INTRODUCTION TO THE THEORY OF ELLIPTIC HYPERGEOMETRIC INTEGRALS

Vyacheslav P. Spiridonov

Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna
and
Laboratory of Mirror Symmetry, NRU HSE, Moscow

DIASTP-LMC Winter School
"Partition functions and automorphic forms"

BLTP JINR, Dubna
29 January 2018

ELLIPTIC HYPERGEOMETRIC INTEGRALS
A new class of transcendental special functions (V.S., 2000, 2003)

Definition (2003), univariate case: contour integrals

$$I := \int_C \Delta(u) du,$$

where $\Delta(u)$ satisfies a first order finite difference equation

$$\Delta(u+\omega_1) = f(u;\omega_2,\omega_3)\Delta(u),$$

with an elliptic function $f(u;\omega_2,\omega_3)$, i.e. meromorphic, double-periodic function:

$$f(u+\omega_2) = f(u+\omega_3) = f(u)$$

for some incommensurate $\omega_{1,2,3} \in \mathbb{C}$, $\mathrm{Im}(\omega_2/\omega_3) \neq 0$.

$$\text{“Bases”:} \qquad q := e^{2\pi i \omega_1/\omega_2}, \quad p := e^{2\pi i \omega_3/\omega_2}.$$

Let $z = e^{2\pi i u/\omega_2}$ and $\rho(z) := \Delta(u)$ be meromorphic in z.
A strong restriction! Then, for $h(z;p) := f(u;\omega_2,\omega_3)$,

$$I = \int \rho(z)\frac{dz}{z}, \qquad \rho(qz) = h(z;p)\rho(z), \quad h(pz) = h(z).$$

An infinite product

$$(z;p)_\infty := \prod_{j=0}^{\infty}(1-zp^j), \quad |p|<1,\ z\in\mathbb{C}.$$

A Jacobi theta function

$$\theta(z;p) := (z;p)_\infty (pz^{-1};p)_\infty = \frac{1}{(p;p)_\infty}\sum_{k\in\mathbb{Z}}(-1)^k p^{k(k-1)/2} z^k,$$

where $z\in\ \mathbb{C}^*$. Important symmetries (**an exercise**):

$$\theta(pz;p) = \theta(z^{-1};p) = -z^{-1}\theta(z;p).$$

Theorem (Abel, Jacobi):

$$h(z) = \prod_{k=1}^{m}\frac{\theta(t_k z;p)}{\theta(w_k z;p)}, \qquad \prod_{k=1}^{m} t_k = \prod_{k=1}^{m} w_k.$$

Proof.

$$\frac{h(pz)}{h(z)} = \prod_{k=1}^{m}\frac{\theta(pt_k z;p)}{\theta(t_k z;p)}\frac{\theta(w_k z;p)}{\theta(pw_k z;p)} = \prod_{k=1}^{m}\frac{-z^{-1}t_k^{-1}}{-z^{-1}w_k^{-1}} = 1.$$

To solve $\rho(qz) = h(z;p)\rho(z)$, it is sufficient to solve

$$\Gamma(qz;p,q) = \theta(z;p)\Gamma(z;p,q)$$

due to the factorization of $h(z;p)$.

A particular solution

$$\Gamma(z;p,q) = \prod_{j,k=0}^{\infty} \frac{1 - z^{-1}p^{j+1}q^{k+1}}{1 - zp^j q^k}, \qquad |p|, |q| < 1,$$

the (standard) **elliptic gamma function**:

Barnes (1904), Jackson (1905), Baxter (1972), **Ruijsenaars** (1997), Felder-Varchenko (2000), V.S. (2003).

An exercise: find this solution from scratch using the factorization of $\theta(z;p)$.

The final result:

$$I(\underline{t}, \underline{w}; p, q) = \int \prod_{k=1}^{m} \frac{\Gamma(t_k z; p, q)}{\Gamma(w_k z; p, q)} \frac{dz}{z}, \qquad \prod_{k=1}^{m} t_k = \prod_{k=1}^{m} w_k.$$

– Many parameters.

– Generalizes **ALL** (old) known univariate ordinary and q-hypergeometric functions.

