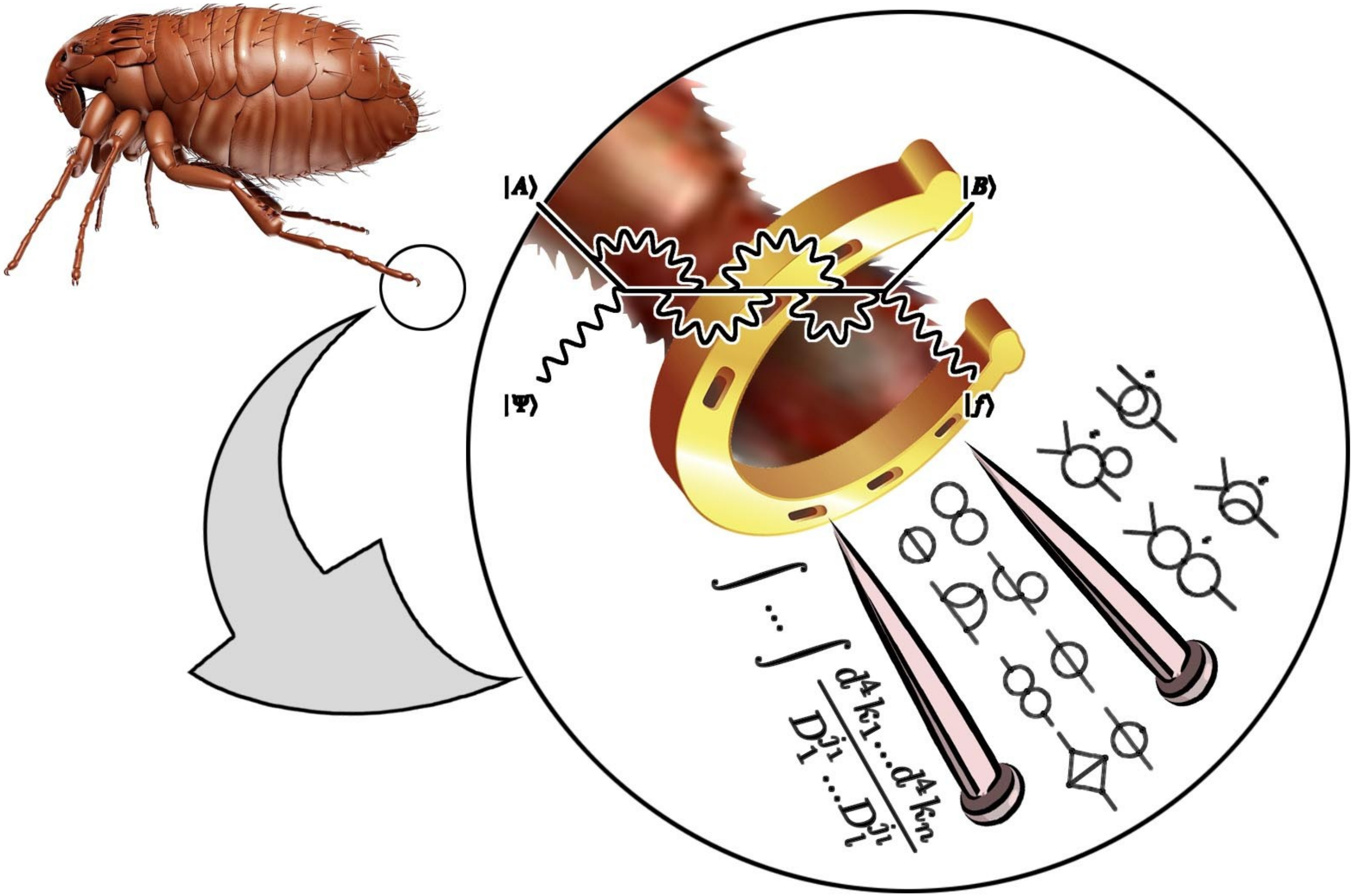
Exact Frobenius solutions for elliptic Feynman integrals

Maxim Bezuglov

Bezuglov, M.A., Onishchenko, A.I. Non-planar elliptic vertex. J. High Energ. Phys. 2022, 45 (2022).

Bezuglov, M.A., Kotikov, A.V. & Onishchenko, A.I. On Series and Integral Representations of Some NRQCD Master Integrals. Jetp Lett. (2022).

