ at fixed y,

$$\epsilon_\infty(y) = \frac{(2y+1)^2}{3(1-y)^2} \quad , \quad \eta_\infty(y) = \frac{2(4y^2+y+1)}{3(1-y)^2} \ . \tag{45}$$

Since the value of y at the end of inflation is determined by the condition $\epsilon(y_{end}) = 1$, y_{end} also approaches a finite limit as $\delta \to +\infty$, which is given by a solution to the equation

$$\epsilon_\infty = \frac{(2y+1)^2}{3(1-y)^2} = 1. \tag{46}$$

This equation has only one positive solution $y_{end}|_{\delta\to+\infty} = 3\sqrt{3} - 5 \approx 0.196$.

When $\delta = 4\sqrt{3}/9$, the function $F_{,\sigma}$ simplifies as

$$F_{,\sigma} = \frac{M_{Pl}^2}{6}\left(\sqrt{\tilde{\sigma}} + \sqrt{3}\right)^2 .$$

Accordingly, the slow-roll parameters are also simplified as

$$\epsilon = \frac{4y^2\left(2\sqrt{y} + y\right)^2}{3\left(\sqrt{y} - y\right)^2\left(3\sqrt{y} + y\right)^2} \tag{47}$$

and

$$\eta = \frac{4y^2\left(2\sqrt{y} + 2y - 1\right)}{3\left(\sqrt{y} - y\right)^2\left(3\sqrt{y} + y\right)} . \tag{48}$$

In this special case we find

$$n_s = 1 - \frac{8y^2\left(7y + 4y\sqrt{y} + y^2 + 3\sqrt{y}\right)}{3\left(\sqrt{y} - y\right)^2\left(3\sqrt{y} + y\right)^2} \tag{49}$$

and

$$r = \frac{64y^2\left(2\sqrt{y} + y\right)^2}{3\left(\sqrt{y} - y\right)^2\left(3\sqrt{y} + y\right)^2} . \tag{50}$$

The inflationary parameters in the special case are also given in Table 1.

Table: The values of y, N_e and r corresponding to $n_s = 0.961$ and $n_s = 0.969$, respectively, and the values of y_{end} for some values of the parameter δ.

δ	y_{end}	$y_{in,\,n_s=0.961}$	$y_{in,\,n_s=0.969}$	$N_{e,\,0.961}$	$N_{e,\,0.969}$	$r_{n_s=0.961}$	$r_{n_s=0.969}$
0	0.464	0.0140	0.0112	49.3	62.3	0.0043	0.0027
0.2	0.395	0.00682	0.00505	45.0	56.8	0.0096	0.0065
$\frac{4\sqrt{3}}{9}$	0.299	0.00146	0.000968	48.1	60.9	0.0152	0.0099
1	0.279	0.000939	0.000616	49.4	62.4	0.0157	0.0102
5	0.205	$4.32 \cdot 10^{-5}$	$2.75 \cdot 10^{-5}$	56.3	69.7	0.0168	0.0108
10	0.199	$1.08 \cdot 10^{-5}$	$6.91 \cdot 10^{-6}$	58.7	72.0	0.0168	0.0108
25	0.197	$1.74 \cdot 10^{-6}$	$1.11 \cdot 10^{-6}$	61.4	74.8	0.0169	0.0108
50	0.196	$4.34 \cdot 10^{-7}$	$2.77 \cdot 10^{-7}$	63.5	76.9	0.0169	0.0108
100	0.196	$1.09 \cdot 10^{-7}$	$6.92 \cdot 10^{-8}$	65.5	79.1	0.0169	0.0108

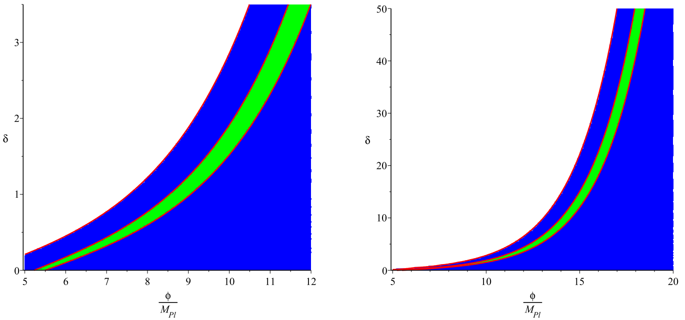


Figure: The inflaton field values against the values of the parameter δ. The green area corresponds to the observational restrictions on n_s and r. The blue area is defined by the restrictions on r only. When $\delta > 5$, the allowed domain is restricted by the lines $\delta\, y^2 = const$.

