Progress in multiloop calculations.

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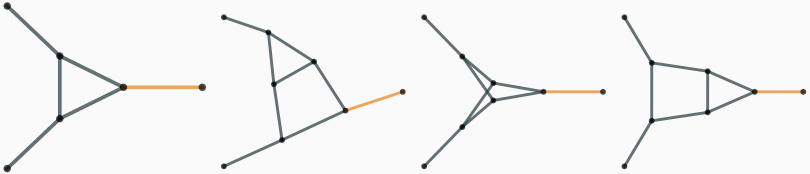
Motivation

- High-precision theoretical description of Standard Model processes is of crucial importance. In particular, the New Physics — new particles and interactions — is likely to appear as small deviations from SM and therefore can be detected only with high precision of theoretical predictions at hand.
- From the computational point of view, our ability to obtain high-precision results depends crucially on multiloop calculation techniques. Complexity grows both qualitatively and quantitatively in an explosive way with the number of loops and/or scales.
- New methods and approaches are always required. Using computer power is a must for at least two last decades. Insights from various fields of mathematics help a lot.

Example: form factors

2 loops:



- Dispersion relation
- Feynman parametrization

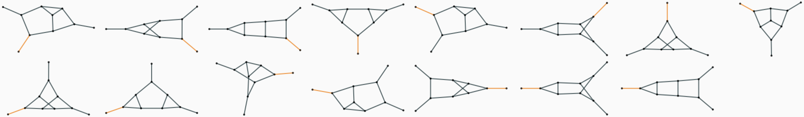
} [Matsuura, van der Marck, and van Neerven, 1989; Harlander, 2000]

- Mellin-Barnes parametrization
- ${}_pF_q$ expansion in indices, `HypExp`

} [Gehrmann, Huber, and Maitre, 2005]

3 loops:

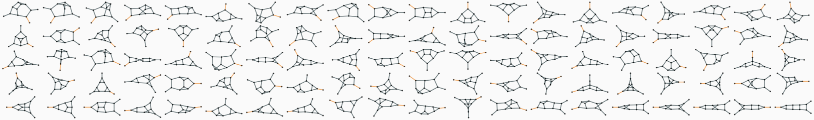
[Gehrmann, Heinrich, Huber, and Studerus, 2006; Heinrich, Huber, and Maître, 2008; RL, Smirnov, and Smirnov, 2010]



- Feynman parametrization
- Mellin-Barnes parametrization, `MB`, `AMBRE` [Czakon, 2006; Gluza et al., 2007]
- Recurrence+analyticity in d, [Tarasov, 1996; RL, 2010]
- PSLQ recognition [Ferguson et al., 1998]

4 loops:

[Henn, Smirnov, Smirnov, and Steinhauser, 2016; RL, Smirnov, Smirnov, and Steinhauser, 2019; RL, von Manteuffel, Schabinger, Smirnov, Smirnov, and Steinhauser, 2021b]



- $\sim$ 100 big topologies.
- Linear reducibility, `HyperInt` [Panzer, 2013]
- Parallelization for IBP reduction,
  finite fields reconstruction [von Manteuffel and Schabinger, 2015; Smirnov and Chuharev, 2020]
- Differential equations, reduction to ϵ-form [Henn, 2013; RL, 2015], `Libra` [RL, 2021]
- PSLQ recognition

5 loops:

- $\sim$ 1000 big topologies.
- It looks like no available techniques can help.

NNLO cross sections

- Massless form factors represent a traditional topic of the multiloop calculations where the "world records" are fixed. But from the experimental point of view less loops and more scales are more important.
- In particular, only very recently multiloop methods have grown to **NNLO differential cross section** calculations of $2 \to 2$ processes with massive particles. NNLO corrections to differential cross sections are not even known for basic QED process: $e^+e^- \to \gamma\gamma$, $e^+e^- \to \mu^+\mu^-$, etc. Partial results start to appear [Duhr, Smirnov, and Tancredi, 2021; Banerjee et al., 2020].
- The complexity of NNLO calculations with massive internal lines is connected with appearance of **non-polylogarithmic integrals**. Effective approach to the calculation of such integrals is, probably, the most hot topic in multiloop calculations.

