Functional reduction of one-loop Feynman integrals with arbitrary masses

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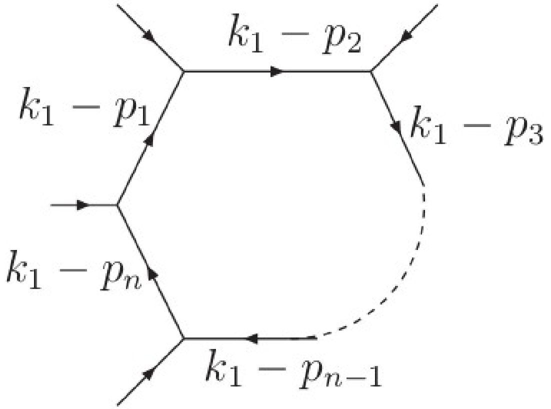
A method of functional reduction for the dimensionally regularized one-loop Feynman integrals with massive propagators is described in detail. The method is based on a repeated application of the functional relations proposed by the author. Explicit formulae are given for reducing one-loop scalar integrals to a simpler ones, the arguments of which are the ratios of polynomials in the masses and kinematic invariants. We show that a general scalar n-point integral, depending on n(n + 1)/2 generic masses and kinematic variables, can be expressed as a linear combination of integrals depending only on n variables. The latter integrals are given explicitly in terms of hypergeometric functions of (n − 1) dimensionless variables. Analytic expressions for the 2-, 3- and 4-point integrals, that depend on the minimal number of variables, were also obtained by solving the dimensional recurrence relations. The resulting expressions for these integrals are given in terms of Gauss' hypergeometric function ${}_2F_1$, the Appell function F_1 and the hypergeometric Lauricella - Saran function F_S . A modification of the functional reduction procedure for some special values of kinematical variables is considered.

Speaker: Maxim Bezuglov

Algebraic relation between propagators

$$I_n^{(d)}(\{m_j^2\};\{s_{ik}\}) = \frac{1}{i\pi^{d/2}} \int \frac{d^d k_1}{D_1 \ldots D_n}$$



$$D_j = (k_1 - p_j)^2 - m_j^2$$

p_j – external momenta

$$\prod_{r=1}^{n} \frac{1}{D_r} = \frac{1}{D_0} \sum_{r=1}^{n} x_r \prod_{\substack{j=1 \\ j \neq r}}^{n} \frac{1}{D_j}.$$

new propagator

p_0, m_0^2 and $x_j (j = 1, \ldots n)$ must be chosen so that the ratio above holds

to do this, they must satisfy the system of equations

Algebraic relation between propagators

$$\prod_{r=1}^{n} \frac{1}{D_r} = \frac{1}{D_0} \sum_{r=1}^{n} x_r \prod_{\substack{j=1 \\ j\neq r}}^{n} \frac{1}{D_j}. \qquad \Big| \qquad \times \prod_{j=0}^{n} D_j$$

$$D_0 = \sum_{r=1}^{n} x_r D_r, \quad \longrightarrow \quad k_1^2 - 2k_1 p_0 + p_0^2 - m_0^2 = \sum_{r=1}^{n} x_r (k_1^2 - 2k_1 p_r + p_r^2 - m_r^2).$$

Differentiating the above by k_1 and given that this is the independent variable of integration we find

$$\left.\begin{aligned} 1 &= \sum_{r=1}^{n} x_r, \\ p_0 &= \sum_{j=1}^{n} x_j p_j. \end{aligned}\right\}$$

$$m_0^2 - \sum_{k=1}^{n} x_k m_k^2 + \sum_{j=2}^{n} \sum_{l=1}^{j-1} x_j x_l s_{lj} = 0,$$

where

$$s_{ij} = s_{i,j} = (p_i - p_j)^2.$$

the solution depends on $(n-2)$ remaning parameters x_i and one arbitrary mass m_0

$$I_n^{(d)}(\{m_r^2\};\{s_{ik}\}) = \sum_{j=1}^{n} x_j \, I_n^{(d)}(\{m_r^2\};\{s_{ik}\})\Big|_{m_j^2 \to m_0^2,\, s_{jk} \to s_{0k}}.$$

Method of functional reduction
Sincov's functional equation

$$f(x, y) = f(x, z) - f(y, z).$$

The general solution

$$f(x, y) = g(x) - g(y),$$

where

$$g(x) = f(x, 0)$$

I.e. the function f (x, y) is a combination of its 'boundary values', which may be completely arbitrary.

