Path integrals in quantum gravity models

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Talk at "International Conference on Quantum Field Theory,
High-Energy Physics, and Cosmology"
18-21 July, 2022. Dubna, Russia

I. Introductory comments

Path integrals in QM and in QFT are considered as the integrals over a Gaussian measure given by a free action $A_0 \sim \int \frac{m\,(x')^2}{2}\,dt$.

In imaginary time, it leads to the Wiener measure

$$w(dx) = \exp\left\{-\frac{1}{2}\int \big(x'(t)\big)^2\,dt\right\}\;dx\,.$$

The similar picture takes place in the common models of fundamental interactions in QFT.

The problem is to choose **the true dynamical variable** that define path integrals measure

and then to find **the substitution** that transforms the measure into the Wiener one.

II. Schwarzian path integrals

JT dilaton 2D gravity and SYK model (a quantum mechanical model of Majorana fermions with a random all-to-all interaction)

lead to an effective theory with the **Schwarzian action**

$$A_{Sch} = -\frac{1}{\sigma^2} \int\limits_{[0,\,1]} \left[\mathcal{S}ch(\varphi,\, t) + 2\pi^2 \left(\varphi'(t)\right)^2\right] dt\,,$$

where

$$\mathcal{S}ch(\varphi,\, t) = \left(\frac{\varphi''(t)}{\varphi'(t)}\right)' - \frac{1}{2}\left(\frac{\varphi''(t)}{\varphi'(t)}\right)^2 .$$

Here $\varphi(t)$ is an orientation preserving ($\varphi'(t) > 0$) diffeomorphism of the interval ($\varphi \in Diff_+^1([0,\, 1])$) .

$\varphi(t)$ plays the role of **the dynamical variable** of the theory.

$-\left(\varphi'(t)\right)^2$ **(wrong sign of the term ?!)**

On the group of diffeomorphisms $Diff_+^1([0,\,1])$, there exists **a countably-additive measure** formally written as

$$\mu_\sigma(d\varphi) = \exp\left\{ \frac{1}{\sigma^2} \int\limits_{[0,\,1]} \mathcal{S}ch(\varphi,\,t)\,dt \right\} d\varphi\,.$$

It is generated by the Wiener measure under some special **substitution of variables**.

If we consider a continuous function on the interval $[0,\,1]$ $\xi(t)$ satisfying the boundary condition $\xi(0)=0$ $(\xi \in C_0([0,\,1]))$, then under the substitution

$$\varphi(t) = \frac{\int\limits_0^t e^{\xi(\tau)}d\tau}{\int\limits_0^1 e^{\xi(\eta)}d\eta}\,, \qquad \xi(t) = \log\varphi'(t) - \log\varphi'(0)\,,$$

the measure $\mu_\sigma(d\varphi)$ on the group $Diff_+^1([0,\,1])$ turns into **the Wiener measure** $w_\sigma(d\xi)$ on $C_0([0,\,1])$, and there is **the equality of functional integrals**

$$\int\limits_{Diff_+^1([0,1])} F\left(\varphi,\,\varphi'\right)\,\mu_\sigma(d\varphi) = \int\limits_{C_0([0,1])} F\left(\varphi(\xi),\,(\varphi(\xi))'\right)\,w_\sigma(d\xi)\,.$$

Polar decomposition of the Wiener measure

The Wiener measure

$$w_\sigma(dx) = \exp\left\{ -\frac{1}{2\sigma^2} \int\limits_0^1 \left(x'(t)\right)^2 dt \right\} dx\,.$$

is quasi-invariant under the following action of the group of diffeomorphisms $Diff_+^3([0,\,1])$ on $C_+([0,\,1])$:

$$x \mapsto fx\,, \quad (fx)(t) = x\left(f^{-1}(t)\right) \frac{1}{\sqrt{\left(f^{-1}(t)\right)'}}\,,$$

$$x \in C_+([0,\,1])\,, \quad f \in Diff_+^3([0,\,1])\,.$$

There is the invariant under the group $Diff_+^3([0,\,1])$. It is given by the integral

$$\frac{1}{\rho^2} = \int\limits_0^1 \frac{1}{x^2(t)} dt\,.$$

Define $\varphi \in Diff^1_+([0,\, 1])$ by the equation

$$\varphi^{-1}(t) = \rho^2 \int\limits_0^t \frac{1}{x^2(\tau)} d\tau\,.$$

Then $x(t)$ is expressed in terms of ρ and $\varphi(t)$

$$x(t) = \rho \frac{1}{\sqrt{(\varphi^{-1}(t))'}}\,.$$

In this case, we have

$$\int\limits_0^1 x^2(t)dt = \rho^2 \int\limits_0^1 \left(\varphi'(\tau)\right)^2 d\tau\,.$$

Therefore, there is a one-to-one correspondence $(\rho\,,\ \varphi) \leftrightarrow x$, and the space $C_+([0,\, 1])$ is stratified into the orbits with different values of the invariant ρ.