State of the art

Complexity crucially depends on # of loops L and on # of scales S.

loops / scales	1 loop	2 loops	3 loops	4 loops	5 loops	>5
1	✓	✓	✓	✓	a few	
2	✓	✓	some	a few		
3	✓	some	a few			
>3	✓	a few				

The following empirical "formula" describes the complexity of calculations:

$$\text{Complexity} = L + S + \delta_m,$$

where $\delta_m = 1$ ($\delta_m = 0$) for diagrams with/without massive internal lines.

- 5-loop massless propagators [Georgoudis, Gonçalves, Panzer, Pereira, Smirnov, and Smirnov, 2021].
- 4-loop $g-2$ integrals (onshell massive propagators) [Laporta, 2017]
- 4-loop form factors [RL, von Manteuffel, Schabinger, Smirnov, Smirnov, and Steinhauser, 2022]
- 3-loop massless boxes [Henn, Mistlberger, Smirnov, and Wasser, 2020]
- 2-loop 5 legs [Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, and Zoia, 2019]

- 3-loop massive form factors: numerical calculation [Fael, Lange, Schönwald, and Steinhauser, 2022].
- Partial results for 2-loop boxes with inner massive lines [Duhr, Smirnov, and Tancredi, 2021].

Calculation path

1. Diagram generation

✓Generate diagrams contributing to the chosen order of perturbation theory.

Tools: `qgraf` [Nogueira, 1993], `FeynArts` [Hahn, 2001], `tapir` [Gerlach et al., 2022],...

2. IBP reduction

Setup IBP reduction, derive differential system for master integrals.

Tools: `FIRE6` [Smirnov and Chuharev, 2020], `Kira2` [Klappert et al., 2021], `LiteRed` [RL, 2012], ...

3. DE Solution

Reduce the system to ϵ-form, write down solution in terms of polylogarithms. Fix boundary conditions by auxiliary methods.

Tools: `Fuchsia` [Gituliar and Magerya, 2017], `epsilon` [Prausa, 2017], `Libra` [RL, 2021]

NB: 3rd step is not always doable.

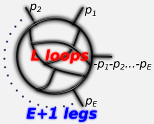
IBP reduction

IBP identities [Chetyrkin and Tkachov, 1981]

Given a Feynman diagram, consider a family

$$j(\boldsymbol{n}) = \int d\mu_L \prod_{k=1}^{N} D_k^{-n_k}, \quad d\mu_L = \prod_{k=i}^{L} d^d l_i$$

$D_1, \ldots, D_M$ — denominators of the diagram,
$D_{M+1}, \ldots, D_N$ — irreducible numerators, such that $N = L(L+1)/2 + L \cdot E$.



From $0 = \int d\mu_L \frac{\partial}{\partial l_i} \cdot q_m \prod_{k=1}^{N} D_k^{-n_k}$ one obtains

IBP identities

$$[c_{kl} B_k A_l + c_l A_l]\, j(\boldsymbol{n}) = 0.$$

Here c_{kl}, c_l are some coefficients.

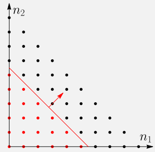
$$A_l j(n_l) = n_l j(n_l + 1),$$
$$B_l j(n_l) = j(n_l - 1)$$

IBP identities allow one to express any integral in the family via a finite number of master integrals. They also allow to construct differential and difference equations for the latter.

IBP reduction

Laporta algorithm (FIRE, Kira, Reduze, ...)

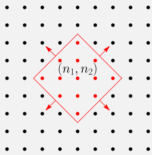
- generate identities for many numeric $\boldsymbol{n} \in \mathbb{Z}^N$.
- use Gauss elimination and collect reduction rules to database.
- twist: mapping to finite fields $\mathbb{F}_p$ + reconstruction. $\Longleftarrow$ naturally parallelizable



Heuristic search (LiteRed)

1. Generate identities for shifts around $\boldsymbol{n}$ with *symbolic* entries.
2. Use Gauss elimination until acceptable rule is found.
3. **Solve Diophantine equations to derive applicability condition.**

Observation: only a small fraction of identities finally contribute to the reduction rule.



IBP reduction in parametric representation

Note that $N = L(L+1)/2 + L \cdot E$ grows quadratically with L, while M, the # of lines in the diagram, grows only linearly. Parametric representation: only M indices.

Parametric representation

$$\tilde{J}^{(d)}(n_1, \ldots n_M) = \int \frac{\prod_{k=1}^{M} dx_k x_k^{n_k - 1}}{G(\boldsymbol{x})^{d/2}}$$

$G = U + F$, where U and F are Feynman graph polynomials.

