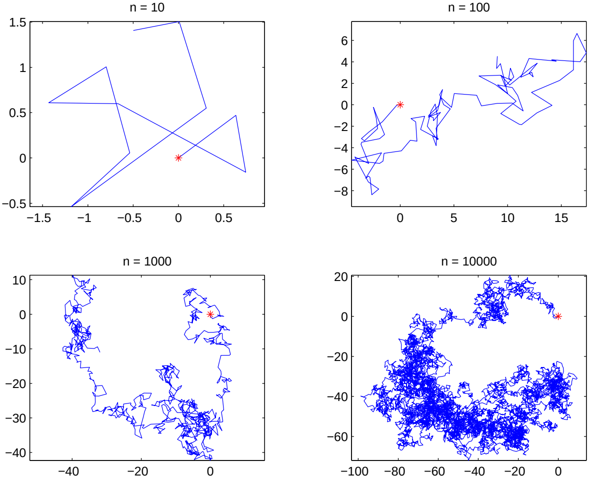
Problems of critical and stochastic dynamics

Michal Hnatič[1]
deputy director of BLTP JINR Dubna

III Summer School OMUS, Alushta

[1]P.J.Šafarik University and IEP Slovak Academy of Science, Košice, Slovakia

Brown motion

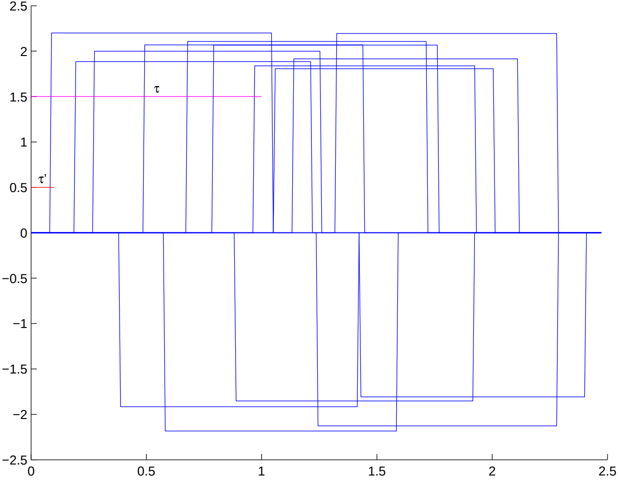


Obr. : Brown motion in select axis

Brown motion

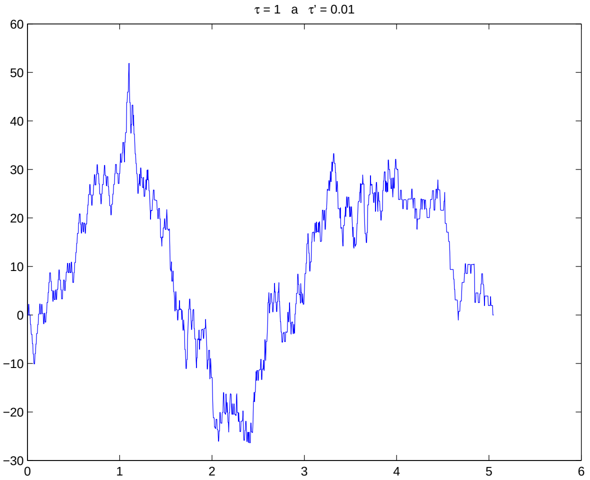
- Random force
- caused by repeated chaotic collisions of macroscopic particle with molecules of environment
- Typical time scales
- time interval between collisions — $\tau' \sim 10^{-17} - -10^{-16}$ s
- collision duration time – $\tau \sim 10^{-12}$ s,
- time scale after the information about initial state is lost — $\tau_M \cong 1/\Gamma \sim 10^{-10}$ s.
- unequalities $\tau' \lll \tau \lll \tau_M$ are valid

Brown motion



Obr. : Geometric mapping of force

Brown motion



Obr. : Geometric mapping of force

Brown motion

random motion of macroskopic particle - Brown motion

$$\frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} = -\gamma\mathbf{v}(t) + \mathbf{f}(t),$$

