The QCD equation of state in background magnetic fields

review of the paper
written by

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$2+1$ QCD

$$f = \frac{\mathcal{F}}{V} = -\frac{T}{V}\ln \mathcal{Z} = \epsilon - Ts = \epsilon^{\text{total}} - Ts - e\vec{B}\cdot\vec{\mathcal{M}} \qquad \text{– free energy density} \qquad (1)$$

ϵ – energy density; s – entropy density; $\vec{\mathcal{M}}$ – magnetization

Thermodynamics in an external magnetic field

$2+1$ QCD

$$f = \frac{\mathcal{F}}{V} = -\frac{T}{V} \ln \mathcal{Z} = \epsilon - Ts = \epsilon^{\text{total}} - Ts - e\vec{B} \cdot \vec{\mathcal{M}} \qquad \text{– free energy density} \tag{1}$$

ϵ – energy density; s – entropy density; $\vec{\mathcal{M}}$ – magnetization

$$p_i = -\frac{1}{V} L_i \frac{\partial \mathcal{F}}{\partial L_i} \qquad \text{– pressure in the } i\text{-th direction} \tag{2}$$

$eB \quad \Rightarrow \quad$ preferred direction $\quad \Rightarrow \quad$ In general, $p_{||} \neq p_{\perp}$

This depends on the magnetic field setup:

- "B-scheme": $eB = \text{const}$
- "Φ-scheme": $\vec{\Phi} = e\vec{B} \cdot L_x L_y = \overrightarrow{\text{const}}$

Thermodynamics in an external magnetic field

$$p_i = -\frac{1}{V} L_i \frac{\partial \mathcal{F}}{\partial L_i}$$

$$e\vec{B} = \frac{\vec{\Phi}}{L_x L_y}$$

B-scheme

$$\mathcal{F} = E^{\text{total}} - TS - e\vec{B} \cdot \vec{\mathcal{M}} \cdot L_x\, L_y\, L_z$$

$$\left.\begin{aligned} p_x &= p^{\text{isotr}} + e\vec{B} \cdot \vec{\mathcal{M}} \\ p_y &= p^{\text{isotr}} + e\vec{B} \cdot \vec{\mathcal{M}} \\ p_z &= p^{\text{isotr}} + e\vec{B} \cdot \vec{\mathcal{M}} \end{aligned}\right\} \Rightarrow \; p_x^{(B)} = p_y^{(B)} = p_z$$

Φ-scheme

$$\mathcal{F} = E^{\text{total}} - TS - \vec{\Phi} \cdot \vec{\mathcal{M}} \cdot L_z$$

$$\left.\begin{aligned} p_x &= p^{\text{isotr}} \\ p_y &= p^{\text{isotr}} \\ p_z &= p^{\text{isotr}} + e\vec{B} \cdot \vec{\mathcal{M}} \end{aligned}\right\} \Rightarrow \; p_x^{(\Phi)} = p_y^{(\Phi)} = p_z - e\vec{B} \cdot \vec{\mathcal{M}}$$

$$p_z = -f$$

Thermodynamics in an external magnetic field

$$\epsilon = -\frac{T}{V} L_t \frac{\partial \ln \mathcal{Z}}{\partial L_t} \qquad p_i = \frac{T}{V} L_i \frac{\partial \ln \mathcal{Z}}{\partial L_i} \qquad \mathcal{M} = \frac{T}{V} \frac{\partial \ln \mathcal{Z}}{\partial (eB)}$$

$$s = -\frac{1}{V} \frac{\partial \mathcal{F}}{\partial T}$$

$$I = \epsilon - p_x - p_y - p_z = -\frac{T}{V} \frac{\mathrm{d} \ln \mathcal{Z}}{\mathrm{d} \ln a}$$ – interaction measure (trace anomaly)

$$\chi_B = \left. \frac{\partial \mathcal{M}}{\partial (eB)} \right|_{B=0} = -\frac{1}{V} \left. \frac{\partial^2 \mathcal{F}}{\partial (eB)^2} \right|_{B=0}$$ – magnetic susceptibility