IBP identities relating integrals with the same d require constructing *syzygy module* for ideal generated by $\langle G, \partial_1 G, \partial_M G \rangle$.

IBP identities from syzygies [RL, 2014]. Baikov rep.: [Zhang, 2014]

Syzygy $QG + Q_1 \partial_1 G + \ldots + Q_M \partial_M G = 0$ leads to IBP identity

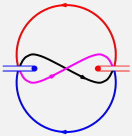
$$[\tfrac{d}{2} Q(A) + Q_k(A) B_k] \tilde{J}(\boldsymbol{n}) = 0$$

Quite promising, but a fast algorithm for constructing a *minimal* (rather than Groebner) basis of syzygy module is very desirable.

IBP reduction with intersection theory? [Mastrolia and Mizera, 2019]

- Integral in parametric representation is understood as bilinear pairing between integration cycle C and differential form ϕ.

$$\int_C G^{-\nu}\phi = \langle\phi|C] \ ,$$



Simplest twisted cycle: Pochhammer contour.

- $\langle\phi|C]$ is invariant under $\phi \to \phi + \nabla_\nu\tilde{\phi}$ and/or $C \to C + \partial\tilde{C}$, where $\nabla_\nu = d - \nu G^{-1}dG$ is twisted differential and $\partial\tilde{C}$ is a boundary (contractable) cycle.
- Therefore, $\langle\cdot|\cdot]$ is defined on the elements of twisted de Rham cohomology and twisted homology. Those are finite-dimensional spaces, therefore we can use basis expansion as IBP.
- Ref. [Cho and Matsumoto, 1995] introduced pairing $\langle\phi_1|\phi_2\rangle$, correctly defined for ∇_ν and $\nabla_{-\nu}$ de Rham cohomologies.
- IBP reduction is simply a basis expansion

$$\langle\phi|C] = \sum_i \langle\phi|\phi_i\rangle\,\langle\phi_i|C] \ ,$$

where $j_i = \langle\phi_i|C]$ are master integrals.
- Unfortunately, $\langle\phi_1|\phi_2\rangle$ is still very difficult to calculate in general. All examples considered so far correspond to integrals with only a few (1 or 2) indexes. Perspectives of this approach are doubtful.

Differential equations and dimensional reucrrences

As a result of IBP reduction we express amplitudes via a finite set of master integrals $\boldsymbol{j} = (j_1, \ldots, j_K)^\intercal$. What is more important, we obtain equations for them:

Differential equations

[Kotikov, 1991; Remiddi, 1997]

$$\partial_x \boldsymbol{j} = M(x, d)\boldsymbol{j}$$

Dimensional recurrences

[Tarasov, 1996; Derkachov et al., 1990]

$$\boldsymbol{j}(d-2) = R(x, d)\boldsymbol{j}(d)$$

Dimensional recurrence relations are especially useful for one-scale integrals, when the differential equations can not help. The approach is very effective when R is triangular. Using the analytical properties wrt d, [RL, 2010] to fix the arbitrary periodic functions, one can obtain the solution in the form of convergent sums. High-precision evaluation of these sums can be done with `SummerTime` package [RL and Mingulov, 2016].

Using PSLQ algorithm, one can turn the obtained numerical results into analytical expressions. This is the approach which was successfully applied to the calculation of the 3-loop form factors.

Differential equations

Differential equations for master integrals

- Differential equations for master integrals have the form

$$\partial_x \boldsymbol{j} = M(x,\epsilon)\boldsymbol{j}$$

- One can try to simplify the equation by transformation $\boldsymbol{j} = T\tilde{\boldsymbol{j}}$, so that

$$\partial_x \tilde{\boldsymbol{j}} = \widetilde{M}\tilde{\boldsymbol{j}}, \qquad \widetilde{M} = T^{-1}\left[MT - \partial_x T\right]$$

- [Henn, 2013]: there is often a "canonical" basis $\boldsymbol{J} = T^{-1}\boldsymbol{j}$ such that

$$\partial_x \boldsymbol{J} = \epsilon S(x)\boldsymbol{J} \qquad (\epsilon\text{-form})$$

- General solution is easily expanded in ϵ:

$$U(x,x_0) = \mathrm{Pexp}\left[\epsilon\int_{x_0}^{x} dx S(x)\right] = \sum_n \epsilon^n \iiint\limits_{x>x_n>\ldots>x_0} dx_n \ldots dx_1 S(x_n)\ldots S(x_1)$$

- We usually want to send the lower limit x_0 to a singular point (say, to 0), so we have to consider the regularized operator $U(x,\underline{0}) = \lim_{x_0\to 0} U(x,x_0)x_0^{\epsilon S_0}$.
- Algorithm of finding transformation to ϵ-form: [RL, 2015]. Implemented in 3 publicly available codes: `Fuchsia` [Gituliar and Magerya, 2017], `epsilon` [Prausa, 2017], and recently in `Libra` [RL, 2021].