The $\left(R + R^2 + R^4\right)$ gravity model of inflation

We consider model defined by

$$F(R_J) = \frac{M_{Pl}^2}{2}\left[R_J + \frac{1}{6m^2}R_J^2 + \frac{\delta_4 R_J^4}{48m^6}\right] \tag{51}$$

with the dimensionless parameter $\delta_4 > 0$, as the natural alternative to the previous model.

We compute the inflaton scalar potential as follows:

$$V = V_0 \frac{\tilde{\sigma}^2(8 + 3\tilde{\sigma}^2\delta_4)}{72y^{-2}} = V_0 \frac{2\tilde{\sigma}^2\left(8 + 3\delta_4\tilde{\sigma}^2\right)}{\left(12 + 4\tilde{\sigma} + \delta_4\tilde{\sigma}^3\right)^2} \quad , \tag{52}$$

where y is related to F'. In the case under consideration we have

$$y^{-1} = 1 + \frac{\tilde{\sigma}}{3} + \frac{\delta_4}{12}\tilde{\sigma}^3 \quad . \tag{53}$$

Solving the cubic equation (53) yields $\tilde{\sigma}$ in terms of y^{-1} ,

$$\tilde{\sigma} = \frac{Z}{3\delta_4} - \frac{4}{Z} \,, \quad \text{where} \tag{54}$$

$$Z = \sqrt[3]{\left(162(y^{-1} - 1) + 6\sqrt{3}\sqrt{243(y^{-1} - 1)^2 + 16\delta_4{}^{-1}}\right)\delta_4^2} \, .$$

Assuming $\delta_4 \ll 1$ and using Eq. (54), we can expand $\tilde{\sigma}^2$ in power series of δ_4 near zero as follows:

$$\tilde{\sigma}^2 = 9(y^{-1}-1)^2 - \frac{81(y^{-1}-1)^4}{2}\delta_4 + \frac{5103(y^{-1}-1)^6}{16}\delta_4^2 + \mathcal{O}\left(\delta_4^3\right) \ . \quad (55)$$

Substituting Eq. (55) into Eq. (52), we get the scalar potential with the same accuracy,

$$\begin{aligned} V(y) &\approx V_0 \left[\frac{\left(y^{-1}-1\right)^2}{y^{-2}} - \frac{9}{8}\frac{\left(y^{-1}-1\right)^4}{y^{-2}}\delta_4 + \frac{81}{16}\frac{\left(y^{-1}-1\right)^6}{y^{-2}}\delta_4^2 \right] \\ &\approx \tilde{V}_{\mathrm{Star.}}(y) \left[1 - \frac{9}{8}\left(y^{-1}-1\right)^2\delta_4 + \frac{81}{16}\left(y^{-1}-1\right)^4\delta_4^2 \right] \ , \quad (56) \end{aligned}$$

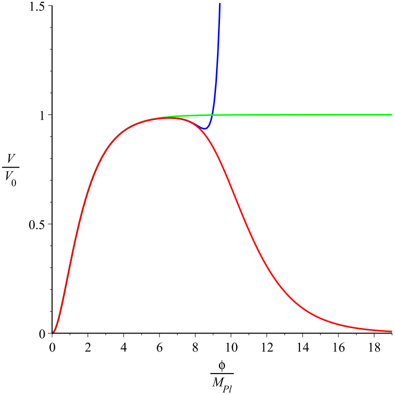


Figure: The scalar potential in the $(R + R^2 + R^4)$ model (red) versus the scalar potential in the Starobinsky $(R + R^2)$ model (green). The parameter value is $\delta_4 = 10^{-7}$. The approximated potential (56) in the $(R + R^2 + R^4)$ model is given by the blue line with the same value of $\delta_4 = 10^{-7}$.

The slow-roll parameters are

$$\epsilon \approx \frac{4}{3\left(y^{-1}-1\right)^2} - 3y^{-1}\delta_4 + \frac{27\left(y^{-1}+14\right)\left(y^{-1}-1\right)^2 y^{-1}}{16}\delta_4^2 \tag{57}$$

and

$$\eta \approx -\frac{4\left(y^{-1}-2\right)}{3\left(y^{-1}-1\right)^2} \tag{58}$$

$$-\frac{3\left(2y^{-1}+3\right)y^{-1}}{2}\delta_4 - \frac{81y^{-1}\left(-10y^{-3}+13y^{-2}+4y^{-1}-7\right)}{16}\delta_4^2 \, .$$