Lattice observables and methods

$N_t \times N_s^3$ lattice $\qquad T = 1/(aN_t), \qquad V = (aN_s)^3$

$$\mathcal{Z} = \int \mathcal{D}[U]\mathcal{D}[\psi,\bar{\psi}]\,\mathrm{e}^{-S(\psi,\bar{\psi},U)} = \int \mathcal{D}[U]\,\mathrm{e}^{-\beta S_g} \prod_{f=u,d,s} \left[\det M(U, q_f B, m_f)\right]^{1/4} \qquad (3)$$

Symanzik improved gauge action:

$$S_g = \frac{1}{3}\sum_n \sum_{\mu\neq\nu} \mathrm{Re}\,\mathrm{Tr}\left\{-\frac{1}{12}\left[\mathbb{1} - U^{2\times1}_{\mu\nu}(n)\right] + \frac{5}{6}\left[\mathbb{1} - U^{1\times1}_{\mu\nu}(n)\right]\right\} \qquad (4)$$

Fermion matrix for staggered quarks with external field[1]:

$$M(n|f) = \frac{1}{2a}\sum_\mu \eta_\mu(n)\left[u_\mu(qB,n)\,U_\mu(n)\delta_{f,n+\hat{\mu}} - u^\star_\mu(qB,f)\,U^\dagger_\mu(f)\delta_{f,n-\hat{\mu}}\right] + m\,\delta_{f,n} \qquad (5)$$

$$u_x(N_x-1, n_y, n_z, n_t) = \mathrm{e}^{-ia^2 qBN_x n_y}, \qquad u_x(n) = 1, \quad n_x \neq N_x - 1$$

$$u_y(n) = \mathrm{e}^{ia^2 qBn_x}$$

$$u_\nu(n) = 1, \quad \nu \notin \{x, y\}$$

PBC $\Rightarrow$ $\Phi = (aN_s)^2 eB = 6\pi N_b, \quad N_b \in \mathbb{Z}$

[1] In the parer stout improved action for staggered fermions is actually used

Lattice observables and methods

The desired quantity: $p_z = \dfrac{T}{V}\ln\mathcal{Z}$

$$\mathcal{Z} = \int \mathcal{D}[U]\mathcal{D}[\psi,\bar\psi]\,\mathrm{e}^{-S(\psi,\bar\psi,U)} = \int \mathcal{D}[U]\,\mathrm{e}^{-\beta S_g} \prod_{f=u,d,s} [\det M(U, q_f B, m_f)]^{1/4} \qquad (3)$$

$$\mathcal{Z} = \mathcal{Z}(\beta, am_f, \Phi)$$

$\dfrac{\partial}{\partial\Phi}$ is not defined

$\Downarrow$

Integration over a constant-Φ trajectory

Desired observables:

$$a^4 s_g = -\frac{1}{N_s^3 N_t}\frac{\partial \ln\mathcal{Z}}{\partial\beta}$$ – gauge action density

$$a^3 \bar\psi_f \psi_f = \frac{1}{N_s^3 N_t}\frac{\partial \ln\mathcal{Z}}{\partial(am_f)}$$ – quark condensate density

$$\Delta X = X|_B - X|_0$$

The integral method at nonzero magnetic field

$$\frac{\Delta p_z(\Phi, T_2; \beta_2)}{T_2^4} - \frac{\Delta p_z(\Phi, T_1; \beta_1)}{T_1^4} = N_t^4 \int_{\beta_1}^{\beta_2} \mathrm{d}\beta \left[-a^4 \Delta s_g + \sum_f \frac{\partial(a m_f^{\mathsf{ph}})}{\partial \beta} \cdot a^3 \Delta \bar{\psi}_f \psi_f \right] \tag{6}$$