Properties of the elliptic gamma function. Symmetry in bases

$$\Gamma(z;p,q)=\Gamma(z;q,p), \qquad \textbf{unexpected}!$$

finite-difference equations

$$\Gamma(qz;p,q)=\theta(z;p)\Gamma(z;p,q), \quad \Gamma(pz;p,q)=\theta(z;q)\Gamma(z;p,q),$$

the inversion relation

$$\Gamma(z;p,q)=\frac{1}{\Gamma(\frac{pq}{z};p,q)},$$

the quadratic transformation

$$\Gamma(z^2;p,q)=\Gamma(\pm z,\pm q^{1/2}z,\pm p^{1/2}z,\pm(pq)^{1/2}z;p,q),$$

with the convention

$$\Gamma(t_1,\ldots,t_k;p,q):=\Gamma(t_1;p,q)\cdots\Gamma(t_k;p,q),$$
$$\Gamma(\pm z;p,q):=\Gamma(z;p,q)\Gamma(-z;p,q),$$
$$\Gamma(tz^{\pm k};p,q):=\Gamma(tz^k;p,q)\Gamma(tz^{-k};p,q).$$

The limiting relation

$$\lim_{z\to 1}(1-z)\Gamma(z;p,q)=\frac{1}{(p;p)_\infty(q;q)_\infty},$$

needed for residue calculus and reduction to terminating elliptic hypergeometric **series** (such infinite series do not converge).

THE ELLIPTIC BETA INTEGRAL (V.S., 2000)

Theorem. Let $|p|, |q|, |t_j| < 1$, $\prod_{j=1}^6 t_j = pq$. Then

$$\kappa \int_{\mathbb{T}} \frac{\prod_{j=1}^6 \Gamma(t_j x^{\pm 1}; p, q)}{\Gamma(x^{\pm 2}; p, q)} \frac{dx}{x} = \prod_{1 \le j < k \le 6} \Gamma(t_j t_k; p, q),$$

where $\mathbb{T}$ is the unit circle and

$$\kappa = \frac{(p;p)_\infty (q;q)_\infty}{4\pi \mathrm{i}}.$$

Proof. Use the Γ-function inversion

$$\Gamma(t_6 x; p, q) = \frac{1}{\Gamma(pq/(t_6 x); p, q)} = \frac{1}{\Gamma(A x^{-1}; p, q)}, \quad A := \prod_{m=1}^5 t_m,$$

and rewrite the integral evaluation as

$$I(t_1, \ldots, t_5; p, q) = \kappa \int_{\mathbb{T}} \Delta(x; t_1, \ldots, t_5; p, q) \frac{dx}{x} = 1,$$

where

$$\Delta = \frac{\prod_{j=1}^5 \Gamma(t_j x^{\pm 1}, t_j^{-1} A; p, q)}{\Gamma(x^{\pm 2}, A x^{\pm 1}; p, q) \prod_{1 \le i < j \le 5} \Gamma(t_i t_j; p, q)}.$$

The q-difference equation:

$$\Delta(x; qt_1) - \Delta(x; t_1) = g(q^{-1}x)\Delta(q^{-1}x; t_1) - g(x)\Delta(x; t_1)$$

with

$$g(x) = \frac{\prod_{m=1}^5 \theta(t_m x; p)}{\prod_{m=2}^5 \theta(t_1 t_m; p)} \frac{\theta(t_1 A; p)}{\theta(x^2, xA; p)} \frac{t_1}{x}$$

is reduced to the elliptic functions identity

$$\frac{\theta(t_1 x^{\pm 1}; p)}{\theta(A x^{\pm 1}; p)} \prod_{m=2}^5 \frac{\theta(A t_m^{-1}; p)}{\theta(t_1 t_m; p)} - 1 = \frac{t_1 \theta(t_1 A; p)}{x\theta(x^2; p) \prod_{m=2}^5 \theta(t_1 t_m; p)}$$
$$\times \left(\frac{x^4 \prod_{m=1}^5 \theta(t_m x^{-1}; p)}{\theta(A x^{-1}; p)} - \frac{\prod_{m=1}^5 \theta(t_m x; p)}{\theta(Ax; p)} \right).$$

An exercise: prove it by comparing poles and their residues in the parallelogram of periods. Integrate the equation for Δ over $x \in \mathbb{T} \Rightarrow$ for $|t_1| < |q|$ one gets

$$I(qt_1) - I(t_1) = 0.$$

Add $p \leftrightarrow q$ symmetry ($|t_1| < |p|$) $\Rightarrow I(pt_1) = I(t_1)$.

