

Weyl Cosmology and Particle Production

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A little bit of mathematics

Differential geometry

Metric tensor $g_{\mu\nu} \Rightarrow$ interval

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \quad (g_{\mu\nu} g^{\nu\lambda} = \delta_\mu^\lambda)$$

Connections $\Gamma_{\mu\nu}^\lambda \Rightarrow$ covariant derivative

$$\nabla_\mu J^\nu = J^\nu_{,\mu} + \Gamma_{\sigma\mu}^\nu J^\sigma, \dots$$

Curvature tensor $R^\mu_{\nu\lambda\sigma}$

$$R^\mu_{\nu\lambda\sigma} = \frac{\partial \Gamma_{\nu\sigma}^\mu}{\partial x^\lambda} - \frac{\partial \Gamma_{\nu\lambda}^\mu}{\partial x^\sigma} + \Gamma_{\kappa\lambda}^\mu \Gamma_{\nu\sigma}^\kappa - \Gamma_{\kappa\sigma}^\mu \Gamma_{\nu\lambda}^\kappa$$

Ricci tensor $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$

Curvature scalar $R = g^{\mu\lambda} R_{\mu\lambda}$

$$g_{\mu\nu}(x), \quad \Gamma_{\mu\nu}^\lambda(x)$$

A little bit of mathematics

Differential geometry II

Three tensors $g_{\mu\nu}(x)$, $S_{\mu\nu}^{\lambda}$, $Q_{\lambda\mu\nu}$

Torsion $S_{\mu\nu}^{\lambda} = \Gamma_{\mu\nu}^{\lambda} - \Gamma_{\nu\mu}^{\lambda}$

Nonmetricity $Q_{\lambda\mu\nu} = \nabla_{\lambda} g_{\mu\nu}$

Connections $\Gamma_{\mu\nu}^{\lambda} = C_{\mu\nu}^{\lambda} + K_{\mu\nu}^{\lambda} + L_{\mu\nu}^{\lambda}$

$$C_{\mu\nu}^{\lambda} = \frac{1}{2} g^{\lambda\kappa} (g_{\kappa\mu,\nu} + g_{\kappa\nu,\mu} - g_{\mu\nu,\kappa})$$

$$Q_{\lambda\mu\nu} = \nabla_{\lambda} g_{\mu\nu} \Rightarrow Q_{\lambda\mu\nu} = Q_{\lambda\nu\mu}$$

$$K_{\mu\nu}^{\lambda} = \frac{1}{2} (S_{\mu\nu}^{\lambda} - S_{\mu}^{\lambda}{}_{\nu} - S_{\nu}^{\lambda}{}_{\mu})$$

$$L_{\mu\nu}^{\lambda} = \frac{1}{2} (Q_{\mu\nu}^{\lambda} - Q_{\mu}^{\lambda}{}_{\nu} - Q_{\nu}^{\lambda}{}_{\mu})$$

A little bit of mathematics

Differential geometry III

Riemann geometry

$$S_{\mu\nu}^{\lambda} = 0, \quad Q_{\lambda\mu\nu} = 0$$

$$S_{\mu\nu}^{\lambda} = 0, \quad Q_{\lambda\mu\nu} = 0, \quad \Rightarrow \quad \Gamma_{\mu\nu}^{\lambda} = C_{\mu\nu}^{\lambda}$$

Weyl geometry

$$S_{\mu\nu}^{\lambda} = 0 \quad \Rightarrow \quad \Gamma_{\mu\nu}^{\lambda} = \Gamma_{\nu\mu}^{\lambda}$$

$$Q_{\lambda\mu\nu} = A_{\lambda}(x)g_{\mu\nu}(x)$$

$$\Gamma_{\mu\nu}^{\lambda} = C_{\mu\nu}^{\lambda} + W_{\mu\nu}^{\lambda}$$

$$W_{\mu\nu}^{\lambda} = -\frac{1}{2}(A_{\mu}\delta_{\nu}^{\lambda} + A_{\nu}\delta_{\mu}^{\lambda} - A^{\lambda}g_{\mu\nu})$$

$A_{\lambda}(x)$ — “Weyl vector”

Local conformal transformation

$$ds^2 = \Omega^2(x) d\hat{s}^2 = \Omega^2 \hat{g}_{\mu\nu}(x) dx^\mu dx^\nu$$

$$g_{\mu\nu} = \Omega^2 \hat{g}_{\mu\nu}, \quad g^{\mu\nu} = \frac{1}{\Omega^2} \hat{g}^{\mu\nu}$$

$$\sqrt{-g} = \Omega^4(x) \sqrt{-\hat{g}}$$

$$C_{\mu\nu}^\lambda = \hat{C}_{\mu\nu}^\lambda + \left(\frac{\Omega_{,\nu}}{\Omega} \delta_\mu^\lambda + \frac{\Omega_{,\mu}}{\Omega} \delta_\nu^\lambda - g^{\lambda\kappa} \frac{\Omega_{,\kappa}}{\Omega} \hat{g}_{\mu\nu} \right)$$