General structure of reduction algorithm

Algorithm proceeds in three major stages, each involving a sequence of "elementary" transformations.

1. *Fuchsification*: Eliminating higher-order poles

Input: Rational matrix $M(x, \epsilon)$
Output: Rational matrix with only simple poles on the extended complex plane, $M(x, \epsilon) = \sum_k \frac{M_k(\epsilon)}{x - a_k}$.

2. *Normalization*: Normalizing eigenvalues

Input: Matrix from the previous step, $M(x, \epsilon) = \sum_k \frac{M_k(\epsilon)}{x - a_k}$.
Output: Matrix of the same form, but with the eigenvalues of all $M_k(\epsilon)$ being proportional to ϵ.

3. *Factorization*: Factoring out ϵ

Input: Matrix from the previous step.
Output: Matrix in ϵ-form, $M(x, \epsilon) = \epsilon S(x) = \epsilon \sum_k \frac{S_k}{x - a_k}$.

Frobenius method

Path-ordered exponent

$$U(x,\underline{0}) = \text{Pexp}\left[\int_{x_0}^{x} M(x)dx\right] x_0^{M_0}, \quad M_0 = \text{res}_{x=0} M(x)$$

can also be expanded in generalized power series when x is small enough.

$$U(x,\underline{0}) = \sum_{\lambda\in S} x^{\lambda} \sum_{n=0}^{\infty} \sum_{k=0}^{K_\lambda} \frac{1}{k!} C\left(n+\lambda,k\right) x^n \ln^k x\,.$$

Note that for expansion around singular point (which we usually want) non-integer powers x^{λ} and $\log x$ might appear.

The convergence radius is the distance to the nearest singularity. However, it is easy to perform analytical continuation to the whole complex plane by matching expansions at different points. Let $x = 1$ is also the singular point, then the continuation of $U(x,\underline{0})$ beyond $x = 1$ is simply

$$U(x>1,\underline{0}) = U(x,\underline{1})U^{-1}(1/2,\underline{1})U(1/2,\underline{0})$$

Frobenius expansion provides a systematic way to obtain numerical results for any family of multiloop integrals, including non-polylogarithmic ones.

`Libra` **program**

- `Libra` is a *Mathematica* package useful for treatment of differential systems which appear in multiloop calculations.
- Tools for reduction to ϵ-form
  - Visual interface
  - Algebraic extensions
  - Birkhoff-Grothendieck factorization
- Tools for constructing solution
  - Determining boundary constants.
  - Constructing ϵ-expansion of Pexp.
  - Constructing Frobenius expansion of Pexp.

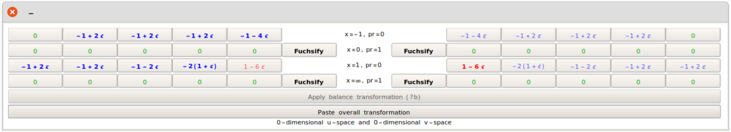
Libra tools for reduction to ϵ-form

- Fuchsification and normalization.
    - Automatic tool (useful for simple cases)

```
In[1]:  t=Rookie[M,x,ϵ];
```

    - Interactive tool (useful for most cases)

```
In[1]:  t=VisTransformation[M,x,ϵ];
```



- Factorization.

```
In[2]:  t=FactorOut[M,x,ϵ,μ];
```

- General solution

```
In[3]:  U=PexpExpansion[{M,6},x];
```

Boundary conditions

Suppose we have found a transformation $T(x) = T(x, \epsilon)$ to ϵ-form, $\mathbf{j} = T\mathbf{J}$. Then we can write

$$\mathbf{J}(x) = U(x, x_0)\mathbf{J}(x_0),$$
$$\mathbf{j}(x) = T(x)U(x, x_0)[T(x_0)]^{-1}\mathbf{j}(x_0)$$

But the point x_0 should be somewhat special to simplify the evaluation of $\mathbf{j}(x_0)$ as compared to $\mathbf{j}(x)$. As a rule, "special" boils down to "singular", i.e., we can expect simplifications for x_0 being a singular point of the differential system. Let it be $x_0 = 0$ for simplicity.