Jacobi theorem on absence of periodic functions with **three** incommensurate periods and t_k-permutation symmetry

$\Rightarrow I = I(p, q) = const.$

Compute the limit $t_1 t_2 \to 1$, when two pairs of residues pinch the contour of integration (this requires computation of a pair of residues) and find that $I = 1$. $\square$

The unique relation:

• An elliptic binomial theorem (series analogue: Frenkel-Turaev sum, 1997)

• An elliptic analogue of the Euler beta integral

$$\int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \mathrm{Re}(x), \mathrm{Re}(y) > 0.$$

• Orthogonality measure for the most general univariate classical special functions

• A germ for infinitely many exact integration formulas and the whole theory of transcendental elliptic hypergeometric functions

• An $W(E_6)$-covariant object

• Generates an elliptic Fourier transform (a Bailey lemma) ⇔ Coxeter relations of a permutation group ⇔ star-triangle relation ⇔ general solution of YBE

• Quantum field theory: proves equality of superconformal indices of two different $4d$ $\mathcal{N} = 1$ supersymmetric theories ⇒ a powerful confirmation of the simplest Seiberg (1994) duality. The process of integral evaluation = transition from UV (weak coupling) to IR (strong coupling) physics (Dolan, Osborn, 2008)

Euler-Gauss hypergeometric function

${}_2F_1$-series:

$$ {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} x^n, \qquad |x| < 1, $$

$(a)_n = a(a+1)\dots(a+n-1)$ the Pochhammer symbol

An integral representation:

$$ {}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a} dt, $$

where $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $x \notin [1,\infty[$, and the gamma function $\Gamma(x)$:

$$ \Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \qquad \mathrm{Re}(x) > 0. $$

The hypergeometric equation:

$$ x(1-x)y''(x) + (c - (a+b+1)x)y'(x) - aby(x) = 0, $$

$y(x) = {}_2F_1(a,b;c;x)$ — solution analytical at $x = 0$.

Barnes representation:

$$ {}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+u)\Gamma(b+u)\Gamma(-u)}{\Gamma(c+u)} (-x)^u du $$

An elliptic analogue of the Euler-Gauss hypergeometric function

$$V(t_1,\ldots,t_8;p,q) := \kappa \int_{\mathbb{T}} \frac{\prod_{j=1}^8 \Gamma(t_j z^{\pm 1};p,q)}{\Gamma(z^{\pm 2};p,q)} \frac{dz}{z},$$

$|t_j|, |p|, |q| < 1$ and $\prod_{j=1}^8 t_j = p^2 q^2$.

For $t_j t_k = pq$, $j \neq k \Rightarrow$ the elliptic beta integral.

Consider a double integral:

$$\int_{\mathbb{T}^2} \frac{\prod_{j=1}^4 \Gamma(a_j z^{\pm 1}, b_j w^{\pm 1};p,q)\; \Gamma(c z^{\pm 1} w^{\pm 1};p,q)}{\Gamma(z^{\pm 2}, w^{\pm 2};p,q)} \frac{dz}{z}\frac{dw}{w},$$

where $a_j, b_j, c \in \mathbb{C}$, $|a_j|, |b_j|, |c| < 1$, and

$$c^2 \prod_{j=1}^4 a_j = c^2 \prod_{j=1}^4 b_j = pq.$$

Computation of the integrals over z or w in different order $\Rightarrow$ $W(E_7)$-group of symmetries (V.S., Rains, 2003)

$$V(t_1,\ldots,t_8;p,q) = \prod_{1\le j<k\le 4} \Gamma(t_j t_k, t_{j+4} t_{k+4};p,q)\, V(s_1,\ldots,s_8;p,q),$$

$$\left\{ \begin{array}{ll} s_j = \varepsilon t_j, & j=1,2,3,4 \\ s_j = \varepsilon^{-1} t_j, & j=5,6,7,8 \end{array} \right. ; \quad \varepsilon = \sqrt{\frac{pq}{t_1 t_2 t_3 t_4}} = \sqrt{\frac{t_5 t_6 t_7 t_8}{pq}}.$$