Quantum field theory



Introduction

Feynman integral: $$\int \dots \int \frac{d^4k_1 \dots d^4k_n}{D_1^{j_1} \dots D_l^{j_l}},$$ $$D_r = \sum_{i \geq j \geq 1} A_r^{ij} p_i p_j - m_r^2$$

Integration by Parts (IBP) $$\int d^dk_1 d^dk_2 \dots \frac{\partial f}{\partial k_i^\mu} = 0$$ In dimensional regularization

F. V. Tkachov, Phys.Lett.B 100 (1981) 65-68

K.G. Chetyrkin, F.V. Tkachov, Nucl.Phys.B 192 (1981) 159-204

Any integral from a given family can be represented as a linear combination of some limited **basis** of integrals, elements of this basis are called **master integrals**.

Methods for calculating loop integrals

Solving a system of equations for the system of master integrals

- System of difference equations
- ***System of differential equations***

Kotikov, A. V., Phys.Lett.B 254 (1991) 158-164

Kotikov, A. V., Phys.Lett.B 267 (1991) 123-127

Kotikov, A. V., Phys.Lett.B 259 (1991) 314-322

Evaluating by direct integration using some parametric representation

- Feynman parametrisation
- Alpha parametrisation
- MB represention
- et al.

$$\frac{d}{dx}\begin{pmatrix} I_1 \\ I_2 \\ \dots \\ I_n \end{pmatrix} = A(x,\varepsilon)\begin{pmatrix} I_1 \\ I_2 \\ \dots \\ I_n \end{pmatrix},$$

«epsilon form»

$$A(x,\varepsilon) = \varepsilon \sum_i \frac{A_i}{x - c_i}, \qquad I_j = \sum_k I_j^{(k)} \varepsilon^k$$

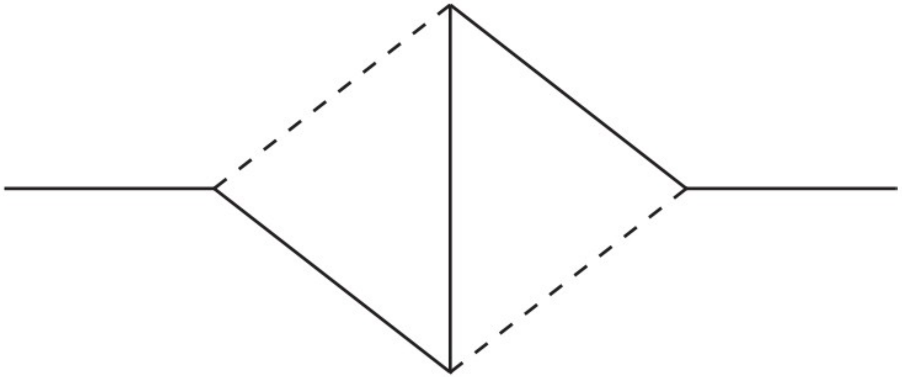
J. M. Henn, Physical review letters, vol. 110, no. 25, p. 251601, 2013.