White noise

$$\langle f(t)f(t')\rangle = \Gamma\delta(t - t'),$$

Stochastic equations of critical dynamics

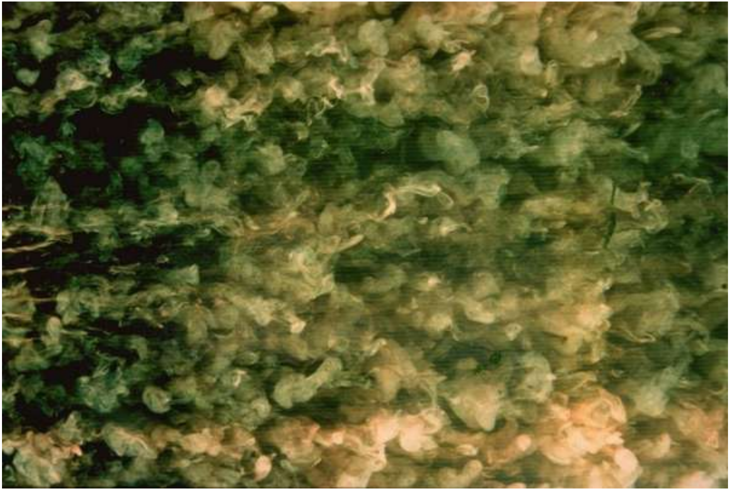
$$\frac{\partial\varphi(t,\mathbf{x})}{\partial t} = -\alpha\frac{\delta S(t,\mathbf{x})}{\delta\varphi} + \eta(t,\mathbf{x}),$$

static action

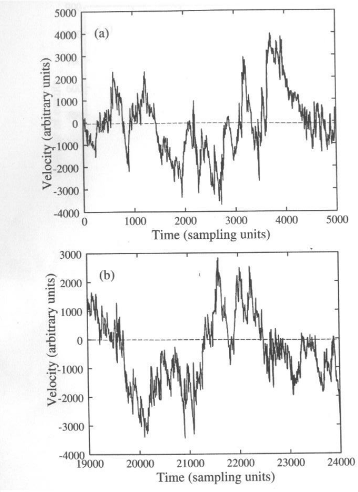
$$S(\varphi) = \int \mathrm{d}t\,\mathrm{d}\mathbf{x}\left[\frac{(\nabla\varphi)^2}{2} + \frac{\tau_0\varphi^2}{2} + \frac{g_0\varphi^4}{24} - \varphi h_0\right]$$

White noise in time and space coordinates

$$\langle\,\eta(t,\mathbf{x})\eta(t'\mathbf{x}')\,\rangle = 2\alpha\delta(\mathbf{x}-\mathbf{x}')\delta(t-t').$$



Obr. : Developed turbulence past grid



Obr. : Velocity fluctuations

Universal properties

- chaos (stochasticity)
- unpredictability of the details of motion
- evidently random process
- starting point - stochastic differential equations

Stochastic Navier-Stokes equation

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \nu_0 \nabla^2 \mathbf{v} + \nabla p = \mathbf{f}$$

$$\langle f_i(x) f_j(x') \rangle \equiv D_{ij}(x, x') = \frac{\delta(t - t')}{(2\pi)^d} \int \mathrm{d}\mathbf{k}\, D(k) P_{ij}(\mathbf{k}) \exp[i\mathbf{k}\mathbf{x}]$$

White noise in time and color in space coordinates

$$D(k) = D_0 k^{4-d-2\varepsilon} F(kL)$$

$$\overline{\mathcal{E}} = -\frac{\nu_0}{2} \langle (\nabla_i v_j + \nabla_j v_i)^2 \rangle$$

$$\overline{\mathcal{E}} = -\frac{d-1}{2(2\pi)^d} \int \mathrm{d}\mathbf{k}\, D(k)$$