Method of functional reduction

$$I_n^{(d)}(\{m_r^2\};\{s_{ik}\}) = \sum_{j=1}^{n} x_j \; I_n^{(d)}(\{m_r^2\};\{s_{ik}\})\Big|_{m_j^2 \to m_0^2, s_{jk} \to s_{0k}} .$$

Additional conditions designed to reduce the number of variables:

$$s_{0j} = 0, \quad s_{0j} - s_{0i} = 0, \quad s_{0j} \pm s_{ik} = 0, \quad s_{0j} \pm m_0^2 = 0, \quad m_j^2 \pm m_0^2 = 0,$$
$$m_0^2 = 0, \qquad s_{0j} \pm m_0^2 \pm m_k^2 = 0, \qquad (i, j, k = 1 \ldots n).$$

Solutions of these systems of equations and analysis of these solutions were performed using computer algebra system MAPLE. The number of these systems depends on n and varied from 10^3 to 10^6. CPU execution time ranged from a few minutes to several hours. Many solutions of these equations have been found. Some of them lead to a simultaneous decrease in the number of variables in all integrals on the right-hand side of the functional equation

Functional reduction of the 2-point integral

$$I_2^{(d)}(m_1^2, m_2^2;\ s_{12}) = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{1}{[(k_1-p_1)^2 - m_1^2][(k_1-p_2)^2 - m_2^2]}.$$

Algebraic relation between propagators:

$$\frac{1}{D_1 D_2} = \frac{x_1}{D_0 D_2} + \frac{x_2}{D_1 D_0}$$

$$x_1 + x_2 = 1, \qquad p_0 = x_1 p_1 + x_2 p_2,$$

Algebraic conditions on parameters :

$$m_0^2 - x_1 m_1^2 - x_2 m_2^2 + x_1 x_2 s_{12} = 0.$$

The only arbitrary parameter will be m_0

$$x_1 = \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} \pm \frac{\sqrt{4s_{12}(m_0^2 - r_{12})}}{2s_{12}}, \qquad x_2 = 1 - x_1$$

$$s_{10} = m_1^2 + m_0^2 - 2r_{12} \pm \frac{m_2^2 - m_1^2 - s_{12}}{2s_{12}} \sqrt{4s_{12}(m_0^2 - r_{12})}, \qquad s_{20} = m_2^2 + m_0^2 - 2r_{12} \pm \frac{m_2^2 - m_1^2 + s_{12}}{2s_{12}} \sqrt{4s_{12}(m_0^2 - r_{12})},$$

$$r_{12} = -\frac{\lambda_{12}}{g_{12}} = \frac{2m_1^2 m_2^2 + 2s_{12}m_1^2 + 2s_{12}m_2^2 - m_1^4 - m_2^4 - s_{12}^2}{4s_{12}}.$$

Notations for determinants

modified Cayley determinant:

$$\Delta_n \equiv \Delta_n(\{p_1, m_1\}, \ldots \{p_n, m_n\}) = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1n} \\ Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix},$$

$$Y_{ij} = m_i^2 + m_j^2 - s_{ij},$$

Gram determinant:

$$G_{n-1} \equiv G_{n-1}(p_1, \ldots, p_n) = -2 \begin{vmatrix} S_{11} & S_{12} & \ldots & S_{1\ n-1} \\ S_{21} & S_{22} & \ldots & S_{2\ n-1} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n-1\ 1} & S_{n-1\ 2} & \ldots & S_{n-1\ n-1} \end{vmatrix}$$

$$S_{ij} = s_{in} + s_{jn} - s_{ij},$$

$$\lambda_{i_1 i_2 \ldots i_n} = \Delta_n(\{p_{i_1}, m_{i_1}\}, \{p_{i_2}, m_{i_2}\}, \ldots, \{p_{i_n}, m_{i_n}\})$$

$$g_{i_1 i_2 \ldots i_n} = G_{n-1}(p_{i_1}, p_{i_2}, \ldots, p_{i_n}).$$

$$r_{ij\ldots k} = -\frac{\lambda_{ij\ldots k}}{g_{ij\ldots k}}.$$

and

$$\kappa_{j_r j_1 \ldots j_{r-1} j_{r+1} \ldots j_n} = \frac{\partial r_{j_1 \ldots j_r \ldots j_n}}{\partial m_{j_r}^2}.$$