Root systems and Weyl groups

Consider $\mathbb{R}^n$ with the orthonormal basis $e_i \in \mathbb{R}^n$, $\langle e_i, e_j \rangle = \delta_{ij}$. For any $x \in \mathbb{R}^n$ define its reflection w.r.t. the hyperplane orthogonal to some $v \in \mathbb{R}^n$:

$$x \to S_v(x) = x - \frac{2\langle v, x \rangle}{\langle v, v \rangle} v, \qquad S_v^2 = 1.$$

If $x = const \cdot v \Rightarrow S_v(x) = -x$. For $\langle v, x \rangle = 0 \Rightarrow S_v(x) = x$.

Define R as some set of vectors $\alpha_1, \ldots, \alpha_m \in \mathbb{R}^n$, forming a basis. $R =$ root system, if for any $\alpha, \beta \in R$, $S_\alpha(\beta) \in R$.

$\Rightarrow$ reflections $W = \{S_\alpha\}$ form a finite subgroup of the rotation group $O(n) =$ Coxeter group.

$n =$ rank of the root system, $\alpha_j =$ roots.

If for all $\alpha, \beta \in R$

$$2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z},$$

$\Rightarrow$ R the crystallographic root system, $W =$ the Weyl group.

$\Rightarrow$ Relation with the classification of semi-simple Lie algebras.

There exist 4 infinite classical series of root systems: A_n, B_n, C_n, D_n, and 5 exceptional cases: G_2, F_4, E_6, E_7, E_8.

Elliptic hypergeometric functions are naturally related to these root systems. Equivalently, superconformal indices = integrals over Haar measures of the compact gauge groups.

Examples of roots systems:

1) A_n: take $U \in \mathbb{R}^{n+1}$ orthogonal to $\sum_{j=1}^{n+1} e_j$, i.e. for $u = \sum_{j=1}^{n+1} u_j e_j \in U$ one has $\sum_{j=1}^{n+1} u_j = 0$.

Then $R_{A_n} = \{e_i - e_j,\, i \neq j\}\big|_{i,j=1,\ldots,n+1}$ and $W(A_n) = S_{n+1}$.

2) C_n: in $\mathbb{R}^n$ the roots $R_{C_n} = \{\pm 2e_i, \pm e_i \pm e_j,\, i < j\}\big|_{i,j=1,\ldots,n}$ and $W = S_n \times \mathbb{Z}_2^n$.

3) E_8: in $\mathbb{R}^8$ choose

$R_{E_8} = \{\pm e_i \pm e_j,\, i < j, \frac{1}{2}\sum_{i=1}^{8}(-1)^{m_i} e_i,\, \sum_{i=1}^{8} m_i = even\}\big|_{i,j=1,\ldots,8}$.

The V-function is $W(A_7) = S_8$ symmetric (permutation of parameters t_j).

Denote $t_j = (pq)^{1/4} e^{2\pi \mathrm{i} x_j}$. Then, $\sum_{j=1}^{8} x_j = 0$ and the function

$$V^{inv}(t_1, \ldots, t_8; p, q) := \frac{V(t_1, \ldots, t_8; p, q)}{\prod_{1 \le k < l \le 8} (t_k t_l; p, q)_\infty},$$

$$(z; p, q)_\infty := \prod_{a,b=0}^{\infty} (1 - z p^a q^b),$$

is invariant w.r.t. the $x \to S_v(x)$-reflection with

$$v = \frac{1}{2}(1, 1, 1, 1, -1, -1, -1, -1), \quad \langle v, v \rangle = 2.$$

Namely,

$$V^{inv}(s_1, \ldots, s_8; p, q) = V^{inv}(t_1, \ldots, t_8).$$

This is the extension $W(A_7) = S_8 \to W(E_7)$.