Kolmogorov scaling

- Structure functions S_p of velocity field $\mathbf{v}$:

$$S_p(r) \equiv \langle [v_r(\mathbf{x}) - v_r(\mathbf{x}')]^p \rangle \,, \quad r \equiv |\mathbf{x} - \mathbf{x}'| \,, \quad v_r \equiv \mathbf{v}\mathbf{r}/r$$

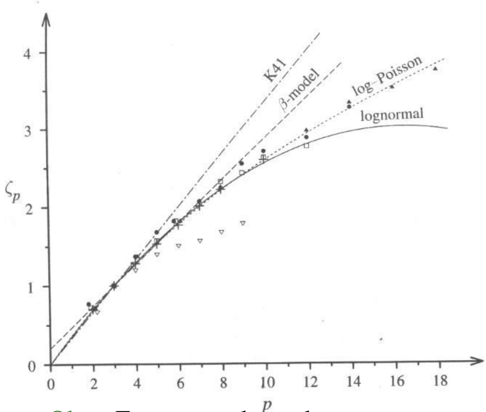
- First Kolmogorov hypothesis

$$S_p(r) = (\overline{\mathcal{E}} r)^{p/3} f_p(r/L) \,, \qquad r \gg l$$

- Kolmogorov power laws (second Kolmogorov hypothesis)

$$S_p(r) = C_p(\overline{\mathcal{E}} r)^{\varsigma_p} \,, \quad \varsigma_p = p/3 \,, \quad l \ll r \ll L$$

- $p = 2, \quad C_2 \simeq C_k$ — Kolmogorov constant $C_k \approx 1.5$



Obr. : Exponents dependence ς_p on p

Obr. : Diffusion and advection processes

Obr. : Diffusion and advection processes

Obr. : Diffusion and advection processes

Another stochastic equations

- Another problems (transport phenomena, diffusion, magnetic field)

$$\partial_t \theta + (\mathbf{v}\nabla)\theta - u\nu\Delta\theta + H(\theta, \mathbf{v}) = \mathbf{f}^\theta$$

- u - inverse Prandtl number
- θ - concentration field, temperature fluctuations, magnetic field
- Additive and multiplicative noises

Stochastic differential equations

$$\partial_t \phi(x) = V(x,\phi) + f(x), \quad \langle f(x) f(x') \rangle = D(x,x')$$

$\phi(x) \equiv \phi(t,\mathbf{x})$ - random fields
random force $f(x)$ - Gaussian distribution $D = D(x)$ - noise
Assumption:

- equation valid for all space coordinates and for time instants $(-\infty\,, +\infty)$
- fields asymptotically vanish: $\phi \to 0$ as $t \to -\infty$ and as $|\mathbf{x}| \to \infty$ for arbitrary tme instants t
- retardation condition

Stochastic differential equations

- $V(x, \varphi)$ - t-local functional, containing regular external force F_r, linear part $L\phi$ and nonlinear part $N(\phi)$:

$$V(\phi) = L\phi + N(\phi) + F_r$$

$$V(\phi) = \alpha \frac{\delta S(\phi)}{\delta \phi(x)}$$

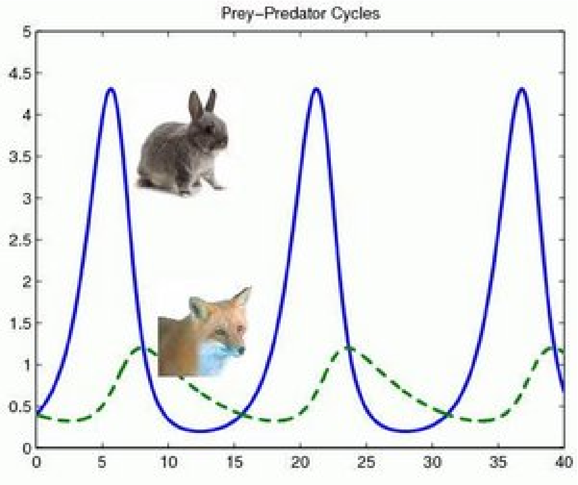
Lotka-Volterra model

- 1926: Vito Volterra
- 1925: Alfred Lotka

$$\frac{dx}{dt} = (b - py)x \qquad \frac{dy}{dt} = (rx - d)y$$

- Periodical solution
- Lotka-Volterra model – oldest model of mathematical ecology