Thus, for the Wiener measure on the space $C_+([0,\, 1])$ the following polar decompositions are valid:

$$w_\sigma(dx) = \mathcal{P}_\sigma(\rho)\ \left(\varphi'(0)\varphi'(1)\right)^{\frac{3}{4}}\ \mu_{\frac{2\sigma}{\rho}}(d\varphi)\, d\rho\,.$$

$$w_\sigma(dx) = \exp\left\{-\frac{\sigma^2}{8\rho^2}\right\}\ \frac{e^{\frac{3}{4}\xi(1)}}{\left(\int\limits_0^1 e^{\xi(\tau)} d\tau\right)^{\frac{3}{2}}}\ w_{\frac{2\sigma}{\rho}}(d\xi)\ d\rho\,.$$

Although $x(t)$ and $\xi(\tau)$ are both Wiener processes , the Markov behaviour of $x(t)$ with respect to the time t of "its own world" obviously does not imply its Markov behaviour with respect to the time τ of the "shadow world", and vice versa.

The proof and the general rules of Schwarzian path integration as well as some properties of Wiener integrals are given in:

VV B and ET Sh, Extraordinary Properties of Functional Integrals and Groups of Diffeomorphisms, Phys. Part. Nucl. **48** (2017) 267.

VV B and ET Sh, Exact solution of the Schwarzian theory, Phys. Rev. **D 96** (2017) 101701(R) , [arXiv:1705.02405 [hep-th]].

VV B and ET Sh, Correlation functions in the Schwarzian theory, JHEP **11** (2018) 036 , [arXiv:1804.00424].

VV B and ET Sh, Unusual view of the Schwarzian theory, Mod. Phys. Lett. A **33** (2018) 1850221 , [arXiv:1806.05605].

VV B and ET Sh, Polar decomposition of the Wiener measure: Schwarzian theory versus conformal quantum mechanics, Theor. Math. Phys. **200** (2019) 1324, [arXiv:1812.04039].

VV B and ET Sh, Functional integration over the factor-space $Diff_+^1(S^1)/SL(2,\mathbf{R})$, Phys. Part. Nucl. **51** (2020) 424, [arXiv:1912.07841].

VV B and ET Sh, Schwarzian functional integrals calculus, J. Phys. A: Math. Theor. **53** (2020) 485201 , [arXiv:1908.10387].

III. PI for quantum 4D quadratic gravity

The idea:

We consider $R + R^2$ theory in the FLRW metric and find
the dynamical variable $g(\tau)$ that is invariant under the group of diffeomorphisms of the time coordinate.

Then we turn the Feynman path integrals

$$\int F(g) \exp\{i\, A(g)\}\, dg$$

into the Euclidean ones

$$\int F(g) \exp\{-A(g)\}\, dg$$

by the corresponding transformation of the space-time metric.

We consider path integrals **not over** the space of metrics $\{\mathcal{G}\}$, as it is usually done,

but over the space of continuous functions $\{g(\tau)\}$ related to the conformal factor of the metric.

The second our novelty is to treat path integrals as the integrals over the functional measure

$$\mu(g) = \exp\{-A_2\}\, dg\,,$$

where A_2 is the part of the action A **quadratic** in R.

The rest part of the action in the exponent stands in the integrand as the "interaction" term.

We prove the measure $\mu(g)$ to be **equivalent to the Wiener measure,** and, as an example, calculate the averaged scale factor in the first nontrivial perturbative order.