R. N. Lee, JHEP, vol. 04, p. 108, 2015.

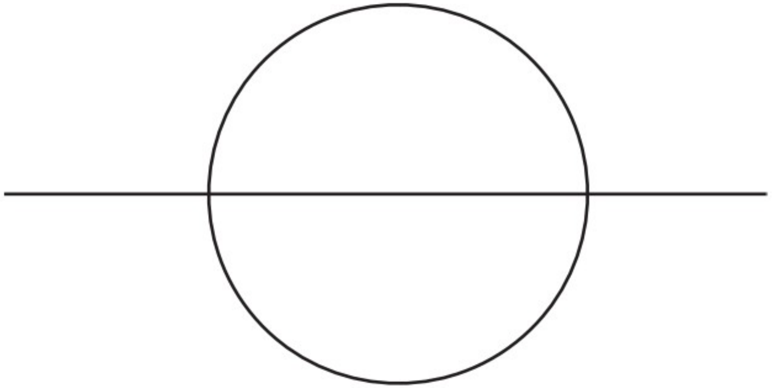
Elliptic loop integrals

«Kite» integral

A. Sabry, Nuclear Physics 33, 401 (1962).



«Sunset» integral



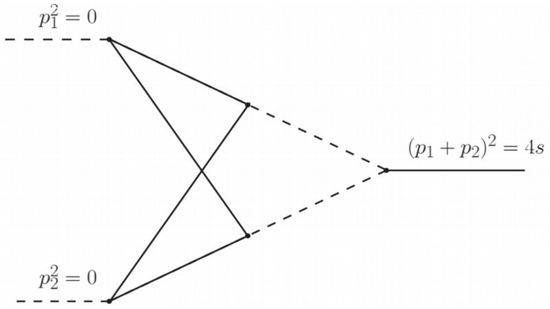
$$\frac{d}{dx}\begin{pmatrix} I_1 \\ I_2 \\ \dots \\ I_n \end{pmatrix} = A(x,\varepsilon)\begin{pmatrix} I_1 \\ I_2 \\ \dots \\ I_n \end{pmatrix}, \qquad A(x,\varepsilon) = \sum_i \frac{A_i(\varepsilon)}{x - c_i},$$

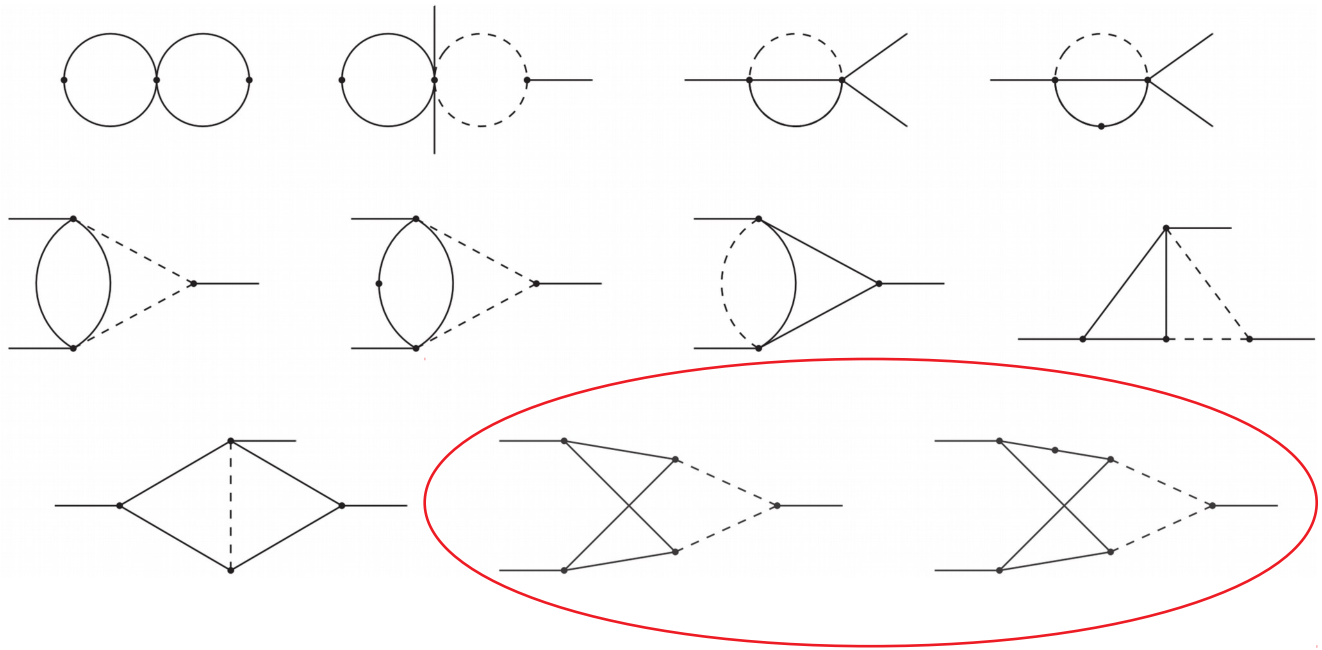
hypothesis:

$$A_i(\varepsilon) = \left(\varepsilon + \frac{1}{2}\right) A_i$$

Lee, R.N. J. High Energ. Phys. 2018, 176 (2018).