Accordingly, the inflationary observables $n_s = 1 - 6\epsilon + 2\eta$ and $r = 16\epsilon$ are given by

$$n_s \approx \frac{3y^{-2} - 14y^{-1} - 5}{3\left(y^{-1} - 1\right)^2} - 3y^{-1}\left(2y^{-1} - 3\right)\delta_4$$

$$-\frac{81y^{-1}\left(-9y^{-3} + 25y^{-2} - 23y^{-1} + 7\right)\delta_4^2}{8}$$

$$r \approx \frac{64}{3\left(y^{-1} - 1\right)^2} - 48y^{-1}\delta_4 + 27y^{-1}\left(y^{-1} + 14\right)\left(y^{-1} - 1\right)^2\delta_4^2 \quad .$$

The e-foldings number N_e as a function of the inflaton field is

$$N_e + N_0 \approx \frac{3}{4}\left(y^{-1} + \ln y\right) + \frac{27}{128}\left(y^{-1} - 1\right)^4 \delta_4 + \frac{243}{256}\left(7^{-1}y^{-7} - 2y^{-6} + 9y^{-5} - 20y^{-4} + 25y^{-3} - 18y^{-2} + 7y^{-1}\right)\delta_4^2 \ , \tag{59}$$

where the integration constant N_0 is fixed by the condition $\epsilon(y_{end}) = 1$. A numerical solution to the condition $\epsilon = 1$ with $\delta_4 \leqslant 10^{-6}$ yields the approximate values of the field and the e-foldings number at the end of inflation as $\phi_{end} \approx 0.9402 M_{Pl}$ and $N_0 \approx 1.040$.

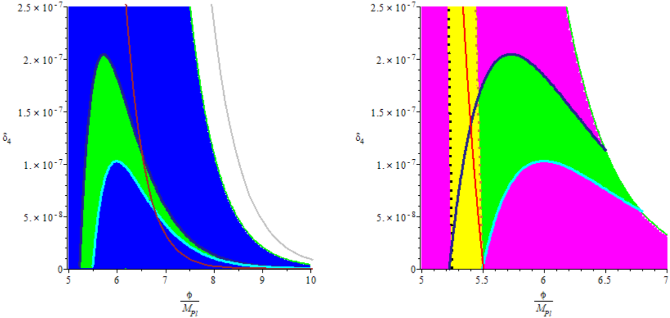


Figure: The green area on these pictures describes the values of the parameter δ_4 and $\phi = \phi_i$ that are in agreement with the observed values of the spectral index n_s and the tensor-to-scalar ratio r. The upper boundary of the green area (dark blue curve) corresponds to $n_s = 0.961$ and the lower boundary (blue curve) corresponds to $n_s = 0.969$. In the right picture, the yellow area corresponds to the e-foldings number N_e of the Starobinsky inflation for the allowed interval $0.961 \leqslant n_s \leqslant 0.969$, the left black dotted line corresponds to $N_e \approx 49.258$, the right orange dotted line corresponds to $N_e \approx 62.335$, and the red line corresponds to $r = 0.0027$.

The red curve in the middle of the left picture in Fig. 6 gives the function $\delta(\phi_{\mathrm{max}})$, where ϕ_{max} is the point of the maximum potential of V. This function can be presented in an analytical form

$$\delta_4(\phi_{max}) = \frac{y_{max}^{-1} + 1 - \sqrt{y_{max}^{-2} - 30y_{max}^{-1} - 15}}{9\left(2y_{max}^{-1} + 1\right)\left(y_{max}^{-1} - 1\right)^2} \quad ,$$

where $y_{max} = y(\phi_{max})$. The scalar field ϕ tends to zero during inflation if its initial value is less than ϕ_{max}.

The maximum allowed value of δ_4 is about 2×10^{-7}. In the case of the Starobinsky inflation, $\delta_4 = 0$, limiting the maximum value of the scalar spectral index n_s leads to a minimum value of the ratio of the tensor to the scalar $r = 0.0027$. When $\delta_4 > 0$, the minimum value of r decreases. In the right picture of Fig. 6, the red line corresponds to the value $r = 0.0027$ for $\delta_4 \neq 0$. To the left of the red line we have $r > 0.0027$, and $r < 0.0027$ to the right.

Our consideration provides quantitative estimation for value of δ_4 for which the parameters n_s and r are consistent with current observations.