Realization of the idea:

We study the gravity model with the action

$$A = A_0 + A_1 + A_2 = \Lambda \int d\tilde{t}\,\sqrt{-\mathcal{G}} - \frac{\kappa}{6} \int d\tilde{t}\,\sqrt{-\mathcal{G}}\,R + \frac{\lambda^2}{72} \int d\tilde{t}\,\sqrt{-\mathcal{G}}\,R^2$$

in FLRW metric

$$ds^2 = N^2(\tilde{t})\,d\tilde{t}^2 - a^2(\tilde{t})\,d\vec{x}^2\,, \qquad N(\tilde{t}) > 0\,, \quad a(\tilde{t}) > 0\,.$$

Now the general coordinate invariance of the action is reduced to its invariance under the group of reparametrizations of the time coordinate. We suppose it to be the group of diffeomorphisms of the real semiaxis including zero $Diff\,(\mathbf{R}^+)$.

The two coordinate systems are the most popular. They are the so-called **cosmological coordinate system** where

$$N(t) = 1\,,$$

and
conformal coordinate system where

$$N(\tau) = a(\tau)\,,$$

with cosmological time t and conformal time τ being the time variable in the corresponding coordinate system.

We define the action of the diffeomorphisms $\varphi \in Diff\,(\mathbf{R}^+)$ on the functions $N(\tilde{t})$ and $a(\tilde{t})$ as follows:

$$(\varphi\,N)\,(\tilde{t}) = \left(\varphi^{-1}(\tilde{t})\right)'\,N\left(\varphi^{-1}(\tilde{t})\right)\ ; \qquad (\varphi\,a)\,(\tilde{t}) = a\left(\varphi^{-1}(\tilde{t})\right)\ .$$

Instead of the laps and the scale factors, it is convenient to use the functions $f(\tilde{t})$ and $h(\tilde{t})$ defined by the following equations:

$$f^{-1}(\tilde{t}) = \int\limits_0^{\tilde{t}} \frac{N(\tilde{t}_1)}{a(\tilde{t}_1)}\,d\tilde{t}_1\,, \qquad h(\tilde{t}) = \int\limits_0^{\tilde{t}} N(\tilde{t}_1)\,d\tilde{t}_1\,,$$

with the transformation rules

$$(\varphi f)(\tilde{t}) = \varphi\left(f(\tilde{t})\right) \equiv (\varphi \circ f)\,(\tilde{t})\,, \qquad (\varphi h)(\tilde{t}) = h\left(\varphi^{-1}(\tilde{t})\right) \equiv \left(h \circ \varphi^{-1}\right)(\tilde{t})\,.$$

The function

$$g(\tau) = (h \circ f)(\tau) = h\left(f(\tau)\right)$$

is **invariant** under the diffeomorphisms φ .

It **turns** the conformal coordinate system time τ **to** the cosmological one t

$$t = g(\tau)\,, \qquad \tau = g^{-1}(t)\,.$$

$$Conformal: \quad ds^2 = \left(g'(\tau)\right)^2 \left[d\tau^2 - d\vec{x}^2\right] \,,$$

$$Cosmological: \quad ds^2 = dt^2 - \left(g'\left(g^{-1}(t)\right)\right)^2 d\vec{x}^2\,.$$

The invariance of the action manifests itself in its **dependence on the only** invariant function g

$$A = A(f,\, h) = A\,(g) = A_0\,(g) + A_1\,(g) + A_2\,(g)\ ,$$

with the explicit form

$$A_0\,(g) = \Lambda \int \left(g'(\tau)\right)^4\, d\tau\,,$$

$$A_1\,(g) = -\kappa \int \left[\left(g''(\tau)\right)^2 - \left(g''(\tau)\, g'(\tau)\right)'\right]\, d\tau\,,$$

and

$$A_2\,(g) = \frac{\lambda^2}{2} \int \left(\frac{g'''(\tau)}{g'(\tau)}\right)^2\, d\tau\,.$$

Therefore, every four-dimensional space-time (and the corresponding space-time FLRW metric) is **determined by its proper function** g, and vice versa, every function g **determines the particular four-dimensional space-time.**

Thus, averaging over the set of functions g means the averaging over the set of possible four-dimensional spaces.

We have found the true dynamical variable!