If $A_\mu =$ gauge field

$$A_\mu = \hat{A}_\mu + 2 \frac{\Omega_{,\mu}}{\Omega} \Rightarrow$$

$$\Gamma_{\mu\nu}^\lambda = \hat{\Gamma}_{\mu\nu}^\lambda \Rightarrow$$

$$R^\mu{}_{\nu\lambda\sigma} = \hat{R}^\mu{}_{\nu\lambda\sigma}, \quad R_{\mu\nu} = \hat{R}_{\mu\nu}$$

Strength tensor

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = A_{\nu,\mu} - A_{\mu,\nu}$$

Weyl gravity

$$\mathcal{L}_W = \alpha_1 R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R^2 + \alpha_4 F_{\mu\nu} F^{\mu\nu}$$

$$S_W = \int \mathcal{L}_W \sqrt{-g} d^4x, \quad \frac{\delta S_W}{\delta \Omega} = 0$$

Dynamical variables: $g_{\mu\nu}(x)$, $A_\mu x$

Matter fields $S_{\text{tot}} = S_W + S_m$, $S_m = \int \mathcal{L}_m \sqrt{-g} d^4x$

S_m is not necessary conformal invariant

δS_m does!

$$\begin{aligned} \delta S_m \stackrel{\text{def}}{=} & -\frac{1}{2} \int T^{\mu\nu} (\delta g_{\mu\nu}) \sqrt{-g} d^4x - \int G^\mu (\delta A_\mu) \sqrt{-g} d^4x \\ & + \int \frac{\delta \mathcal{L}_m}{\delta \Psi} (\delta \Psi) \sqrt{-g} d^4x = 0 \end{aligned}$$

Ψ — collective dynamical variable for matter fields, $\delta \mathcal{L}_m / \delta \Psi = 0$

$T^{\mu\nu}$ — energy-momentum tensor, G^μ — “Weyl current”

Weyl gravity II

$$\delta g_{\mu\nu} = \frac{2}{\Omega} g_{\mu\nu} (\delta\Omega) + \Omega^2 (\delta \hat{g}_{\mu\nu})$$

$$\delta A_\mu = (\delta A_\mu) + 2(\delta(\log \Omega))_{,\mu}$$

$$\delta \hat{g}_{\mu\nu} : \frac{\delta S_m}{\delta \hat{g}_{\mu\nu}} \stackrel{\text{def}}{=} -\frac{1}{2} \hat{T}_{\mu\nu} \Rightarrow \hat{T}_{\mu\nu} = \Omega^2 T_{\mu\nu}$$

$$(\hat{T}_\mu^\nu = \Omega^4 T_\mu^\nu, \hat{T}^{\mu\nu} = \Omega^6 T^{\mu\nu})$$

$$\delta \hat{A}_\mu : \frac{\delta S_m}{\delta \hat{A}_\mu} \stackrel{\text{def}}{=} -\hat{G}^\mu \Rightarrow \hat{G}^\mu = \Omega^4 G^\mu$$

Self-consistency condition

$$\delta\Omega : 2G_{;\mu}^\mu = \text{Trace}[T^{\mu\nu}]$$

“;” — covariant derivative with $C_{\mu\nu}^\lambda$

Cosmology

Cosmology = homogeneity and isotropy
Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right), \quad k = 0, \pm 1$$

$$A_\mu = (A_0(t), 0, 0, 0) \Rightarrow$$

$$F_{\mu\nu} \equiv 0$$

$$T_\nu^\mu = (T_0^0(t), T_1^1(t) = T_2^2 = T_3^3)$$

Special gauge

$$A_0(t) = 0$$

Field equations

$$\delta A_\mu : \quad -6\gamma\dot{R} = G^0, \quad \gamma = \frac{1}{3}(\alpha_1 + \alpha_2 + 3\alpha_3)$$

$$R = -6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} \right)$$

$$\delta g_{\mu\nu} : \quad -12\gamma \left\{ \frac{\dot{a}}{a} \dot{R} + R \left(\frac{R}{12} + \frac{\dot{a}^2 + k}{a^2} \right) \right\} = T_0^0$$

$$-4\gamma \left\{ \ddot{R} + 2\frac{\dot{a}}{a} \dot{R} - R \left(\frac{R}{12} + \frac{\dot{a}^2 + k}{a^2} \right) \right\} = T_1^1$$