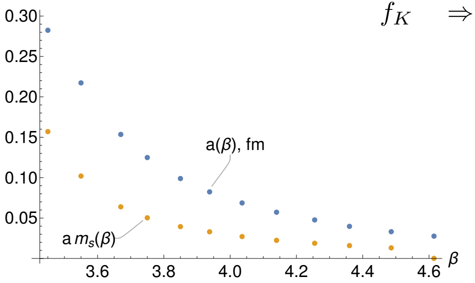
The integral method at nonzero magnetic field

$$\frac{\Delta p_z(\Phi, T_2; \beta_2)}{T_2^4} - \frac{\Delta p_z(\Phi, T_1; \beta_1)}{T_1^4} = N_t^4 \int_{\beta_1}^{\beta_2} \mathrm{d}\beta \left[-a^4 \Delta s_g + \sum_f \frac{\partial(am_f^{\mathrm{ph}})}{\partial \beta} \cdot a^3 \Delta \bar{\psi}_f \psi_f \right] \qquad (6)$$

am_f^{ph} is tuned along the LCP:

$\frac{f_K}{M_\pi}$ and $\frac{f_K}{M_K}$ are fixed $\Rightarrow \quad m_u = m_d = \frac{m_s}{28.15}$

$f_K \quad \Rightarrow \quad a(\beta)$



[15] S. Borsányi et al., "The QCD equation of state with dynamical quarks", JHEP 11 (2010) [arXiv:1007.2580]

The integral method at nonzero magnetic field

$$\frac{\Delta p_z(\Phi,T_2;\beta_2)}{T_2^4}-\frac{\Delta p_z(\Phi,T_1;\beta_1)}{T_1^4}=N_t^4\int_{\beta_1}^{\beta_2}\mathrm{d}\beta\left[-a^4\Delta s_g+\sum_f\frac{\partial(am_f^{\text{ph}})}{\partial\beta}\cdot a^3\Delta\bar{\psi}_f\psi_f\right] \quad (6)$$

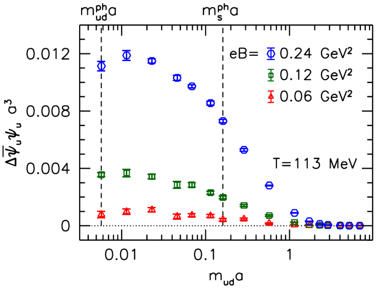
Determination of the integration constant:

At $B=0$: $p/T^4 \xrightarrow{T\to 0} 0$

$\Delta p_z \to 0 \quad \forall\ \Phi<\infty,\ T; \qquad m_f^2 \gg qB$

At $B\neq 0$ $\lim_{T\to 0}(p/T^4)\neq 0;$ $\quad a\,m_f=\infty \quad \Rightarrow$ pure SU(3) + free quarks

No contribution to Δp_z



$$\frac{\Delta p_z(\Phi,T;\beta)}{T^4}=-N_t^4\sum_f\int_{a\,m_f^{\text{ph}}}^{\infty}\mathrm{d}(a\,m_f)\,a^3\Delta\bar{\psi}_f\psi_f \quad (7)$$

In practice the integration is over m_u and m_d up to the point where $m_u=m_d=m_s$ ($N_f=3$ theory with different q_f). Then the integration is over the quark masses simultaneously up to $a\,m_f=\infty$.