Functional reduction of the 2-point integral

$$I_2^{(d)}(m_1^2, m_2^2;\ s_{12}) = \int \frac{d^d k_1}{i\pi^{d/2}} \frac{1}{[(k_1-p_1)^2 - m_1^2][(k_1-p_2)^2 - m_2^2]}.$$

Algebraic relation between propagators:

$$\frac{1}{D_1 D_2} = \frac{x_1}{D_0 D_2} + \frac{x_2}{D_1 D_0}$$

Integrating algebraic relation over momentum k_1 we get the functional relation

$$I_2^{(d)}(m_1^2, m_2^2; s_{12}) = x_1 I_2^{(d)}(m_2^2, m_0^2; s_{20}) + x_2 I_2^{(d)}(m_1^2, m_0^2; s_{10}).$$

The only arbitrary parameter will be m_0

- Case 1. $m_0^2 = 0$
- Case 2. $m_0^2 = r_{12}$
- Case 3. Combination of two equations

Case 1. $m_0^2 = 0$

$$I_2^{(d)}(m_1^2, m_2^2; s_{12}) = \overline{x}_1 I_2^{(d)}(m_2^2, 0; \overline{s}_{20}) + \overline{x}_2 I_2^{(d)}(m_1^2, 0; \overline{s}_{10}),$$

$$\overline{x}_{1,2} = x_{1,2}\big|_{m_0^2=0}\,, \qquad \overline{s}_{01} = s_{01}\big|_{m_0^2=0}\,, \qquad \overline{s}_{02} = s_{02}\big|_{m_0^2=0}$$

$$I_2^{(d)}(m^2, 0;\; p^2) = -\Gamma\left(1 - \frac{d}{2}\right) m^{d-4}\; {}_2F_1\left[\begin{matrix}1, 2 - \frac{d}{2}\,; \\ \frac{d}{2}\,;\end{matrix}\; \frac{p^2}{m^2}\right]$$

Case 2. $m_0^2 = r_{12}$

$$I_2^{(d)}(m_1^2, m_2^2; s_{12}) = \kappa_{12} I_2^{(d)}(r_{12}, r_2; r_2 - r_{12}) + \kappa_{21} I_2^{(d)}(r_{12}, r_1; r_1 - r_{12})$$

where

$$\kappa_{12} = \frac{\partial r_{12}}{\partial m_1^2}, \qquad \kappa_{21} = \frac{\partial r_{12}}{\partial m_2^2}, \qquad r_i = m_i^2.$$

$$(d-1) I_2^{(d+2)}(r_{12}, r_j; r_j - r_{12}) = -2 r_{12} I_2^{(d)}(r_{12}, r_j; r_j - r_{12}) - I_1^{(d)}(r_j).$$

$$I_2^{(d)}(r_{12}, r_j; r_j - r_{12}) = \frac{-\pi^{\frac{3}{2}} r_{12}^{\frac{d}{2}-2}}{2 \sin \frac{\pi d}{2} \Gamma\left(\frac{d-1}{2}\right)} \sqrt{\frac{r_{12}}{r_{12} - r_j}} + \frac{\pi}{2 r_{12}} \frac{r_j^{\frac{d}{2}-1}}{\sin \frac{\pi d}{2} \Gamma\left(\frac{d}{2}\right)} \, {}_2F_1\left[\begin{matrix} 1, \frac{d-1}{2} \\ \frac{d}{2} \end{matrix}; \frac{r_j}{r_{12}}\right]$$