Another random processes



Verhulst (logistical) model

- 1845: P.V. Verhulst

$$\frac{dn}{dt} = -\beta n + \lambda n - \gamma n^2,$$

- $n(t)$ - mean number of individuals at time instant t, β - measure of mortality , λ - measure of natality
- quadratic delimiting term

description of tumour growth, autocatalic reactions

Another random processes

Reaction-diffusion process

- chemical reaction of species A: $A + A \to \emptyset$
- Equation for density of particles $n(t, \mathbf{x})$

$$\frac{\partial n}{\partial t} = D\nabla^2 n - 2\lambda n^2$$

- How to include elements of randomness?
- Important feature: number of species is not preserved, creation and annihilation processes
- direct inclusion of random force is not suitable

Another random processes

Master equations

$$\frac{dP_f(t)}{dt} = \sum_i [w(i \to f)P_i - w(f \to i)P_f]$$

- $w(i \to f)$ -probability of transition of the system from state i to state f per time unit
- $P_f(t)$ - a probability that the system is in state f at time instant t

Verhuest model

$$\begin{aligned} \frac{dP(n,t)}{dt} &= [\beta(n+1) + \gamma(n+1)^2]P(n+1,t) + \lambda(n-1)P(n-1,t) \\ &- (\beta n + \gamma n^2 + \lambda n)P(n,t) \end{aligned}$$

Stochastic differential equations

$$\partial_t \phi(x) = V(x, \phi) + f(x), \quad \langle f(x) f(x') \rangle = D(x, x') \,, \tag{1}$$

$\phi(x) \equiv \phi(t, \mathbf{x})$ random fields
random force $f(x)$ - Gaussian distribution $D = D(x)$ - noise
Assumption:

- equation valid for all space coordinates and for time instants $(-\infty\,, +\infty)$
- fields asymptotically vanish: $\phi \to 0$ as $t \to -\infty$ and as $|\mathbf{x}| \to \infty$ for arbitrary tme instants t
- retardation condition

- $V(x, \varphi)$ - t-local functional, containing regular external force F_r, linear part $L\phi$ and nonlinear part $N(\phi)$:

$$V(\phi) = L\phi + N(\phi) + F_r$$

$$V(\phi) = \alpha \frac{\delta S(\phi)}{\delta \phi(x)}$$

Statistical averages

Statistical averages of field ϕ

- *correlation functions*

$$\langle \phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle$$

- *response functions*

$$\left\langle \frac{\delta^m \left[\phi(x_1)\ldots\phi(x_n)\right]}{\delta f(x'_1)\ldots\delta f(x'_m)} \right\rangle$$

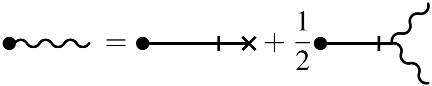
- average over statistics (gaussian) of random force

Solution of stochastic equation

- perturbation approach
- integral form

$$\phi = \Delta_{12} \left[F_r + f + N(\phi) \right] , \tag{2}$$

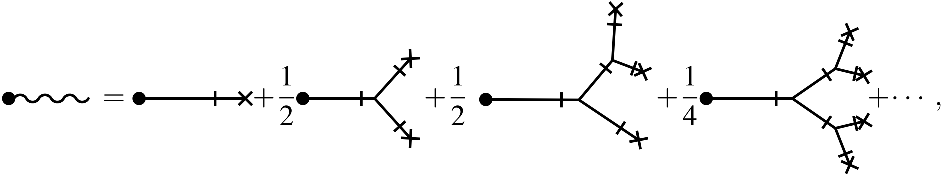
- $\Delta_{12} = \Delta_{12}(x, x') \equiv (\partial_t - L)^{-1}$ retardation Green function of linear operator $(\partial_t - L)$, $\Delta_{12}(x, x') = 0$ pre $t < t'$.
- perturbation solution with $N(\phi) = g\phi^2(x)/2$
- graphic representation