Now we represent path integrals in the theory

$$\int F(g)\, \exp\{-A(g)\}\, dg$$

as the integrals of the form

$$\int F(g)\, \exp\{-A_0(g) - A_1(g)\}\, \mu_\lambda(dg)\,,$$

over the functional measure

$$\mu_\lambda(dg) = \exp\{-A_2(g)\}\ dg = \exp\left\{-\frac{\lambda^2}{2}\int \left(\frac{g'''(\tau)}{g'(\tau)}\right)^2 d\tau\right\}\ dg\,.$$

If we substitute

$$q(\tau) = \frac{g''(\tau)}{g'(\tau)} \,,$$

we can rewrite the integral in the exponent in the measure density as

$$-\frac{\lambda^2}{2} \int \left(\frac{g'''(\tau)}{g'(\tau)} \right)^2 d\tau = -\frac{\lambda^2}{2} \int (p'(\tau))^2 \, d\tau \,,$$

where p is given by the nonlinear nonlocal substitution

$$p(\tau) = q(\tau) + \int\limits_0^{\tau} q^2(\tau_1) \, d\tau_1 \,.$$

There is the one-to-one correspondence between the function $g(\tau)$ and the Wiener variable $p(\tau)$, and the measure $\mu_\lambda(dg)$ written in terms of $p(\tau)$ is the Wiener measure $w_{\frac{1}{\lambda}}(dp)$.

The nonlinear nonlocal substitutions in the Wiener measure were studied in

VV B and ET Sh : J. Math. Sci. **248** (2020) 544, (and also arXiv:1112.3899v2).

and it was demonstrated that the paths $p(\tau)$ form the space of all continuous functions on the interval $[0,\, T]$, while the paths $q(\tau)$ are continuous almost at all points of the interval but may have singularities of the form

$$q(\tau) \sim \frac{1}{\tau - \tau_j^*}$$

at a finite number of points of the finite interval $[0,\, T]$.

It gives us the possibility to study the quantum corrections to the classical solutions of the form

$$g_{cl}(\tau) = \frac{1}{2}\sigma\,\tau^2\,, \qquad a_{cl}(t) = g'_{cl}\left(g_{cl}^{-1}(t)\right) = \sqrt{2\sigma\,t}\,.$$

Calculating the path integrals in the first nontrivial order $\left(\sim \frac{1}{\lambda^2}\right)$

$$< a(t) >_g = \mathcal{Z}^{-1} \int g'\left(g^{-1}(t)\right)$$

$$\times \exp\left\{ \int\limits_0^{g^{-1}(t)} \left[-\Lambda \, \left(g'(\tau_1)\right)^4 + \kappa \, \left(g''(\tau_1)\right)^2\right] d\tau_1 - \kappa g''(g^{-1}(t))\, g'(g^{-1}(t)) \right\} \mu_\lambda(dg)$$

where the normalizing factor is

$$\mathcal{Z} = \int \exp\left\{ \int\limits_0^{g^{-1}(t)} \left[-\Lambda \, \left(g'(\tau_1)\right)^4 + \kappa \, \left(g''(\tau_1)\right)^2\right] d\tau_1 - \kappa g''(g^{-1}(t))\, g'(g^{-1}(t)) \right\} \mu_\lambda(dg)$$

we obtain the result for the averaged scale factor as the function of cosmological time t

$$< a(t) >_g = \sqrt{2\sigma\, t} \left\{1 + \frac{1}{\lambda^2} \left[-\frac{59}{63} \left(\frac{2t}{\sigma}\right)^{\frac{3}{2}} + \frac{11}{120}\, \kappa \, (2t)^2 - \frac{1423}{2800}\, \Lambda \, (2t)^4\right]\right\} .$$

VV B and ET Sh, **Path integrals in quadratic gravity**, JHEP **02** (2022) 112 , [arXiv:2110.06041];

IV. PI for quantum 2D quadratic gravity

The general form of the 2D gravity action up to the terms quadratic in curvature K is

$$\tilde{\mathcal{A}} = c_0 \int \sqrt{\mathcal{G}}\, d^2x + c_1 \int K \sqrt{\mathcal{G}}\, d^2x + c_2 \int K^2 \sqrt{\mathcal{G}}\, d^2x \,.$$

We consider the action restricted to the conformal gauge, where the metric of the 2D surface looks like

$$dl^2 = g(u,\, v)\, \left(du^2 + dv^2\right) = g(z,\, \bar{z})\, dz\, d\bar{z} \qquad \sqrt{\mathcal{G}} = g \,.$$

The Gaussian curvature of the surface is

$$K = -\frac{1}{2g}\, \Delta\, \log g \,,$$

where Δ stands for the Laplacian.