Case 3. Combination of two equations

We take $m_0^2 = m_2^2$

$$I_2(m_1^2, m_2^2; s_{12}) = \frac{m_1^2 - m_2^2}{s_{12}} I_2^{(d)}\left(m_1^2, m_2^2;\ \frac{(m_1^2 - m_2^2)^2}{s_{12}}\right) + \frac{s_{12} - m_1^2 + m_2^2}{s_{12}} I_2^{(d)}\left(m_2^2, m_2^2;\ \frac{(s_{12} - m_1^2 + m_2^2)^2}{s_{12}}\right)$$

invariant under interchange $m_1^2 \longleftrightarrow m_2^2$

$$I_2(m_1^2, m_2^2;\ s_{12}) = \frac{s_{12} + m_1^2 - m_2^2}{2s_{12}} I_2^{(d)}\left(m_1^2, m_1^2;\ \frac{(s_{12} + m_1^2 - m_2^2)^2}{s_{12}}\right)$$

$$+ \frac{s_{12} - m_1^2 + m_2^2}{2s_{12}} I_2^{(d)}\left(m_2^2, m_2^2;\ \frac{(s_{12} - m_1^2 + m_2^2)^2}{s_{12}}\right)$$

$$I_2^{(d)}(m^2, m^2; p^2) = m^{d-4}\ \Gamma\left(2 - \frac{d}{2}\right)\ {}_2F_1\left[\begin{matrix}1, 2 - \frac{d}{2}\ ; \\ \frac{3}{2}\ ;\end{matrix}\ \frac{p^2}{4m^2}\right]$$

Functional reduction of the 3-point integral

$$I_3^{(d)}(m_1^2, m_2^2, m_3^2; s_{23}, s_{13}, s_{12}) = \frac{1}{i\pi^{d/2}} \int \frac{d^d k_1}{D_1 D_2 D_3}.$$

Algebraic relation between propagators:
$$\frac{1}{D_1 D_2 D_3} = \frac{x_1}{D_0 D_2 D_3} + \frac{x_2}{D_1 D_0 D_3} + \frac{x_3}{D_1 D_2 D_0}.$$

Algebraic conditions on parameters :
$$\begin{cases} p_0 = x_1 p_1 + x_2 p_2 + x_3 p_3, \\ x_1 + x_2 + x_3 = 1, \\ x_1 x_2 s_{12} + x_1 x_3 s_{13} + x_2 x_3 s_{23} - x_1 m_1^2 - x_2 m_2^2 - x_3 m_3^2 + m_0^2 = 0. \end{cases}$$

The functional equation depends on two arbitrary parameters

$$I_3^{(d)}(m_1^2, m_2^2, m_3^2; s_{23}, s_{13}, s_{12}) = x_1 I_3^{(d)}(m_0^2, m_2^2, m_3^2; s_{23}, s_{03}, s_{02}) + x_2 I_3^{(d)}(m_1^2, m_0^2, m_3^2; s_{03}, s_{13}, s_{01}) + x_3 I_3^{(d)}(m_1^2, m_2^2, m_0^2; s_{02}, s_{01}, s_{12})$$

Functional reduction goes in two steps

Step 1

$$
\begin{aligned}
I_3\left(m_1^2, m_2^2, m_3^2; s_{23}, s_{13}, s_{12}\right)
&= \kappa_{123} I_3\left(r_{123}, r_2, r_3; s_{23}, r_3 - r_{123}, r_2 - r_{123}\right) \\
&+ \kappa_{213} I_3\left(r_{123}, r_1, r_3; s_{13}, r_3 - r_{123}, r_1 - r_{123}\right) \\
&+ \kappa_{312} I_3 \underbrace{\left(r_{123}, r_2, r_1; s_{12}, r_1 - r_{123}, r_2 - r_{123}\right)}_{\text{four independent variables}},
\end{aligned}
$$

where

$$
r_{123} = -\frac{\lambda_{123}}{g_{123}}, \qquad r_i = m_i^2,
$$

$$
\kappa_{123} = \frac{\partial r_{123}}{\partial m_1^2}, \quad \kappa_{213} = \frac{\partial r_{123}}{\partial m_2^2}, \quad \kappa_{312} = \frac{\partial r_{123}}{\partial m_3^2},
$$