The elliptic hypergeometric equation

Addition formula for theta-functions (**an exercise**)

$$t_8\theta(t_7t_8^{\pm 1}, t_6z^{\pm 1}; p) + t_6\theta(t_8t_6^{\pm 1}, t_7z^{\pm 1}; p) + t_7\theta(t_6t_7^{\pm 1}, t_8z^{\pm 1}; p) = 0$$

induces the contiguous relation

$$\frac{t_6V(qt_6)}{\theta(t_6t_7^{\pm 1}, t_6t_8^{\pm 1}; p)} + \frac{t_7V(qt_7)}{\theta(t_7t_6^{\pm 1}, t_7t_8^{\pm 1}; p)} + \frac{t_8V(qt_8)}{\theta(t_8t_6^{\pm 1}, t_8t_7^{\pm 1}; p)} = 0,$$

where $\prod_{j=1}^{8} t_j = p^2q$.

Apply $W(E_7)$-group $\to$ many contiguous relations, one of which has the form

$$\mathcal{A}(t_1, \ldots, t_6, t_7, t_8, q; p)\Big(U(qt_6, q^{-1}t_7) - U(\underline{t})\Big)$$
$$+\mathcal{A}(t_1, \ldots, t_7, t_6, t_8, q; p)\Big(U(q^{-1}t_6, qt_7) - U(\underline{t})\Big) + U(\underline{t}) = 0,$$

where

$$\mathcal{A}(t_1, \ldots, t_8, q; p) := \frac{\theta(t_6/qt_8, t_8t_6, t_8/t_6; p)}{\theta(t_6/t_7, t_7/qt_6, t_6t_7/q; p)} \prod_{k=1}^{5} \frac{\theta(t_7t_k/q; p)}{\theta(t_8t_k; p)}$$

and

$$U(\underline{t}) = \frac{V(\underline{t}; p, q)}{\prod_{k=6}^{7} \Gamma(t_kt_8^{\pm 1}; p, q)}.$$

$\mathcal{A}(t_1, \ldots, t_8, q; p)$ – an elliptic function of all parameters.

This elliptic hypergeometric equation can be written as

$$A(x)\left(f(qx)-f(x)\right)+A(x^{-1})\left(f(q^{-1}x)-f(x)\right)+\nu f(x)=0,$$

$$A(x)=\frac{\prod_{k=1}^{8}\theta(\varepsilon_k x;p)}{\theta(x^2,qx^2;p)},\qquad \nu=\prod_{k=1}^{6}\theta\left(\frac{\varepsilon_k\varepsilon_8}{q};p\right),$$

with the constraints $\prod_{k=1}^{8}\varepsilon_k=p^2q^2$ and $\varepsilon_7=\frac{\varepsilon_8}{q}$.

Solutions $\propto V(\ldots)$.