Problem

$U(x, x_0)$ diverges when x_0 tends to zero. Therefore, we have to consider not the values, but the asymptotics of $\mathbf{j}(x_0)$ at $x = 0$.

`Libra` can determine which asymptotic coefficients, $\mathbf{c}$, are sufficient to calculate and find the "adapter" matrix L relating those with the column of boundary constants, $\mathbf{C} = L\mathbf{c}$.

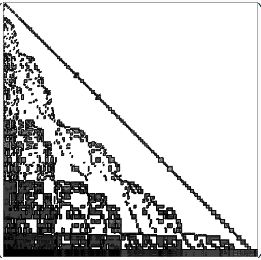
```
In[4]:  {L,cs}=GetLcs[M,T,{x,0}];
```

Example of using `Libra`

One of many 4-loop massless vertex topologies with two off-shell legs.

- Differential system

$$\partial_x j = \underbrace{\left(\text{matrix} \right)}_{374 \times 374 \text{ matrix}} j, \qquad \text{where } j = \begin{pmatrix} \cdot \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \cdot \end{pmatrix}$$



- Maximum size of the diagonal blocks is “only” 11×11.
- No global rationalizing variable. Three algebraic extensions are needed for the reduction to ϵ-form:

$$x_1 = \sqrt{x}, \qquad x_2 = \sqrt{x - 1/4}, \qquad x_3 = \sqrt{1/x - 1/4}$$

Simplifications with symbol map

There is a standard approach to the simplification of the polylogarithmic expressions using symbol map. One might think of symbols as a cleaner way to represent iterated (or path-ordered) integrals with logarithmic weights (with some reservations, though):

$$I = \int\!\!\cdots\!\!\int\limits_{1>\tau_n>\ldots>\tau_1>0} d\ln p_n(\tau_n)\ldots d\ln p_1(\tau_1) \xrightarrow{\quad\mathcal{S}\quad} p_n\otimes\ldots\otimes p_1$$

Formal symbol manipulation rules then easily follow, e.g.

$$d\ln(pq) = d\ln p + d\ln q \qquad \Longrightarrow \qquad (\ldots\otimes pq\otimes\ldots) = (\ldots\otimes p\otimes\ldots) + (\ldots\otimes q\otimes\ldots)$$

Similarly, by ordering the integration variables in the product of integrals, we get $\mathcal{S}(I_1 I_2) = \mathcal{S}(I_1) \sqcup\!\sqcup\, \mathcal{S}(I_2)$, where $\sqcup\!\sqcup$ denotes a shuffle product, e.g.

$$(a\otimes b)\sqcup\!\sqcup(c\otimes d) = a\otimes b\otimes c\otimes d + a\otimes c\otimes b\otimes d + a\otimes c\otimes d\otimes b + c\otimes a\otimes b\otimes d + c\otimes a\otimes d\otimes b + c\otimes d\otimes a\otimes b$$

We have, in particular, symbols for classical polylogarithms

$$\mathcal{S}(\mathrm{Li_n}(x)) = \underbrace{x\otimes\ldots\otimes x}_{n-1}\otimes(x-1)$$

Simplifications with symbol map

Symbols are good for checking the identities, e.g., using $\mathcal{S}$ it is easy to establish[1]