Step 2

$$I_3^{(d)}\left(r_{123}, r_2, r_3; s_{23}, r_3 - r_{123}, r_2 - r_{123}\right) = \kappa_{23} I_3^{(d)}(r_{123}, r_{23}, r_3; r_3 - r_{23}, r_3 - r_{123}, r_{23} - r_{123}) + \kappa_{32} I_3^{(d)}(r_{123}, r_{23}, r_2; r_2 - r_{23}, r_2 - r_{123}, r_{23} - r_{123})$$

$$\begin{aligned}
I_3^{(d)}(m_1^2, m_2^2, m_3^2; s_{23}, s_{13}, s_{12}) \\
&= \kappa_{123}\kappa_{23} I_3^{(d)}(r_{123}, r_{23}, r_3; r_3 - r_{23}, r_3 - r_{123}, r_{23} - r_{123}) \\
&+ \kappa_{123}\kappa_{32} I_3^{(d)}(r_{123}, r_{23}, r_2; r_2 - r_{23}, r_2 - r_{123}, r_{23} - r_{123}) \\
&+ \kappa_{213}\kappa_{31} I_3^{(d)}(r_{123}, r_{13}, r_1; r_1 - r_{13}, r_1 - r_{123}, r_{13} - r_{123}) \\
&+ \kappa_{213}\kappa_{13} I_3^{(d)}(r_{123}, r_{13}, r_3; r_3 - r_{13}, r_3 - r_{123}, r_{13} - r_{123}) \\
&+ \kappa_{312}\kappa_{12} I_3^{(d)}(r_{123}, r_{12}, r_2; r_2 - r_{12}, r_2 - r_{123}, r_{12} - r_{123}) \\
&+ \kappa_{312}\kappa_{21} I_3^{(d)}\underbrace{(r_{123}, r_{12}, r_1; r_1 - r_{12}, r_1 - r_{123}, r_{12} - r_{123})}_{\text{three independent variables}}.
\end{aligned}$$

Analytic results for integrals depending on the MNV

$$(d-2)I_3^{(d+2)}(r_{123}, r_{23}, r_3; r_3 - r_{23}, r_3 - r_{123}, r_{23} - r_{123}) =$$
$$- 2r_{123}I_3^{(d)}(r_{123}, r_{23}, r_3; r_3 - r_{23}, r_3 - r_{123}, r_{23} - r_{123}) - I_2^{(d)}(r_{23}, r_3; r_3 - r_{23})$$

Solution

$$I_3^{(d)}(r_{123}, r_{23}, r_3; r_3 - r_{23}, r_3 - r_{123}, r_{23} - r_{123}) =$$
$$\frac{1}{\sin\frac{\pi d}{2}}\left\{\frac{r_{123}^{\frac{d-6}{2}}}{\Gamma\left(\frac{d-2}{2}\right)}C_3(x,y) + \frac{\pi^{\frac{3}{2}}r_{23}^{\frac{d-4}{2}}}{4r_{123}\Gamma\left(\frac{d-1}{2}\right)}\sqrt{\frac{r_{23}}{r_{23}-r_3}}\ {}_2F_1\left[\begin{matrix}1, \frac{d-2}{2}\ ;\\ \frac{d-1}{2}\ ;\end{matrix}\ \frac{r_{23}}{r_{123}}\right]\right.$$
$$\left. - \frac{\pi r_3^{\frac{d-2}{2}}}{4\Gamma\left(\frac{d}{2}\right)(r_{23}-r_3)r_{123}}\sqrt{1-\frac{r_3}{r_{23}}}\ F_1\left(\frac{d-2}{2}, 1, \frac{1}{2}, \frac{d}{2}; \frac{r_3}{r_{123}}, \frac{r_3}{r_{23}}\right)\right\}$$

where

$$C_3(x,y) = \frac{\pi x y^2}{4(x^2-y^2)^{\frac{1}{2}}}\ln\left(\frac{x-(x^2-y^2)^{\frac{1}{2}}}{x+(x^2-y^2)^{\frac{1}{2}}}\right) \qquad x = \sqrt{\frac{r_{123}}{r_{123}-r_3}}, \qquad y = \sqrt{\frac{r_{123}}{r_{123}-r_{23}}}.$$