- basic diagrammatic elements - times, wavy and straight lines, interaction vertex
- infinity series of three graphs

Solution of stochastic equation

- Perturbative solution with g^3 precision



- correlation functions - mutual multiplying of graphs for corresponding numbers of field ϕ and averaging over all realizations of random force f

Solution of stochastic equation

- contraction of pairs creating noise D
- new graphical element pair correlator function of field ϕ in leading approximation $\langle\phi\phi\rangle_0$

$\Delta_{11} \equiv \langle\phi\phi\rangle_0 = \Delta_{12} D \Delta_{21} =$ 

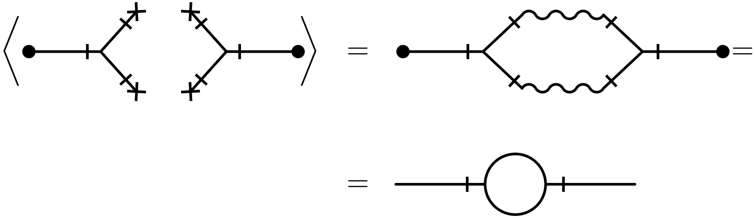
- wavy line bounded by vertical dash - noise D

$$\Delta_{21}(x,x') \equiv \Delta_{12}^T(x,x') = \Delta_{12}(x',x),$$

- all $\times$ have to be contracted

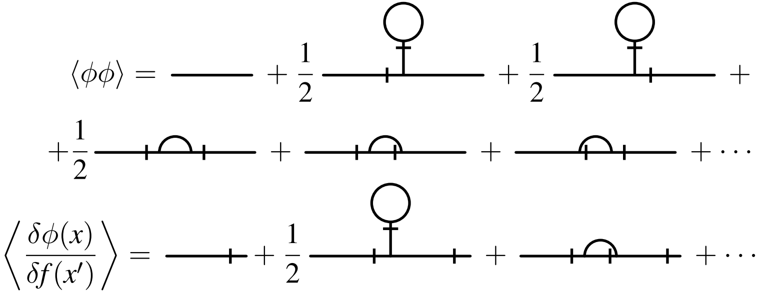
Solution of stochastic equation

- example of graph for pair correlation function of field ϕ in next to leading order (one loop) approximation



Solution of stochastic equation

- illustration of perturbative scheme: a few first graphs for correlation pair function $\langle\phi\phi\rangle$ and response function $\langle\delta\phi(x)/\delta f(x')\rangle$:



- graphs do not contained closed loops of response function (line with vertical dash) $\Delta_{12}(x,x')$

Transition to quantum field model

- solution of SDE: $\tilde{\phi} = \tilde{\phi}(x; f)$
- generating functional $G(A^\phi)$

$$G(A^\phi) = \int \mathrm{D}f \exp\left[-\frac{fD^{-1}f}{2} + A^\phi \tilde{\phi}\right]$$

- $\mathrm{D}f$ - functional integral measure,

$$A^\phi \tilde{\phi} = \int \mathrm{d}x A^\phi(x) \tilde{\phi}(x)$$

$$fD^{-1}f = \iint \mathrm{d}x\, \mathrm{d}x' f(x) D^{-1}(x, x') f(x')$$

Transition to quantum field model

- Useful identity

$$\exp(A^\phi \tilde{\phi}) = \int D\phi \delta(\phi - \tilde{\phi}) \exp(A^\phi \phi)$$

- Functional δ-function

$$\delta(\phi - \tilde{\phi}) \equiv \prod_x \delta \left[\phi(x) - \tilde{\phi}(x)\right]$$

$$\phi = \tilde{\phi} \Leftrightarrow Q(\phi, f) \equiv -\partial_t \phi + V(\phi) + f = 0$$

$$\delta(\phi - \tilde{\phi}) = \delta\left[Q(\phi, f)\right] \det M\,, \quad M = \frac{\delta Q}{\delta \phi}$$

$$M(x, x') = \delta Q(x)/\delta\phi(x')$$

$$\delta\left[Q(\phi, f)\right] = \int D\phi'\, e^{[\phi'\, Q(\phi, f)]}$$

- ϕ' - auxiliary field

Transition to quantum field model

- Generating functional

$$G(A^{\phi}) = \int\int D\phi D\phi' \,\det M \exp\left[\phi' D\phi'/2 + \phi'\left(-\partial_t\phi + V(\phi)\right) + A^{\phi}\phi\right]$$