$$
\begin{aligned}
&7\mathrm{Li}_2\left(\tfrac{1+\varepsilon/z}{1-i\varepsilon}\right)-7\mathrm{Li}_2\left(\tfrac{1+\bar\varepsilon/z}{1+i\bar\varepsilon}\right)+7\mathrm{Li}_2\left(\tfrac{z+\bar\varepsilon}{\bar\varepsilon-i}\right)-7\mathrm{Li}_2\left(\tfrac{z+\varepsilon}{\varepsilon+i}\right)+11\mathrm{Li}_2\left(\tfrac{z+\varepsilon}{\varepsilon-i}\right)-11\mathrm{Li}_2\left(\tfrac{z+\bar\varepsilon}{\bar\varepsilon+i}\right)\\
&+4\mathrm{Li}_2(1+z\varepsilon)-4\mathrm{Li}_2(1+z\bar\varepsilon)+18\mathrm{Li}_2(-iz)-18\mathrm{Li}_2(iz)+11\mathrm{Li}_2\left(\tfrac{1+\bar\varepsilon/z}{1-i\bar\varepsilon}\right)-11\mathrm{Li}_2\left(\tfrac{1+\varepsilon/z}{1+i\varepsilon}\right)\\
&=\tfrac{2i\pi^2}{5\sqrt{3}}-\tfrac{23}{3}i\pi\ln z+6i\pi\ln\left(2-\sqrt{3}\right)-\tfrac{i\psi'\left(\frac{1}{6}\right)}{5\sqrt{3}}-24iG,\qquad \text{where } \varepsilon=1/\bar\varepsilon=e^{2\pi i/3}.
\end{aligned}
$$

However, strictly speaking, they are much less powerful in simplifying expressions. E.g., if we omit in the left-hand side a couple of dilogs with not so simple arguments, we could have failed to recognize in the symbol of the resulting expression that of the sum of the omitted dilogs.

Simplification algorithm idea (stay tuned)

For a given expression:

1. find all possible arguments of Li_n which might enter the simplified form.
2. find equivalent form with the minimal number of polylogs.

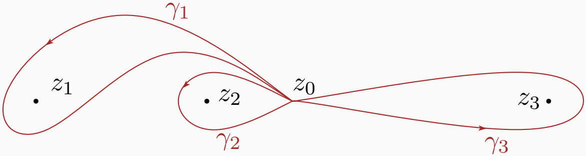
[1] NB: This identity was used in real life (as well as some yet more complicated identities) for the simplification of the total cross section of Compton scattering @NLO [RL, Schwartz, and Zhang, 2021a].

Non-polylogarithmic integrals

Non-polylogarithmic integrals: "Systematic" approach

1. "Systematic" approach.
   - Reduce the system to $(A + \epsilon B)$-form:
     $$\partial_x j = (A + \epsilon B) j.$$
   - "Integrate out" the ϵ^0 form: make substitution $j = U_0 J$, where U_0 is a fundamental matrix for the unperturbed system $\partial_x U_0 = A U_0$.
   - The system for J is in ϵ-form:
     $$\partial_x J = \epsilon \widetilde{B} J, \qquad \widetilde{B} = U_0^{-1} B U_0.$$
   - The general solution $U_1 = \mathrm{Pexp}\left[\epsilon \int dx \widetilde{B}(x)\right]$ is expanded in terms of iterated integrals with weights being the elements of $\widetilde{B}$.
     NB: irreducibility to ϵ-form means that elements of $\widetilde{B}$ are transcendental functions. In particular, the weights might be possible to represent in terms of modular forms.
   - Pros: to some extent decouples the solution of unperturbed equation and ϵ-expansion.
   - Cons: Iterated integrals with transcendental weights are poorly investigated as compared to polylogarithms. When it comes to numerical evaluation, it is often necessary to reside to some sort of power series expansion.
2. Meanwhile, the Frobenius method can be applied directly to the differential system. It seems to be the most effective approach for numerical evaluation. In particular, it works for 3-loop massive form factors [Fael, Lange, Schönwald, and Steinhauser, 2022].
3. For many cases of non-polylogarithmic integrals there exists a one-fold integral representation in terms of polylogarithms and algebraic functions.

Monodromy group



- Monodromy group $\mathcal{G}_{\circlearrowleft} \subset GL(n, \mathbb{C})$ of the differential system $\partial_z \boldsymbol{j} = M\boldsymbol{j}$ with $\boldsymbol{j} = (j_1, \ldots j_n)^\intercal$ determines how the solution space transforms under analytical continuation along nonequivalent closed paths[2]. It is generated by the monodromies around the loops encircling each singular point of the system.
- Monodromy group captures all nontrivial properties of the differential system while being blind to a specific realazation (in particular, $\mathcal{G}_{\circlearrowleft}$ is invariant wrt rational transformations of the system).
  Hilbert's 21st problem: **Proof of the existence of linear differential equations having a prescribed monodromic group**.