Functional reduction of the 4-point integral

The integral depending on ten variables is rewritten through a combination of integrals depending on four independent variables

$$
\begin{aligned}
&I_4(m_1^2, m_2^2, m_3^2, m_4^2; s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) \\
&= \kappa_{1234}\kappa_{234}\kappa_{34} I_4^{(d)}(r_{1234}, r_{234}, r_{34}, r_4; \\
&\qquad r_{234}-r_{1234}, r_{34}-r_{234}, r_4-r_{34}, r_4-r_{1234}, r_4-r_{234}, r_{34}-r_{1234}) \\
&+\kappa_{1234}\kappa_{234}\kappa_{43} I_4^{(d)}(r_{1234}, r_{234}, r_{34}, r_3; \\
&\qquad r_{234}-r_{1234}, r_{34}-r_{234}, r_3-r_{34}, r_3-r_{1234}, r_3-r_{234}, r_{34}-r_{1234}) \\
&+\kappa_{1234}\kappa_{324}\kappa_{24} I_4^{(d)}(r_{1234}, r_{234}, r_{24}, r_4; \\
&\qquad r_{234}-r_{1234}, r_{24}-r_{234}, r_4-r_{24}, r_4-r_{1234}, r_4-r_{234}, r_{24}-r_{1234}) \\
&+\kappa_{1234}\kappa_{324}\kappa_{42} I_4^{(d)}(r_{1234}, r_{234}, r_{24}, r_2; \\
&\qquad r_{234}-r_{1234}, r_{24}-r_{234}, r_2-r_{24}, r_2-r_{1234}, r_2-r_{234}, r_{24}-r_{1234}) \quad + \ldots
\end{aligned}
$$

The full answer is almost 2 pages long

Functional reduction of the 5 and 6-point integrals

The I_5 integral depending on 15 variables can be represented as a linear combination of 120 integrals, each of which depends on only 5 variables

$$I_5^{(d)}(m_i^2, m_j^2, m_k^2, m_l^2, m_r^2;\ m_j^2 - m_i^2, m_k^2 - m_j^2, m_l^2 - m_k^2, m_r^2 - m_l^2, m_r^2 - m_i^2,$$
$$m_k^2 - m_i^2, m_l^2 - m_i^2, m_l^2 - m_j^2, m_r^2 - m_j^2, m_r^2 - m_k^2),$$

where m_i^2, m_j^2,m_k^2,m_l^2,m_r^2 are ratios of polynomials in masses and kinematic invariants.

The I_6 integral depending on 21 variables can be represented as a linear combination of 720 integrals, each of which depends on only 6 variables

General algorithm of the functional reduction

Final functional reduction formulae for the integrals $I_2^{(d)}, \ldots, I_6^{(d)}$ can be obtained by exploiting the following algorithm:

- write down the term

$$\kappa_{1\ldots n}\kappa_{2\ldots n}\ldots\kappa_{n-1\ n} \quad I_n^{(d)}(m_1^2, m_2^2, \ldots m_n^2; s_{12}, s_{23}, \ldots) \tag{10.1}$$

- replace in the integral $s_{ij} \to m_j^2 - m_i^2 \ \ (j > i)$
- replace in the integral $m_1^2 \to r_{1\ldots n},\ m_2^2 \to r_{2\ldots n}, \ldots, m_n^2 \to r_n$
- replace $\kappa_{ij\ldots} \to \frac{\partial r_{ij\ldots}}{\partial m_i^2}$
- generate $n! - 1$ terms by symmetrizing the term (10.1) with respect to the indices $1, 2, \ldots n$ and add all these terms to (10.1).

All steps are very straightforward and easily achieved with a computer program. This algorithm works perfectly for integrals $I_2^{(d)},\ldots,I_6^{(d)}$. We verified numerically that it is also valid for integrals $I_7^{(d)}$, $I_8^{(d)}$. Notice that the number of terms in the final reduction formula for massless integrals is $n!/2$.

Conclusions

- Author provided a systematic approach for reducing a generic n-point one-loop integral with arbitrary masses and kinematic invariants to a linear combination of integrals that depend on n variables.
- The integrals depending on the MNV encountered at the last stage of the reduction were expressed in terms of multiple hypergeometric series depending on $n-1$ dimensionless variables.
- Analytic results for integrals with the MNV can be derived by solving dimensional recurrence relations.
- Obtained representations of one-loop integrals can be helpful for deriv-
- ing $\varepsilon = (4 - d)/2$ expansion of these integrals.

Thank you for your attention!