The integral method at nonzero magnetic field

$$\frac{\Delta p_z(\Phi, T_2; \beta_2)}{T_2^4} - \frac{\Delta p_z(\Phi, T_1; \beta_1)}{T_1^4} = N_t^4 \int_{\beta_1}^{\beta_2} \mathrm{d}\beta \left[-a^4 \Delta s_g + \sum_f \frac{\partial (a m_f^{\mathsf{ph}})}{\partial \beta} \cdot a^3 \Delta \bar{\psi}_f \psi_f \right] \quad (6)$$

$$\frac{\Delta p_z(\Phi, T; \beta)}{T^4} = -N_t^4 \sum_f \int_{a\, m_f^{\mathsf{ph}}}^{\infty} \mathrm{d}(a\, m_f)\, a^3 \Delta \bar{\psi}_f \psi_f \quad (7)$$

The order of calculation:

- With eq. (7) determine the integration constant (at fixed lattice, Φ, $\beta\,(\Rightarrow T)$)
- With eq. (6) determine Δp_z for various T and Φ (lattice is fixed)
- Renormalization of $\Delta p_z \quad \Rightarrow \quad \Delta p_{z,r}(\Phi, T)$
- Interpolation of resulting curves $\quad \Rightarrow \quad \Delta p_{z,r}(eB, T)$ for any B and T
- Shifting by the zero-field pressure taken from [43] $\quad \Rightarrow \quad$ full pressure for a range of T and B

[43] S. Borsányi et al., "Full result for the QCD equation of state with 2+1 flavors", Phys. Lett. B 730 (2014) [arXiv:1309.5258]

Renormalization

f contains additive divergences in the cutoff – in $1/a$.

They are independent of eB, except for $[-b_1(eB)^2 \ln(\mu\, a)]$, which is canceled through

$$\frac{B^2}{2} \to \frac{B_r^2}{2} + b_1(eB)^2 \ln(\mu\, a) \qquad (8)$$

Added as counter-term to Δf

Can be ommited from the Lagrangian

$$Z_e = 1 + 2b_1 e_r^2 \ln(\mu\, a), \qquad B^2 = Z_e B_r^2, \qquad e^2 = Z_e^{-1} e_r^2, \qquad eB = e_r B_r$$

b_1 is related to the QED β-function

eB is external $\Rightarrow$ no U(1) degrees of freedom $\Rightarrow$ only the lowest order coeff. b_1 in Z_e

Renormalization

Charge renormalization

Free case:

$$b_1^{\text{free}} = \sum_f b_{1f}^{\text{free}}, \quad b_{1f}^{\text{free}} = \frac{N_c}{12\pi^2} \cdot \left(\frac{q_f}{e}\right)^2 \tag{9}$$

$$\Delta \ln \mathcal{Z}_r^{\text{free}} = b_{1f}^{\text{free}} \cdot \Phi^2 \cdot \ln(m_f\, a) + \mathcal{O}(\Phi^4) - b_1^{\text{free}} \cdot \Phi^2 \cdot \ln(\mu\, a) \tag{10}$$

$$\Delta \bar{\psi}_f \psi_f^{\text{free}} = \frac{1}{L^4} \frac{\partial \Delta \ln \mathcal{Z}_r^{\text{free}}}{\partial m_f} = b_{1f}^{\text{free}} \frac{(eB)^2}{m_f} + \mathcal{O}[(eB)^4] \tag{11}$$

Full QCD:

$$b_1(a) = b_1^{\text{free}} \cdot \left[1 + \sum_{i \geq 1} c_i g^{2i}(1/a)\right] \xrightarrow{a \to 0} b_1^{\text{free}} \tag{12}$$

$$\Delta \ln \mathcal{Z}_r = b_1(a) \cdot \Phi^2 \cdot \ln(\Lambda_H\, a) + \mathcal{O}(\Phi^4) - b_1(a) \cdot \Phi^2 \cdot \ln(\mu\, a) \tag{13}$$

$$\Delta \bar{\psi}_f \psi_f = b_1(a) \cdot \frac{(eB)^2}{\Lambda_H} \cdot \frac{\partial \Lambda_H}{\partial m_f} + \mathcal{O}[(eB)^4] \tag{14}$$

Renormalization

$\mu = \Lambda_H$ is chosen.

$$f_r = (1 - \mathcal{P})[f], \qquad p_{z,r} = (1 - \mathcal{P})[p_z] \tag{15}$$

$\mathcal{P}$ is the operator that projects out the $\mathcal{O}[(eB)^2]$ term:

$$\mathcal{P}[X] = (eB)^2 \cdot \lim_{eB \to 0} \frac{X}{(eB)^2}\bigg|_{T=0} \tag{16}$$

At finite T additional finite terms appear here $\Rightarrow$ subtraction of $\mathcal{P}[X]$ is to be performed at $T = 0$.