Frobenius method





**Manteuffel, A., Tancredi, L.
J. High Energ. Phys.
2017, 127 (2017)**

$$\frac{d\tilde{I}_{canonical}}{ds} = \left[\frac{\mathcal{M}_0}{s} + \frac{\mathcal{M}_1}{s-1} + \frac{\mathcal{M}_{-1}}{s+1} + \frac{\mathcal{M}_4}{s+4} + \mathcal{M}_0^1 + \mathcal{M}_0^2 s\right] \tilde{I}_{canonical}$$

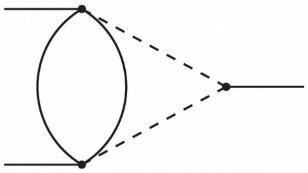
Earlier, the exact solution in the elliptic case was known only for an integral of the sunset type

Tarasov, O. V. Physics Letters B, 638(2-3), 195-201. (2006).

M.Y. Kalmykov and B.A. Kniehl, Nucl. Phys. B 809(2009) 365

Solutions for non-elliptic master integrals

$$J_5 = \varepsilon^3(-1+2\varepsilon)$$



$$\frac{1}{4}\varepsilon(1-2\varepsilon)\left(J_1+2J_2\right)-\frac{1}{2}\left(1+(1+4s)\varepsilon-2\varepsilon^2\right)J_5-$$
$$-\frac{1}{2}s(5+4s(1+\varepsilon))\frac{dJ_5}{ds}-s^2(1+s)\frac{d^2J_5}{ds^2}=0.$$

We will look for solutions in the form:

$$J_5=\sum_{n=0}^{\infty}\sum_{\lambda}a_n^{(\lambda)}s^{\lambda+n}=\sum_{n=0}^{\infty}\left(c_n+d_n s^{-\varepsilon}\right)s^n.$$

$$d_{n+1}=\frac{2(n-\varepsilon+1)(n+\varepsilon)}{(3+2n)(n-2\varepsilon+2)}d_n,\qquad c_{n+1}=\frac{2(n+1)(n+2\varepsilon)}{(n+2-\varepsilon)(2n+3+2\varepsilon)}c_n.$$

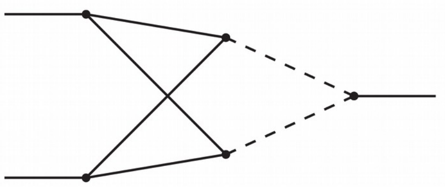
$$f(n+1)=H(n)f(n)+Q(n)$$

$$f(n)=\prod_{m=l}^{n-1}H(m)\left(\sum_{k=l}^{n-1}\frac{Q(k)}{\prod_{m=l}^{k}H(m)}+C\right)$$

$$\frac{2^{2\varepsilon+1}\sin(\pi\varepsilon)}{\pi^{3/2}}J_5=\frac{2\sqrt{\pi}(-s)^{-\varepsilon}}{\Gamma(2-2\varepsilon)}\,{}_3F_2\left[\begin{matrix}1-\varepsilon\ \ \varepsilon\ \ 1\\ 2-2\varepsilon\ \ \frac{3}{2}\end{matrix};-s\right]-\frac{\Gamma(2\varepsilon)}{\Gamma(2-\varepsilon)\Gamma(3/2+\varepsilon)}\,{}_3F_2\left[\begin{matrix}1\ \ 2\varepsilon\ \ 1\\ 2-\varepsilon\ \ \frac{3}{2}+\varepsilon\end{matrix};-s\right]$$

Solutions for elliptic master integrals

$$J_{10} = \varepsilon^3(1+2\varepsilon)s^2$$



$$s^2(4+s)\frac{d^2J_{10}(s)}{ds^2} + s(s+2(2+s)\varepsilon)\frac{dJ_{10}(s)}{ds} + (1-2\varepsilon)J_{10}(s) + J^{inhom}(s) = 0$$

The homogeneous part of the DE has the same structure as in the non-elliptic case

we will again look for solutions in the form:

$$J_5 = \sum_{n=0}^{\infty}\sum_{\lambda} a_n^{(\lambda)} s^{\lambda+n} = \sum_{n=0}^{\infty}\left(c_n + d_n s^{-\varepsilon}\right)s^n.$$

The difference equations for the coefficients c_n and d_n will again be of the first order!