The action is invariant under the complex analytic substitutions. Therefore, we reduce the region of integration to the disc $d\, :\, (|z| \leq 1)\,.$

We consider the specific form of the action

$$A = \frac{\lambda^2}{2} \int\limits_d (K+4)^2 \; g(z,\, \bar{z})\, dz\, d\bar{z} = \frac{\lambda^2}{2} \int\limits_d (\Delta\psi)^2 \;\; dz\, d\bar{z}$$

where

$$\Delta\psi = q\, \Delta \log q + \frac{4}{q}\,, \qquad q = \frac{1}{\sqrt{g}}\,,$$

and study path integrals

$$\int F(\psi)\, \exp\{-A(\psi)\}\, d\psi = \int F(\psi)\, \mu_\lambda(d\psi)$$

over the Gaussian functional measure

$$\mu_\lambda(d\psi) = \frac{\exp\{-A(\psi)\}\, d\psi}{\int \exp\{-A(\psi)\}\, d\psi}\,.$$

The extremum of the action is given by the equation

$$\Delta\psi = 0\,, \qquad q\,\Delta\log q + \frac{4}{q} = 0\,.$$

We choose the boundary condition corresponding to the Poincare model of the Lobachevsky plane

$$q_0|_{|z|=1} = 0\,.$$

The unique solution in the disk $d\,(|z| \leq 1)$ satisfying the boundary condition is

$$q_0 = 1 - z\bar{z}\,.$$

Let us rewrite the action substituting $\psi \to f$ with

$$\Delta\psi = \frac{1}{q_0} T^{-1} \, [f]$$

where

$$T^{-1} \equiv \left(q_0^2 \, \Delta - 8\right)$$

is the Casimir operator of $SL(2, \mathbb{R})$.

Now the action is written as the integral over the measure

$$\frac{dz \, d\bar{z}}{(1 - z\bar{z})^2}$$

invariant under the action of the group $SL(2, \mathbb{R})$ in the disk

$$A = \frac{\lambda^2}{2} \int\limits_d \left(T^{-1} \, [f]\right)^2 \, \frac{dz \, d\bar{z}}{q_0^2} \, .$$

Therefore, we obtain the $SL(2, \mathbb{R})$ invariant Gaussian functional measure

$$\mu_\lambda(df) = \frac{\exp\{-A(f)\} \, df}{\int \exp\{-A(f)\} \, df} \, .$$

We reduce path integrals over the measure $\mu_\lambda(df)$
to the products of Wiener integrals.
First, we consider the Fourier series (we use polar coordinates)

$$\Delta\psi(\varrho,\,\varphi) = x_0(\varrho) + \sum_{n=1}^{\infty} (x_n\,\cos n\varphi + y_n\,\sin n\varphi)\ .$$

Now the action is written as

$$A = \frac{\lambda^2}{2}\, 2\pi \int_0^1 (x_0(\varrho))^2\ \varrho\, d\varrho + \frac{\lambda^2}{2}\, \pi \sum_{n=1}^{\infty} \left[\int_0^1 (x_n(\varrho))^2\ \varrho\, d\varrho + \int_0^1 (y_n(\varrho))^2\ \varrho\, d\varrho \right] .$$

Then, using the relation between ψ and f, we express x_n , y_n in terms of the coefficients of the Fourier series

$$f(\varrho,\,\varphi) = a_0(\varrho) + \sum_{n=1}^{\infty} (a_n\,\cos n\varphi + b_n\,\sin n\varphi)\ .$$

Therefore we have

$$\int\limits_0^1 \left(x_n(\varrho)\right)^2 \, \varrho \, d\varrho = \int\limits_0^{+\infty} \left(U_n'(\tau_n)\right)^2 \, d\tau_n$$

$$U_n(\tau_n) = \varrho^{2n+1} \, \frac{\left(1+n+(1-n)\varrho^2\right)^2}{\left(1-\varrho^2\right)^2} \, \left(\frac{\left(1-\varrho^2\right) \, a_n(\varrho)}{\varrho^n \, \left(1+n+(1-n)\varrho^2\right)}\right)' \, .$$

$$\tau_n = \int\limits_0^{\varrho} \frac{\varrho_1^{2n+1}}{\left(1-\varrho_1^2\right)^2} \, \left(\frac{2}{\left(1-\varrho_1^2\right)} + n - 1\right)^2 \, d\varrho_1 \, .$$

The same equations are valid for the other terms

$$\int\limits_0^1 \left(y_n(\varrho)\right)^2 \, \varrho \, d\varrho = \int\limits_0^{+\infty} \left(\tilde{U}_n'(\tau_n)\right)^2 \, d\tau_n \, .$$