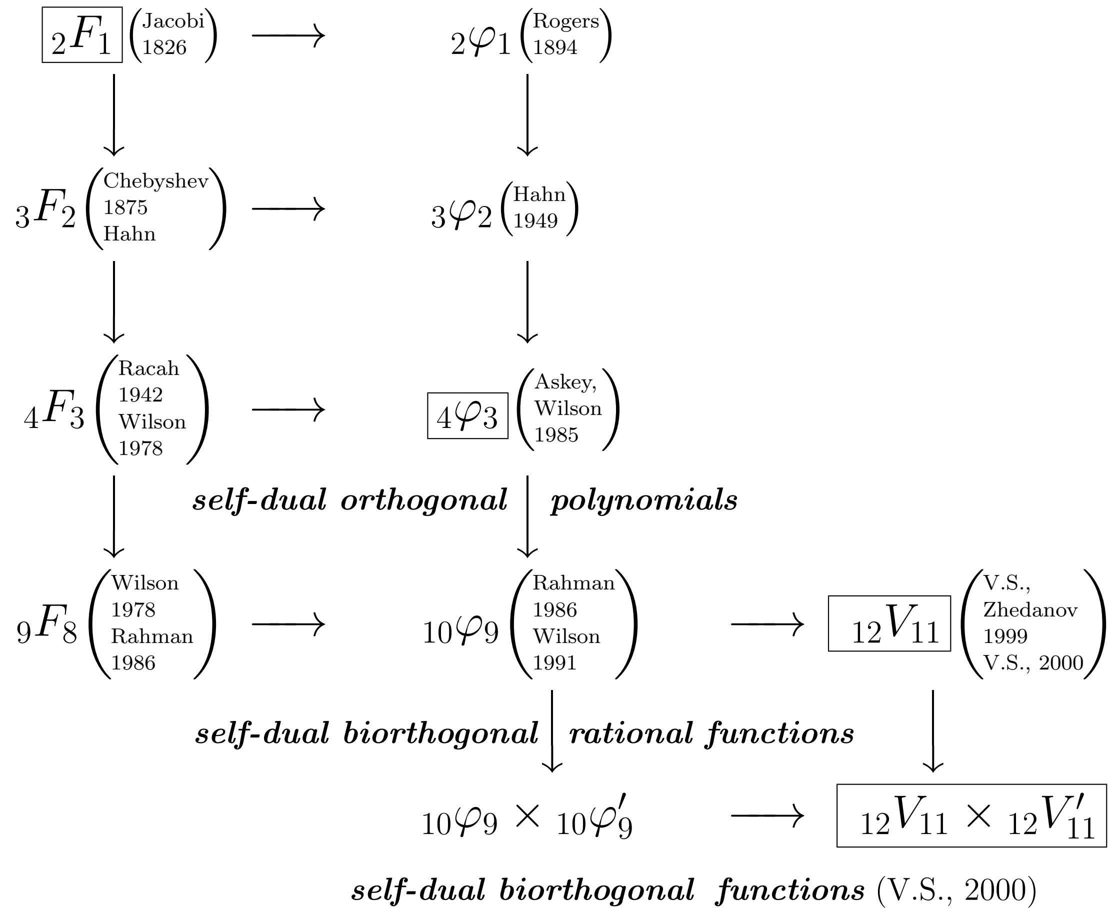
Conclusion: elliptic hypergeometric integrals solve some particular finite-difference equations with elliptic function coefficients.

Orthogonality relations

$$\langle R_n(x),R_m(x)\rangle=\delta_{nm},\quad n,m=0,1,2,\ldots,$$

where R_n are either polynomials of x or rational functions, the scalar product $\langle\cdot,\cdot\rangle$ is an integral w.r.t. some measure.

CLASSICAL ORTHOGONAL POLYNOMIALS AND THEIR GENERALIZATIONS



C_n elliptic beta integral of type I
(van Diejen, V.S., 2001; Rains, 2003, V.S., 2004)

Let $|p|, |q|, |t_j| < 1$ and $\prod_{j=1}^{2n+4} t_j = pq$, then

$$\kappa_n \int_{\mathbb{T}^n} \prod_{1\leq j<k\leq n} \frac{1}{\Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)} \prod_{j=1}^{n} \frac{\prod_{k=1}^{2n+4} \Gamma(t_k z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_j}{z_j}$$

$$= \prod_{1\leq k<s\leq 2n+4} \Gamma(t_k t_s; p, q), \qquad \kappa_n = \frac{(p;p)_\infty^n (q;q)_\infty^n}{(4\pi \mathrm{i})^n n!}.$$

The simplest proof: a straightforward generalization of $n = 1$ case.

Analytical properties of general integrals:

$$\prod_{1\leq k<l\leq 2n+2m+4} (t_k t_l; p, q)_\infty \int_{\mathbb{T}^n} \prod_{1\leq j<k\leq n} \frac{1}{\Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)}$$

$$\times \prod_{j=1}^{n} \frac{\prod_{k=1}^{2n+2m+4} \Gamma(t_k z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \frac{dz_j}{z_j}, \qquad \prod_{k=1}^{2n+2m+4} t_k = (pq)^{m+1},$$

is holomorphic in parameters (Rains, 2010).