Lattice ensembles

	$m_{ud}/m_{ud}^{\rm ph}$	$24^3 \times 6$	$24^3 \times 8$	$28^3 \times 10$	$36^3 \times 12$	$48^3 \times 16$	$24^3 \times 32$
low-T	$1 \dots 1200$	$\beta = 3.45$	$\beta = 3.55$	$\beta = 3.625$	$\beta = 3.695$	$\beta = 3.81$	$\beta = 3.45, 3.55$
high-T	1	$\beta = 3.45 \dots 3.81$	$\beta = 3.55 \dots 3.94$	$\beta = 3.625 \dots 4.06$			

$$113\,\text{MeV} < T < 300\,\text{MeV}$$

eB up to 0.7 GeV^2

Condensates, the β-function and a commant on magnetic catalysis

Dashed line:

$m_f \to \infty \quad \longrightarrow \quad$ pure SU(3) + free quarks

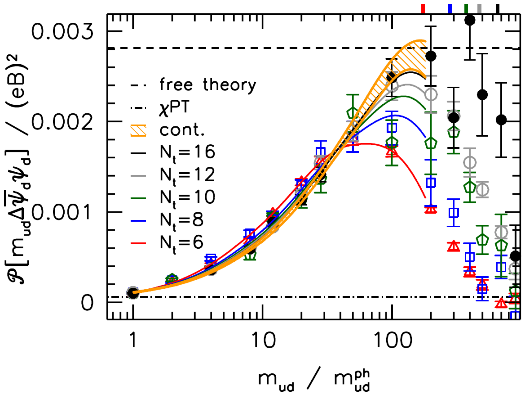
$$\Delta\bar\psi_f\psi_f \longrightarrow \Delta\bar\psi_f\psi_f^{\text{free}}$$

From eq. (11):

$$\frac{\mathcal{P}[m_f\cdot\Delta\bar\psi_f\psi_f^{\text{free}}]}{(eB)^2} = b_{1f}^{\text{free}} \tag{17}$$

Dashed-dotted line:

$$\frac{\mathcal{P}[m_f\cdot\Delta\bar\psi_f\psi_f^{\chi^{PT}}]}{(eB)^2} = \frac{b_{1f}^{\text{free}}}{16N_c}$$



Cont. limit at large m_f is consistent with eq. (17) $\Rightarrow$

$\Rightarrow$ in cont. limit at $m_f \gg \Lambda_{QCD}$ $\Delta\bar\psi_f\psi_f \sim 1/m_f$ $\Rightarrow$ logarithmic divergence in eq. (7).

On the lattice this is removed through integration up to $a\,m_f \sim 1$.

In cont. limit this divergence reappears and should be subtracted via charge renormalization.

$$\frac{\Delta p_z(\Phi,T;\beta)}{T^4} = -N_t^4\sum_f\int_{a\,m_f^{\text{ph}}}^{\infty} \mathrm{d}(a\,m_f)\,a^3\Delta\bar\psi_f\psi_f \tag{7}$$