[2] Reminder: Let $U(z)$ is a fundamental matrix, $\partial_z U = MU$ determined in the vicinity of a regular point z_0, and let $U(z)|_\gamma$ denotes its analytical continuation along the closed path γ starting and ending in this vicinity. Then $U(z)|_\gamma = U(z)g(\gamma)$, where $g(\gamma)$ is a complex $n \times n$ matrix (i.e. $g(\gamma) \in GL(n, \mathbb{C})$). In fact, this matrix depends only on homotopy class $[\gamma]$ (they form a fundamental group $\pi_1(\overline{\mathbb{C}})$). Thus the monodromy group $\mathcal{G}_{\circlearrowleft} = \{g([\gamma])|\ [\gamma] \in \pi_1(\overline{\mathbb{C}})\}$ is a representation of the fundamental group $\pi_1(\overline{\mathbb{C}})$.

Monodromy group at $\epsilon = 0$ and (ir)reducibility

The ϵ-reducible and ϵ-irreducible systems differ intrinsically by the type of their monodromy groups at $\epsilon = 0$:

- ϵ-reducible with rational transformations: monodromy group is trivial, $\mathcal{G}_{\circlearrowleft} = \{1\}$.
- ϵ-reducible with algebraic transformations: monodromy group is finite, $|\mathcal{G}_{\circlearrowleft}| < \infty$. Monodromy group becomes trivial on the corresponding covering space.
- ϵ-irreducible: monodromy group is (isomorphic to) a subgroup of $GL(n, \mathbb{Z})$.

In particular, for elliptic cases $\mathcal{G}_{\circlearrowleft}$ is a congruence subgroup of $SL(2, \mathbb{Z})$, see [Broedel et al., 2022] for the case of 2-loop sunrise and 3-loop banana graph. This fact allows one to express the integration kernels via modular forms.

Monodromy group can be obtained numerically from Frobenius expansion, so it is not so easy to see the structure from, e.g.,

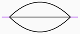
$$g_1 = \begin{pmatrix} 1 & 0. & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, g_2 = \begin{pmatrix} -2. & -5.6325 & -4.11456 \\ 0.618343 & 2.16094 & 0.84807 \\ -0.117344 & -0.220313 & 0.83906 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} -8. + 0.i & -16.8975 + 19.5116i & -12.3437 + 102.816i \\ 1.85503 - 0.296943i & 3.83906 - 4.57912i & -0.84807 - 21.5991i \\ -0.352031 + 0.406491i & 0.220313 + 1.52637i & 5.16094 + 4.57912i \end{pmatrix}.$$

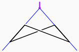
We neead to find a matrix t such that $t^{-1} g_k t$ are all integer matrices. One needs some experimentation to find such a matrix. However, it appears to be possible! We find that $t = \begin{pmatrix} 1 & 0 & 3 \\ -3c - \frac{1}{32c} & \frac{i\left(1-96c^2\right)}{16\sqrt{3}c} & -3c - \frac{1}{32c} \\ c & \frac{2ic}{\sqrt{3}} & c \end{pmatrix}$ with $c = 0.11734382...$ being some unrecognized constant, renders

$$t^{-1} g_1 t = \begin{pmatrix} -2 & 0 & -3 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}, t^{-1} g_2 t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, t^{-1} g_3 t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 6 \\ 0 & -4 & -5 \end{pmatrix}.$$

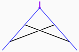
Examples of monodromy groups



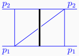
Two-loop sunrise[3]: $\mathcal{G}_{\circlearrowleft} \cong \Gamma_1(6) \subset SL(2,\mathbb{Z})$



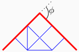
Two-loop massive vertex [von Manteuffel and Tancredi, 2017]: $\mathcal{G}_{\circlearrowleft} \cong \Gamma(2) \subset SL(2,\mathbb{Z})$.



Two-loop EW vertex [Broedel, Duhr, Dulat, Penante, and Tancredi, 2019]: $\mathcal{G}_{\circlearrowleft} \cong \Gamma_1(6) \subset SL(2,\mathbb{Z})$.



3-loop forward box [Mistlberger, 2018]: $\mathcal{G}_{\circlearrowleft} \cong \Gamma_1(5) \subset SL(2,\mathbb{Z})$.



4-loop HQET vertex [Brüser, Dlapa, Henn, and Yan, 2020] : $\mathcal{G}_{\circlearrowleft} \cong \Gamma(3) \subset SL(2,\mathbb{Z})$.