C_n elliptic beta integral of type II

(van Diejen, V.S., 2000, 2001)

Let $|p|, |q|, |t|, |t_j| < 1$ and $t^{2n-2}\prod_{m=1}^{6} t_m = pq$, then

$$\kappa_n \int_{\mathbb{T}^n} \prod_{1\leq j<k\leq n} \frac{\Gamma(tz_j^{\pm 1}z_k^{\pm 1};p,q)}{\Gamma(z_j^{\pm 1}z_k^{\pm 1};p,q)} \prod_{j=1}^{n} \frac{\prod_{m=1}^{6}\Gamma(t_m z_j^{\pm 1};p,q)}{\Gamma(z_j^{\pm 2};p,q)} \frac{dz_j}{z_j}$$

$$= \prod_{j=1}^{n} \left(\frac{\Gamma(t^j;p,q)}{\Gamma(t;p,q)} \prod_{1\leq m<s\leq 6} \Gamma(t^{j-1}t_m t_s;p,q) \right).$$

Proof. Denote the integral as $I_n(t, t_1, \ldots, t_5)$ and consider

$$\int_{\mathbb{T}^{2n-1}} \prod_{1\leq j<k\leq n} \frac{1}{\Gamma(z_j^{\pm 1}z_k^{\pm 1};p,q)} \prod_{l=1}^{n} \frac{\prod_{r=0}^{5}\Gamma(t_r z_l^{\pm 1};p,q)}{\Gamma(z_l^{\pm 2};p,q)} \frac{dz_l}{z_l}$$

$$\times \prod_{\substack{1\leq j\leq n\\ 1\leq k\leq n-1}} \Gamma(t^{1/2}z_j^{\pm 1}w_k^{\pm 1};p,q) \prod_{1\leq j<k\leq n-1} \frac{1}{\Gamma(w_j^{\pm 1}w_k^{\pm 1};p,q)}$$

$$\times \prod_{j=1}^{n-1} \frac{\Gamma(w_j^{\pm 1}t^{n-3/2}\prod_{s=1}^{5}t_s;p,q)}{\Gamma(w_j^{\pm 2}, w_j^{\pm 1}t^{2n-3/2}\prod_{s=1}^{5}t_s;p,q)} \frac{dw_j}{w_j},$$

where $t^{n-1}\prod_{r=0}^{5} t_r = pq$.

Integration over w_j or z_j using the type I C_n-integral yields the recurrence relation:

$$I_n(t,t_1,\ldots,t_5) = \frac{\Gamma(t^n;p,q)}{\Gamma(t;p,q)} \prod_{0\le r<s\le 5} \Gamma(t_r t_s;p,q)\ I_{n-1}(t,t^{1/2}t_1,\ldots,t^{1/2}t_5)$$

with known $n=1$ initial condition. □

This is an elliptic analogue of the Selberg beta integral.

Relation to root systems R: the integrands contains the products

$$\prod_{\alpha\in R} \Gamma(e^{\alpha};p,q), \quad z_j = \exp(e_j),$$

where e_j are orthonormal $\mathbb{R}^n$ basis vectors. This is related to Haar measures for compact (gauge) groups.

Full Seiberg duality (1995) for $\mathcal{N}=1$ $4d$ SUSY theories:

"Electric" theory:

	$SU(N_c)$	$SU(N_f)_l$	$SU(N_f)_r$	$U(1)_B$	$U(1)_R$
Q	f	f	1	1	$\widetilde{N}_c/N_f$
$\widetilde{Q}$	$\overline{f}$	1	$\overline{f}$	-1	$\widetilde{N}_c/N_f$
V	adj	1	1	0	1

"Magnetic" theory:

	$SU(\widetilde{N}_c)$	$SU(N_f)_l$	$SU(N_f)_r$	$U(1)_B$	$U(1)_R$
q	f	$\overline{f}$	1	$N_c/\widetilde{N}_c$	N_c/N_f
$\widetilde{q}$	$\overline{f}$	1	f	$-N_c/\widetilde{N}_c$	N_c/N_f
M	1	f	$\overline{f}$	0	$2\widetilde{N}_c/N_f$
$\widetilde{V}$	adj	1	1	0	1

where $\widetilde{N}_c = N_f - N_c$ and $3N_c/2 < N_f < 3N_c$ (conformal window)

Seiberg conjecture: these two $\mathcal{N}=1$ SYM theories have the same physics at their IR fixed points

Consistency checks:

- The global anomalies match ('t Hooft anomaly matching)
- Matching of the reductions $N_f \to N_f - 1$
- The moduli spaces have the same dimensions and the gauge invariant operators match