Self-consistency condition:

$$2 \frac{(G^0 a^3)'}{a^3} = T_0^0 + 3T_1^1$$

Perfect fluid, Riemann geometry I

J.R. Ray (1972):

$$S_m = - \int \varepsilon(X, n) \sqrt{-g} d^4x + \int \lambda_0 (u_\mu u^\mu - 1) \sqrt{-g} d^4x \\ + \int \lambda_1 (n u^\mu)_{;\mu} \sqrt{-g} d^4x + \int \lambda_2 X_{,\mu} u^\mu \sqrt{-g} d^4x$$

$\varepsilon(X, n)$ — energy density

$u^\mu(x)$ — four-velocity

$n(x)$ — particle number density,

$X(x)$ — auxiliary variable,

$\lambda_i(x)$ — Lagrange multipliers

Constraints

$u^\mu u_\mu = 1$ — normalization

$(n u^\mu)_{;\mu} = 0$ — particle number conservation

$X_{,\mu} u^\mu = 0$ — numbering of the trajectories

Perfect fluid, Riemann geometry II

Energy-momentum tensor

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - p g^{\mu\nu}$$

Hydrodynamical pressure

$$p = n \frac{\partial \varepsilon}{\partial n} - \varepsilon$$

V.A.Berezin (1987):

Phenomenological description of particle creation

$$(nu^\mu)_{;\mu} = 0 \quad \Rightarrow \quad (nu^\mu)_{;\mu} - \Phi(inv) = 0$$

Perfect fluid, Weyl geometry: G^μ — ? How to incorporate A_μ ?



New possibilities

Single particle u^μ

Riemann geometry

$$S_{\text{part}} = -m \int ds$$

Weyl geometry — new invariant $B = A_{,\mu} u^\mu$

$$S_{\text{part}} = \int f_1(B) ds + \int f_2(B) d\tau = \int \{ f_1(B) \sqrt{g_{\mu\nu} u^\mu u^\nu} + f_2(B) \} d\tau$$

Equations of motion

$$\begin{aligned} f_1(B) u_{\lambda;\mu} u^\mu &= \left((f_1'(B) + f_2''(B)) A_\lambda - f_1'(B) u_\lambda \right) B_{,\mu} u^\mu \\ &+ (f_1'(B) + f_2'(B)) F_{\lambda\mu} u^\mu \end{aligned}$$

Since $F_{\lambda\mu} u^\lambda u^\mu \equiv 0$ and $u_{\lambda;\sigma} u^\lambda \equiv 0$

Either $(f_1' + f_2'')B = f_1'$ or $B_{,\mu} u^\mu = 0$

The invariant $B = A_{\mu}u^{\mu}$ is closely tied to the number density n .
Hence,

$$\varepsilon(X, n) \Rightarrow \varepsilon(X, \varphi(B)n)$$

And how about the constraint $(nu^{\mu})_{;\mu} = \Phi(inv)$?

Conformal transformation

$$(nu^{\mu})_{;\mu} \sqrt{-g} = (nu^{\mu} \sqrt{-g})_{;\mu}$$

$$n = \frac{\hat{n}}{\Omega^3}, \quad u^{\mu} = \frac{\hat{u}^{\mu}}{\Omega}, \quad \sqrt{-g} = \Omega^4 \sqrt{-\hat{g}} \Rightarrow$$

$$(nu^{\mu} \sqrt{-g})_{;\mu} = (\hat{n} \hat{u}^{\mu} \sqrt{-\hat{g}})_{;\mu} \Rightarrow$$

$\Phi \sqrt{-g}$ — conformal invariant \Rightarrow

In the absence of the classical fields, i.e., in the case of the particle creation by the vacuum fluctuations

$$\Phi = \alpha'_1 R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + \alpha'_2 R_{\mu\nu} R^{\mu\nu} + \alpha'_3 R^2 + \alpha'_4 F_{\mu\nu} F^{\mu\nu}$$

Riemann geometry — the only possibility

$$\Phi \propto C^2$$

$C^2 = C_{\mu\nu\lambda\sigma} C^{\mu\nu\lambda\sigma}$ — square of the Weyl tensor.

Y.B.Zel'dovich and A.A.Starobinsky (1977):

Particle creation by the vacuum fluctuation of the massless scalar field on the homogeneous slightly anisotropic cosmological background.

Now it becomes fundamental!