Quadratic contribution to the EoS

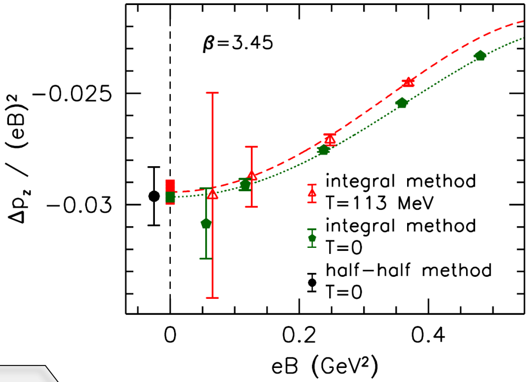
Calculation of the $\mathcal{O}[(eB)^2]$ contribution to p_z at $T \simeq 0$

- Performing integration in eq. (7) at $T = 0$ for various Φ
- Fit data with quadratic in $(eB)^2$ curve and extrapolation to obtain

$$\lim_{eB \to 0} \frac{\Delta p_z}{(eB)^2}$$

- Do the same at $T = 113$ MeV

$T = 113$ MeV may be used instead of $T = 0$ for the quadratic subtraction



$$\frac{\Delta p_z(\Phi, T; \beta)}{T^4} = -N_t^4 \sum_f \int_{a\, m_f^{\text{ph}}}^{\infty} \mathrm{d}(a\, m_f)\, a^3 \Delta \bar{\psi}_f \psi_f \qquad (7)$$

Quadratic contribution to the EoS

Calculation of the $\mathcal{O}[(eB)^2]$ contribution to p_z at $T \simeq 0$

- Consider obtained values of $\lim_{eB\to 0} \frac{\Delta p_z}{(eB)^2}$ at $T = 113$ MeV for different lattices as function of $\ln(a/a_0)$,
  $a_0 = 1.47$ GeV^{-1} – the largest used lattice spacing (at $N_t = 6$)

From eq. (13)

$$\frac{\Delta p_z}{(eB)^2} = b_1 \ln(\Lambda_H a)$$

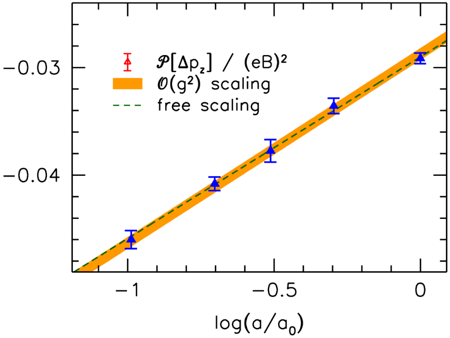
The data are fitted with this function:

1. b_1 from eq. (3.14) with $i = 1$:

$$b_1(a) = b_1^{\text{free}}[1 + c_1 g^2(1/a)],$$

$$g^2(1/a) = \frac{6}{\beta(a)}; \quad c_1 \text{ from [38]}$$

2. $b_1 = b_1^{\text{free}}$



Two fits agree $\Rightarrow$ b_1^{free} may be used

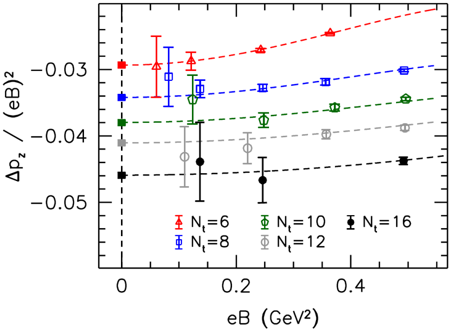
[38] P.A. Baikov, K.G. Chetyrkin, J.H. Kuhn and J. Rittinger, "Vector correlator in massless QCD at order $\mathcal{O}(\alpha_s^4)$ and the QED β-function at five loop", JHEP 07 (2012) [arXiv:1206.1284]