Deforming the scalar potential in the Starobinsky model with analytic F-functions

The field y is *small* during slow-roll inflation. The inflaton potential (12) as a function of y is

$$V(\phi) = \tilde{V}(y) = V_0 \left[1 - 2y + y^2\right] \ , \tag{60}$$

where only the first two terms are essential for the CMB observables. The inflaton potential (12) can therefore be modified as

$$\tilde{V}(y) = V_0 \left[1 - 2y + y^2 \omega(y)\right] \ , \tag{61}$$

with arbitrary analytic function $\omega(y)$ *without* changing the CMB observables predicted by the Starobinsky model, at least for those values of ω that are not very large. The Starobinsky model appears at $\omega = 1$.

Equation (23) reads

$$\tilde{\sigma} \equiv \frac{R_J}{m^2} = 3\left(\frac{1}{y} - 1 - \frac{1}{2}y^2\frac{d\omega}{dy}\right) \quad , \tag{62}$$

and Eq. (24) is given by

$$F = V_0\left(\frac{1}{y^2} - \omega - y\frac{d\omega}{dy}\right) \quad . \tag{63}$$

As a check, in the Starobinsky case, $\omega = 1$ and $V = V_0(1-y)^2$, and Eq. (23) gives

$$y = \left(1 + \frac{R_J}{3m^2}\right)^{-1} \quad . \tag{64}$$

Substituting it into Eq. (24), we get

$$F_{Star.}(R_J) = \frac{M_{Pl}^2}{2}\left(R_J + \frac{R_J^2}{6m^2}\right) \quad , \tag{65}$$

as it should be. Moreover, when ω is an arbitrary constant, we find

$$F(R_J) = F_{Star.}(R_J) - \Lambda \ , \tag{66}$$

where $\Lambda = V_0(1-\omega)$ is a cosmological constant.

Let us consider the case of

$$\omega(y) = \omega_0 + \omega_1 y \ , \tag{67}$$

where $\omega_0 \leqslant 1$ and $\omega_1 > 0$ are constants.
The constant ω_1 should be positive for the potential V bounded from below. The inequality $\omega_0 \leqslant 1$ is needed for positivity of a cosmological constant, see Eq. (66).
Equation (62) leads to the depressed cubic equation

$$y^3 + \frac{2}{\omega_1}\left(1 + \frac{R_J}{3m^2}\right) y - \frac{2}{\omega_1} \equiv y^3 + py + q = 0 \tag{68}$$

with the negative discriminant

$$\Delta = -\left(4p^3 + 27q^2\right) = -\frac{32}{\omega_1^3}\left(1 + \frac{R_J}{3m^2}\right)^3 - \frac{108}{\omega_1^2} < 0 \ ,$$

so that it has only one real root.

The explicit $F(R)$ function in this case is as follows:

$$
\begin{aligned}
\frac{F}{V_0} &= \frac{1}{y^2} - \omega_0 - 2\omega_1 y \\
&= \omega_1^{\frac{2}{3}} \left[\left(1 + \sqrt{1 + \frac{8}{27\omega_1}\left(1 + \frac{R_J}{3m^2}\right)^3} \right)^{\frac{1}{3}} + \left(1 - \sqrt{1 + \frac{8}{27\omega_1}\left(1 + \frac{R_J}{3m^2}\right)^3} \right)^{\frac{1}{3}} \right]^{-2} \\
&- 2\omega_1^{\frac{2}{3}} \left[\left(1 + \sqrt{1 + \frac{8}{27\omega_1}\left(1 + \frac{R_J}{3m^2}\right)^3} \right)^{\frac{1}{3}} + \left(1 - \sqrt{1 + \frac{8}{27\omega_1}\left(1 + \frac{R_J}{3m^2}\right)^3} \right)^{\frac{1}{3}} \right] \\
&- \omega_0 \ ,
\end{aligned}
$$

where

$$
\begin{aligned}
\omega_0 = \omega_1^{2/3} &\left[\left(1 + \sqrt{1 + \frac{8}{27\omega_1}} \right)^{1/3} + \left(1 - \sqrt{1 + \frac{8}{27\omega_1}} \right)^{1/3} \right]^{-2} \\
- 2\omega_1^{2/3} &\left[\left(1 + \sqrt{1 + \frac{8}{27\omega_1}} \right)^{1/3} + \left(1 - \sqrt{1 + \frac{8}{27\omega_1}} \right)^{1/3} \right] .
\end{aligned}
$$

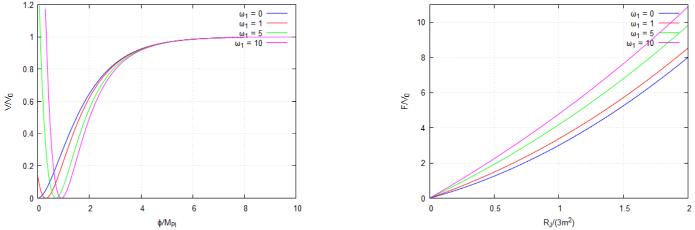


Figure: The scalar potential of the canonical inflaton field ϕ (left) and the related $F(R)$-function (right) in the case of the deformation of the Starobinsky model for some values of the parameter ω_1: 0, 1, 5, and 10.