The electric theory index:

$$I_E = \kappa_{N_c} \int_{\mathbb{T}^{N_c-1}} \frac{\prod_{i=1}^{N_f} \prod_{j=1}^{N_c} \Gamma(s_i z_j, t_i^{-1} z_j^{-1})}{\prod_{1\leq i<j\leq N_c} \Gamma(z_i z_j^{-1}, z_i^{-1} z_j)} \prod_{j=1}^{N_c-1} \frac{dz_j}{z_j},$$

$$\prod_{j=1}^{N_c} z_j = 1, \qquad \kappa_N = \frac{(p;p)_\infty^{N-1} (q;q)_\infty^{N-1}}{N!(2\pi \mathrm{i})^{N-1}}.$$

This is an elliptic hypergeometric integral for A_{N_c-1} root system (i.e. $SU(N_c)$ gauge group).

The magnetic theory: $\qquad I_M = \kappa_{\tilde{N}_c} \prod_{i,j=1}^{N_f} \Gamma(s_i t_j^{-1}) \times$

$$\times \int_{\mathbb{T}^{\tilde{N}_c-1}} \frac{\prod_{i=1}^{N_f} \prod_{j=1}^{\tilde{N}_c} \Gamma(S^{\frac{1}{\tilde{N}_c}} s_i^{-1} x_j, T^{-\frac{1}{\tilde{N}_c}} t_i x_j^{-1})}{\prod_{1\leq i<j\leq \tilde{N}_c} \Gamma(x_i x_j^{-1}, x_i^{-1} x_j)} \prod_{j=1}^{\tilde{N}_c-1} \frac{dx_j}{x_j},$$

where $\prod_{j=1}^{\tilde{N}_c} x_j = 1, \quad \tilde{N}_c = N_f - N_c,$

$S = \prod_{i=1}^{N_f} s_i, \quad T = \prod_{i=1}^{N_f} t_i, \quad ST^{-1} = (pq)^{N_f - N_c}.$

Theorem: $I_E = I_M$ V.S. (special cases), Rains

Proof (Rains, 2003). For special discrete values of parameters the integral I_E is reduced to computation of determinants of matrices with entries described by univariate integrals of combinations of theta functions. The latter determinants can be transformed to multivariate integrals of the products of two computable theta-function determinants, equivalent to I_M for special values of parameters. The set of such discrete parameter values is dense $\Rightarrow$ the integration formula is true for I_E, I_M analytically continued in parameters.

Better proofs should be found !

This mathematically rigorous statement absorbs all previously known criteria of Seiberg and gives even more (equality of BPS states spectra).

An interplay between Physics and Mathematics

Physics $\Rightarrow$ very many new identities

Mathematics $\Rightarrow$ many new dualities; new duality tests

Miscellaneous results:

- ’t Hooft anomaly matching $\longleftrightarrow$ $SL(3,\mathbb{Z})$-modular covariance (skipped)
- $N_f \to N_f - 1$ reduction in SCIs: $s_k t_k^{-1} = pq$ (a simple substitution for I_E and a residue calculus for I_M)
- Conjectures: a hundred new computable elliptic beta integrals and symmetry transformations for higher order elliptic hypergeometric integrals on root systems ($SU(N)$, $SP(2N)$, $SO(N)$, G_2, E_6, F_4 gauge groups)
- Tens of new $\mathcal{N}=1$ dualities, new confining theories
- Discovery of many relations between dualities (some of them are deducible from the others)
- There are infinitely many quiver dualities whose SCIs are generated by integral Bailey lemma

- Conjecture: there are infinitely (countably) many qualitatively different supersymmetric dualities for simple gauge groups and corresponding elliptic hypergeometric integral identities
- Reduction of $4d$ SCIs/dualities $\to$ $3d$ and $2d$ partition functions/dualities
- Dirac delta function behavior of integrals/superconformal indices $\longleftrightarrow$ chiral symmetry breaking
- $4d$ $\mathcal{N} = 2$ SCIs and $2d$ topological field theories (Gadde, Pomoni, Rastelli, Razamat, Yan, ...)
- SCIs on lens spaces $\Rightarrow$ finite sums of elliptic hypergeometric integrals, reduction to $3d$ SCIs with monopoles (Benini, Nishioka, Yamazaki, Razamat, Willet, Kels, V.S).

A lot of work ahead !