Quadratic contribution to the EoS

Calculation of the $\mathcal{O}[(eB)^2]$ contribution to p_z at $T \simeq 0$

- Combined fit of $\Delta p_z/(eB)^2$ as function eB at $T = 113$ MeV

$$\frac{\Delta p_z}{(eB)^2} = c_0 + b_1^{\text{free}} \ln(a/a_0) + (eB)^2 \cdot (c_1 + c_1' a^2 + c_1'' a^4) + (eB)^4 \cdot (c_2 + c_2' a^2) \qquad (18)$$



c_0	c_1	c_1'	c_1''
$-0.0294(5)$	$0.006(6)$ GeV^{-4}	$0.011(7)$ GeV^{-2}	$0.003(2)$
c_2	c_2'	b_1^{free}	a_0
$0.007(8)$ GeV^{-8}	$-0.025(9)$ GeV^{-6}	0.0169	1.47 GeV^{-1}

$$\lim_{eB\to 0} \frac{\Delta p_z}{(eB)^2}\bigg|_{T=0} = c_0 + b_1^{\text{free}} \ln(a/a_0)$$

$$\lim_{eB\to 0} \frac{\Delta p_z}{(eB)^2}\bigg|_{T=0} = b_1^{\text{free}} \ln(\Lambda_H\, a)$$

$$\Lambda_H(m_{ud}^{\text{ph}}) = (1/a_0)\, e^{c_0/b_1^{\text{free}}} = 0.120(9) \text{ GeV}$$

Λ_H depends on the regularization scheme (on lattice action), but it is expected to be mild.
Λ_H is not free parameter but is automatically determined by the lattice implementation of the renormalization prescription.

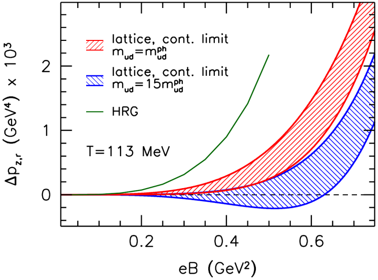
Quadratic contribution to the EoS

Calculation of the $\mathcal{O}[(eB)^2]$ contribution to p_z at $T \simeq 0$

- Calculation of

$$\Delta p_{z,r} = (1-\mathcal{P})[\Delta p_z]$$

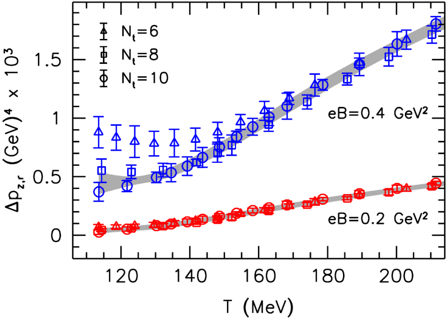
and taking cont. limit at $T = 113$ MeV

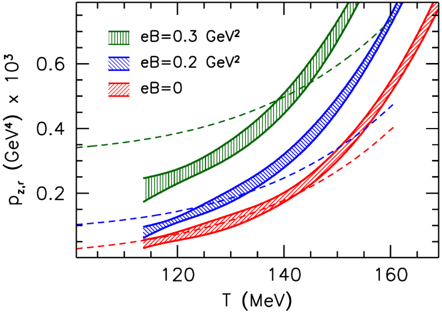


$$\frac{\Delta p_z(\Phi, T_2; \beta_2)}{T_2^4} - \frac{\Delta p_z(\Phi, T_1; \beta_1)}{T_1^4} = N_t^4 \int_{\beta_1}^{\beta_2} \mathrm{d}\beta \left[-a^4 \Delta s_g + \sum_f \frac{\partial(a m_f^{\text{ph}})}{\partial \beta} \cdot a^3 \Delta \bar{\psi}_f \psi_f \right] \quad (6)$$

Found for eq. (6):
$T_1 = 113$ MeV; $\beta_1 \Leftrightarrow T_1$; Δp_z for different Φ

Complete magnetic field dependence of the EoS





Summary

- Using a novel 'generalized integral method', the QCD EoS is determined for range of temperatures and magnetic fields.
- The tabulated data are available.
- The thermodynamic structure of QCD is altered by magn. field:
  - vacuum term
  - pressure anisotropy
- ...