3-loop equal-mass sunrise [Broedel, Duhr, and Matthes, 2022]:

$$\mathcal{G}_{\circlearrowleft} \cong \left\langle \begin{pmatrix} 1 & 6 & -5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & -2 \\ 4 & 4 & -3 \end{pmatrix}, \begin{pmatrix} -3 & -10 & 7 \\ 12 & 31 & -21 \\ 16 & 40 & -27 \end{pmatrix} \right\rangle \subset GL(3,\mathbb{Z}).$$

3-loop HQET sunrise $\mathcal{G}_{\circlearrowleft} \cong \left\langle \begin{pmatrix} -2 & 0 & -3 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 6 \\ 0 & -4 & -5 \end{pmatrix} \right\rangle \subset GL(3,\mathbb{Z})$

[3] Here

$$\Gamma_1(N) = \left\{ g \in SL(2,\mathbb{Z}) \middle| \, g = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \text{ mod } N \right\}, \qquad \Gamma(N) = \left\{ g \in SL(2,\mathbb{Z}) \middle| \, g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ mod } N \right\}$$

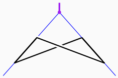
More ideas of treating non-polylogarithmic integrals

- Use ϵ-regular basis [RL and Onishchenko, 2019].
- Use Feynman parametrization to gather two denominators into one [Bezuglov and Onishchenko, 2022].
- Introduce suitable cut denominator $\delta(s - D)$ to later integrate wrt s.
- For the integrals expressible via hypergeometric functions ${}_pF_q$ use integral representation and expand in ϵ under the integral sign [Bezuglov, Kotikov, and Onishchenko, 2022].

These methods seem to be not universal, but may help in real-life calculations.

Example: maximal cut of non-planar vertex

Consider one solution of the homogeneous differential system,
$J_1 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2} + 2\epsilon, 1+\epsilon | x\right)$. Integrating out ϵ^0 gives



$$J_1 = \sum_k \epsilon^k \sum_{\boldsymbol{i}\in\{1,2\}^{k+1}} \frac{2\mathrm{K}\left(x_{i_0}\right)}{\pi} \mathcal{I}(\Omega_{i_0 i_1}, \Omega_{i_1 i_2}, \ldots, \Omega_{i_{k-1} i_k}, \Omega_{i_k 1} | x),$$

where $\mathcal{I}$ denotes iterated integral, $x_1 = x, x_2 = \overline{x} = 1-x$, and

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = \widetilde{B}(x)dx = \begin{pmatrix} u(\overline{x})v(x) & -u(\overline{x})v(\overline{x}) \\ u(x)v(x) & -u(x)v(\overline{x}) \end{pmatrix} \frac{dx}{\pi x \overline{x}},$$

$$u(x) = \mathrm{K}(x) - 2\mathrm{E}(x), \; v(x) = 2\overline{x}\mathrm{K}(x) - 2\mathrm{E}(x).$$

Ω can be expressed via modular forms. Meanwhile, there are much simpler representations in terms of one-fold integrals:

$$J_1 = \frac{\Gamma(\epsilon+1)}{\sqrt{\pi}\Gamma\left(\epsilon+\frac{1}{2}\right)} \sum_k \epsilon^k \int_0^1 \frac{dt}{\sqrt{t(1-t)(1-tx)}} \frac{\ln^k \frac{1-t}{(1-tx)^2}}{k!}$$

$$J_1 = \frac{1}{i\pi^2} \oint_{\sqrt{x}<|t|<1} \frac{dt\,\mathrm{K}(x/t^2)}{t(1-t^2)} \left[1 - 2\epsilon(1-2t)H_1 + 2\epsilon^2 \left[2H_{0,1} - (1-2t)\left(3H_{1,-1} + H_{1,1}\right)\right] + \ldots\right],$$

where $H_{\boldsymbol{n}} = H_{\boldsymbol{n}}(t)$ is harmonic polylogarithm.

- Each step towards increasing the # of loops and/or # of scales requires new methods. Those involve both technological advances (e.g. massive parallelization) and new algorithms coming various fields of mathematics.
- IBP reduction still remains a bottleneck for some calculations. New ideas of IBP reduction appear, whether they will be successful is yet to find out.
- Differential equations method is already in a very good shape. However, there is still no regular approach to the computation of non-polylogarithmic integrals.
- From the practical point of view, there is always a Frobenius method which might be used to obtain numerical high-precision results.

Thank you!

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