Gravitational cosmological equations

Vector:

$$-6\gamma\dot{R} = G^0$$

Tensor:

$$-\gamma \left(12 \frac{\dot{a}}{a} \dot{R} + R(4R_0^0 - R) \right) = T_0^0$$

$$-12\gamma \left(\ddot{R} + 3 \frac{\dot{a}}{a} \dot{R} \right) = T$$

Self-consistency condition

$$2 \frac{(G^0 a^3)'}{a^3} = T_0^0 + 3T_1^1 = T$$

It is quite clear that the self-consistency condition is just the consequence of the vector and trace equations. At last, let us write down the expressions for the $\binom{0}{0}$ -component of Ricci tensor and the scalar curvature in terms of the scale factor $a(t)$

$$R_0^0 = -3\frac{\ddot{a}}{a}$$

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right), \quad k = 0, \pm 1$$

Note that in cosmology we need to know only $T^{00} = T_0^0$ and $T = \text{Trace } T^{\mu\nu}$, since $T^{0i} = 0$ and $T^{ij} = T_1^1 g^{ij}$, $T_1^1 = (1/3)(T - T_0^0)$. Thus,

$$T = T[\text{part}] + T[\text{cr}]$$

$$T[\text{part}] = \varepsilon - 3p$$

$$T[\text{cr}] = \ddot{\lambda}_1(8\beta'R_0^0 - 4\beta'R - 12\gamma'R)$$

$$-4\dot{\lambda}_1\left(\beta'\frac{\dot{a}}{a}(R + 2R_0^0) + 6\gamma'\dot{R} + 9\gamma'\frac{\dot{a}}{a}R\right) - 12\lambda_1\gamma'(\ddot{R} + 3\frac{\dot{a}}{a}R)$$

$$T_0^0 = T_0^0[\text{part}] + T_0^0[\text{cr}], \quad T_0^0[\text{part}] = \varepsilon$$

$$T_0^0[\text{cr}] = 8\gamma'\dot{\lambda}_1\frac{\dot{a}}{a}R_0^0 - 4(\beta' + 3\gamma')\dot{\lambda}_1\frac{\dot{a}}{a}R \\ - \gamma'\lambda_1\left(12\frac{\dot{a}}{a}\dot{R} + R(4R_0^0 - R)\right)$$

How about the equations of motion for the cosmological perfect fluid? We are left with only one equation plus the law of the particle creation, namely

$$\begin{cases} \dot{\lambda}_1 = -\frac{\varepsilon+p}{n} \\ \frac{(na^3)^\cdot}{a^3} = \Phi(\text{inv}) \end{cases}$$

$$\Phi(\text{inv}) = -\frac{4}{3}\beta'R_0^0(2R_0^0 - R) + \gamma'R^2.$$

Creation of the universe from nothing

Empty from the very beginning

Vacuum is not absolute, but physical

May or may not it persists?

“Pregnant” vacuum

$$\Phi(\text{inv}) = 0, \quad |\beta'| + |\gamma'| \neq 0$$

$$\frac{4}{3}\beta' R_0^0 (2R_0^0 - R) = \gamma' R^2$$

$$G^0[\text{part}] = T_0^0[\text{part}] = T[\text{part}] = 0$$

$$n = 0$$

$$\dot{\lambda}_1 = -\frac{\varepsilon + p}{n} \Rightarrow$$

1 Non-dust matter

$$\dot{\lambda}_1 = 0 \Rightarrow \lambda_1 = \text{const}$$

2 Dust matter

$$\dot{\lambda}_1 = -\phi(0) = \text{const} \Rightarrow \lambda_1 = -\phi(0)(t - t_0)$$

Non-dust pregnancy I

General case: $\beta', \gamma' \neq 0$

$$\lambda_1 = \text{const}$$

$$R = \xi R_0^0 \Rightarrow$$

$$(3\gamma'\xi^2 + 4\beta'(\xi - 2))R_0^0 = 0$$

$$-6(\gamma - \gamma'\lambda_1)\dot{R} = 0$$

$$-12(\gamma - \gamma'\lambda_1)\frac{(\dot{R}a^3)'}{a^3} = 0$$

$$-(\gamma - \gamma'\lambda_1) \left\{ 12\frac{\dot{a}}{a}\dot{R} + R(R - 4R_0^0) \right\} = 0$$

Let, first, $\gamma \neq \gamma'\lambda_1 \Rightarrow \dot{R} = 0$

Either $R = 0 \Rightarrow R_0^0 = 0 \Rightarrow$ Milne universe

Or $\xi = 4 \Rightarrow$ de Sitter for $\beta + 6\gamma' = 0$

Non-dust pregnancy II

General case: $\beta', \gamma' \neq 0$

$$\gamma = \gamma' \lambda_1$$

It is not a special condition, but the solution λ_1

$$\dot{a}^2 + k = C_0 a^{\frac{4}{\xi-2}}$$

Instability and so on...

Dust pregnancy

$$\lambda_1 = -\phi(0)(t - t_0) \Rightarrow \dot{\lambda}_1 = -\phi < 0, \quad \ddot{\lambda}_1 = 0$$

Surely, there exists the solution with $R = R_0^0 = 0$, i. e., Milne universe.

It can be shown that the only other solution is just the de Sitter universe, with $\xi = 4$ ($R, R_0^0 = \text{const}$) for $\beta' + 6\gamma' = 0$.

If it is not so, the universe emerged from the vacuum foam, like Aphrodite, immediately starts to produce dust particles !!!

The End