- contribution of the determinant

$$M = -\partial_t + L + \frac{\delta N(\phi)}{\delta\phi} = -\Delta_{12}^{-1}\left[1 - \Delta_{12}\frac{\delta N(\phi)}{\delta\phi}\right],$$

- $det\, M = exp[tr\;\; lnM]$

$$detM \approx e^{\left[1-\Delta_{12}\frac{\delta N(\phi)}{\delta\phi}\right]}$$

Transition to quantum field model

- generating functional

$$G(A^{\phi}) = \iint \mathrm{D}\phi\, \mathrm{D}\phi'\, \mathrm{e}^{S(\phi,\phi')+A^{\phi}\phi},$$

- action

$$S(\phi,\phi') = \operatorname{tr}\ln\left(1 - \Delta_{12}\frac{\delta N(\phi)}{\delta\phi}\right) + \frac{\phi' D\phi'}{2} + \phi'\left(-\partial_t\phi + L\phi + N(\phi)\right)$$

$$\operatorname{tr}\ln\left(1 - \Delta_{12}\frac{\delta N(\phi)}{\delta\phi}\right) = -\operatorname{tr}\left[\Delta_{12}\frac{\delta N(\phi)}{\delta\phi} + \frac{1}{2}\Delta_{12}\frac{\delta N(\phi)}{\delta\phi}\Delta_{12}\frac{\delta N(\phi)}{\delta\phi} + \cdots\right]$$

Transition to quantum field model

- excercise: to find the solution and form of equal-time response function

$$(\partial_t - L)\Delta_{12}(x, x') = \delta(x - x')$$

$$S(\phi, \phi') = \frac{\phi' D \phi'}{2} + \phi' \left(-\partial_t \phi + L\phi + N(\phi)\right)$$

Transition to quantum field model

- Final action

$$S(\phi, \phi') = \iint \mathrm{d}x\, \mathrm{d}x' \, \frac{\phi'(x) D(x, x') \phi'(x')}{2} + \int \mathrm{d}x \, \phi'(x) \left[-\partial_t \phi(x) + V(\phi(x)) \right]$$

- Final generating functional

$$G(A) = \int \mathrm{D}\Phi \, \exp\left[S(\Phi) + A\Phi \right], \, A\Phi \equiv \int \mathrm{d}x \left[A^{\phi}(x) \phi(x) + A^{\phi'}(x) \phi'(x) \right]$$

- Green functions - correlation and response functions

$$\langle \phi(x) \phi'(x') \rangle = \frac{\delta^2 G(A)}{\delta A^{\phi}(x) \, \delta A^{\phi'}(x')} |_{A=0} = \int \mathrm{D}\Phi \, \phi(x) \phi'(x') e^{S(\Phi)}$$

Transition to quantum field model

- Wick theorem and Feynman graphs

$$G(A) = \exp\left(\frac{1}{2}\frac{\delta}{\delta\Phi}\Delta\frac{\delta}{\delta\Phi}\right) \exp[S_I(\Phi) + A\Phi]|_{\Phi=0}]$$

- nonlinear part of action S_I

$$S_I = \int \mathrm{d}x\, \phi'(x) N\left(\phi(x)\right)$$

Transition to quantum field model

- QFT model of annihilation reaction

$$S = -\int_0^t dt \int d\mathbf{x} \left\{ \psi^\dagger \partial_t \psi - D\psi^\dagger \nabla^2 \psi + \lambda D[2\psi^\dagger + (\psi^\dagger)^2]\psi^2 + n_0 \int d\mathbf{x} \psi^\dagger(\mathbf{x}, 0) \right\}$$