The second derivative $F_{,\sigma\sigma}$ is

$$F_{,\sigma\sigma}(y) = \frac{M_{Pl}^2}{6m^2\left(1+\omega_1 y^3\right)} \quad , \tag{69}$$

so that the considering $F(R)$ gravity model satisfies the stability conditions at $\omega_1 \geqslant 0$.

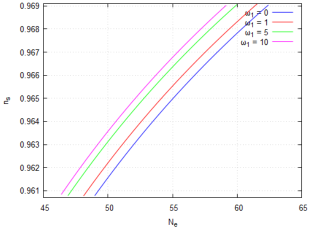


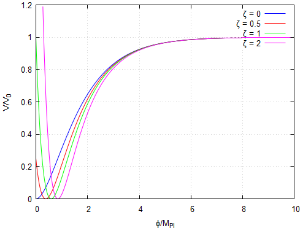
Figure: The index n_s (left) of scalar perturbations and the tensor-to-scalar ratio r (right) as the functions of e-folds N_e in the case I of the deformation of the Starobinsky model for some values of the parameter ω_1: 0, 1, 5, and 10.

Let us consider another one-parametric deformation of the Starobinsky potential as follows:

$$V(\phi) = V_0 \left[1 - e^{-\sqrt{2/3}\phi/M_{Pl}} - \zeta e^{-2\sqrt{2/3}\phi/M_{Pl}} \right]^2 = V_0 \left(1 - y - \zeta y^2 \right)^2 , \tag{70}$$

where we assume the parameter $\zeta \geqslant 0$. This potential can be realized in supergravity [5] We find a restriction on the deformation parameter $\zeta < \left(1 - e^{-\sqrt{2/3}} \right) e^{2\sqrt{2/3}} \approx 2.9$.

[5] S.V. Ketov, On the equivalence of Starobinsky and Higgs inflationary models in gravity and supergravity, J. Phys. A 53 (2020) 084001 [1911.01008]



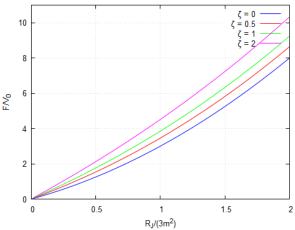


Figure: The scalar potential of the canonical inflaton field ϕ, and the related $F(R)$ function in case II.

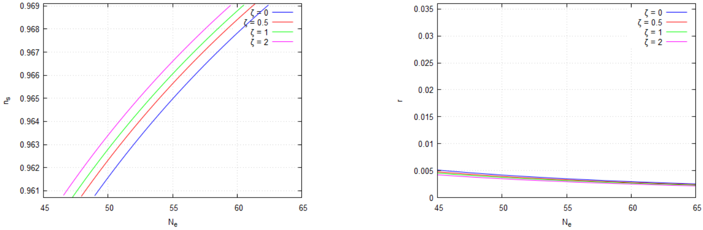


Figure: The index n_s (left) of scalar perturbations and the tensor-to-scalar ratio r (right) as the functions of e-folds N_e in case II for some values of the parameter ζ: 0, 1/2, 1, and 2.

The critical points of the potential correspond to three solutions,

$$y_{1,2} = \frac{1 \mp \sqrt{1 - 4\,|\zeta|}}{2\,|\zeta|} \ , \quad y_3 = \frac{1}{2\,|\zeta|} \ . \tag{71}$$

To get slow-roll inflation, we need $|\zeta| < 1/4$. It leads to the scalar potential with two minima and one maximum, while only one minimum has $y < 1$, thus leading to hilltop inflation with restricted initial conditions.

CONCLUSIONS

• We studied several extensions of the Starobinsky inflation model of the $(R + R^2)$ gravity in the context of $F(R)$ gravity and scalar-tensor gravity.

• By deforming the scalar potential of the Starobinsky model, and derived the corresponding F-function in the analytic form in a new model with a single parameter, and found the lower and upper bounds on the values of the parameters. The new model is very close to the original Starobinsky model of inflation, as regards their inflationary parameters.

• The modification of the Starobinsky model by the $R^{3/2}$ term does not lead to significant constraints on its coefficient in slow-roll inflation, at least for $0 < \delta < 100$.

Thank for your attention