Now the measure $\mu_\lambda(df)$ is represented as the product of the Wiener measures

$$\mu_\lambda(df) = w_{\frac{1}{\lambda\sqrt{2\pi}}}\,(dU_0)\,\prod_{n=1}^{\infty} w_{\frac{1}{\lambda\sqrt{\pi}}}\,(dU_n)\;w_{\frac{1}{\lambda\sqrt{\pi}}}\left(d\tilde{U}_n\right)$$

where

$$w_\sigma\,(dU) = \exp\left\{-\frac{1}{2\sigma^2}\int\limits_0^{+\infty}\left(U'(\tau)\right)^2\,d\tau\right\}\,dU\,.$$

For path integrals with integrands that depend on the modes with definite numbers, the product of the measures is reduced because of the cancelation of the same terms in the nominator and in the denominator.

$SL(2,\,\mathbb{R})$ invariance of the measure simplifies the calculations.

In particular, due to the $SL(2,\,\mathbb{R})$ invariance,

$$<\,g(\varrho,\,\varphi)\,>_\mu = \int\limits_{C(d)} g(\varrho,\,\varphi)\;\mu_\lambda(df) = \frac{1}{q_0^2(\varrho)}\;<\,g(0,\,0)\,>_\mu = \frac{1}{q_0^2(\varrho)}\,.$$

As an example, we have calculated the correlation function of the metric given by the integral

$$< g(\varrho_1, \varphi_1)\, g(\varrho_2, \varphi_2) >_\mu = \int\limits_{C(d)} g(\varrho_1, \varphi_1)\, g(\varrho_2, \varphi_2)\; \mu_\lambda(df)$$

in the first nontrivial perturbative order.

Due to the $SL(2, \mathbb{R})$ invariance, the correlation function of the metric can be rewritten as

$$< g(\varrho_1, \varphi_1)\, g(\varrho_2, \varphi_2) >_\mu = \frac{q_0^2(\varrho_*)}{q_0^2(\varrho_1)\, q_0^2(\varrho_2)} \int\limits_{C(d)} g(0, 0)\, g(\varrho_*, \varphi_*)\; \mu_\lambda(df)$$

$$= \frac{q_0^2(\varrho_*)}{q_0^2(\varrho_1)\, q_0^2(\varrho_2)} < g(0, 0)\, g(\varrho_*, \varphi_*) >_\mu ,$$

where $(0, 0)$ and (ϱ_*, φ_*) are the results of the shift at the Lobachevsky plane of the coordinates (ϱ_1, φ_1) and (ϱ_2, φ_2)

$$\varrho_* = \frac{\sqrt{\varrho_1^2 + \varrho_2^2 - 2\varrho_1\varrho_2 \cos(\varphi_2 - \varphi_1)}}{\sqrt{1 + \varrho_1^2\varrho_2^2 - 4\varrho_1\varrho_2 \cos(\varphi_2 - \varphi_1)}} .$$

Thus,

$$< g(\varrho_1, \varphi_1)\, g(\varrho_2, \varphi_2) >_\mu$$

$$= \frac{2}{q_0^2(\varrho_1)\, q_0^2(\varrho_2)} \left\{ 1 + \frac{1}{6\pi\lambda^2} \left[-\frac{1}{2} \int\limits_{\varrho_*}^{1} \frac{\log t}{1-t} dt + 2\log 2 + \frac{1}{2} - \frac{1}{1+\varrho_*^2} \right.\right.$$

$$\left.\left. + \frac{1}{1+\varrho_*^2} \log\left(\frac{1-\varrho_*^2}{\varrho_*} \right) + \log \varrho_* - 2\log(1+\varrho_*^2) - \log\sqrt{1-\varrho_*^2} + \log\varrho_* \, \log\sqrt{1-\varrho_*^2} \right] \right\} .$$

where

$$\varrho_* = \frac{\sqrt{\varrho_1^2 + \varrho_2^2 - 2\varrho_1\varrho_2 \cos(\varphi_2 - \varphi_1)}}{\sqrt{1 + \varrho_1^2\varrho_2^2 - 4\varrho_1\varrho_2\cos(\varphi_2 - \varphi_1)}} .$$

VV B and ET Sh, **An approach to quantum 2D gravity**, arXiv:2206.05172.