Solutions for elliptic master integrals

$$J^{inhom}(s) = -\frac{3\left(6\varepsilon^2+\varepsilon-1\right)s^2 J_5}{4(s+1)} + \frac{(2\varepsilon+1)s^2(4\varepsilon(2s+5)-3)J_6}{2(s+1)} + \ldots$$

The sums in the solutions for elliptic integrals will be at least double

"Rough" approach

$$\frac{\sum_{n=0}^{\infty} c_n s^n}{s+1} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{k+n} c_k s^n$$

$$\frac{\sum_{n=0}^{\infty} c_n s^n}{s-1} = -\sum_{n=0}^{\infty}\sum_{k=0}^{n} c_k s^n$$

The solution will contain triple sums

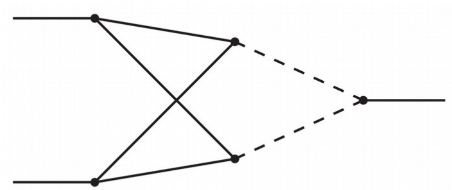
"Smart" approach

$$\frac{dJ_6}{ds} = \frac{(1-3\varepsilon)J_5}{4s(s+1)} - \frac{(3+2s-4\varepsilon)J_6}{2s(s+1)} + \ldots$$

$$J^{inhom}(s) = -3s^3(1+2\varepsilon)\frac{dJ_6}{ds} - s^2(1+2\varepsilon)(3+4\varepsilon)J_6 + \ldots$$

The solution will only contain double sums

Exact solution for non-planar vertex



$$
= -\frac{\pi^{3/2}(\varepsilon+6)\csc(\pi\varepsilon)\Gamma(2\varepsilon+2)}{2^{2\varepsilon+3}(\varepsilon+2)\Gamma(3-\varepsilon)\Gamma\left(\varepsilon+\frac{5}{2}\right)}\left\{F^{2:1:5}_{2:0:4}\left[\begin{matrix}2 & 2(1+\varepsilon)\\ \frac{5}{2} & \frac{5}{2}+\varepsilon\end{matrix}\,\middle|\,\begin{matrix}1\\ -\end{matrix}\,\middle|\,\begin{matrix}1 & 2 & \frac{5}{2} & 2+\varepsilon & 3+\frac{\varepsilon}{3}\\ 3 & 3-\varepsilon & 3+\varepsilon & 2-\frac{\varepsilon}{3} & \end{matrix};\frac{-s}{4};-s\right]\right.
$$

$$
\left.-\frac{\varepsilon(\varepsilon+1)(3\varepsilon+10)}{2(\varepsilon+6)}F^{2:1:6}_{2:0:5}\left[\begin{matrix}2 & 2(1+\varepsilon)\\ \frac{5}{2} & \frac{5}{2}+\varepsilon\end{matrix}\,\middle|\,\begin{matrix}1\\ -\end{matrix}\,\middle|\,\begin{matrix}1 & 2 & \frac{5}{2} & 2+\varepsilon & 2+\varepsilon & 3+\frac{3\varepsilon}{5}\\ 3 & 3-\varepsilon & 3+\varepsilon & 3 & 2+\frac{3\varepsilon}{5} & \end{matrix};-\frac{s}{4};-s\right]\right\}
$$

$$
+\frac{\pi\cot(\pi\varepsilon)\Gamma(2\varepsilon+3)}{3(-s)^{\varepsilon}4^{\varepsilon+1}(1-4\varepsilon^2)}F^{2:1:5}_{2:0:4}\left[\begin{matrix}2+\varepsilon & 2-\varepsilon\\ \frac{5}{2} & \frac{5}{2}-\varepsilon\end{matrix}\,\middle|\,\begin{matrix}1\\ -\end{matrix}\,\middle|\,\begin{matrix}1 & 1 & 1-\varepsilon & \frac{3}{2}-\varepsilon & 2-\frac{2\varepsilon}{3}\\ 2 & 2-2\varepsilon & 2-\varepsilon & 1-\frac{2\varepsilon}{3} & \end{matrix};-\frac{s}{4};-s\right]
$$

$$
+\frac{\pi^{3/2}2^{-2\varepsilon-3}(\varepsilon+6)\csc(\pi\varepsilon)\Gamma(2\varepsilon+2)}{(\varepsilon+2)\Gamma(3-\varepsilon)\Gamma\left(\varepsilon+\frac{5}{2}\right)}\,{}_6F_5\left[\begin{matrix}1 & 2 & 2 & 2(1+\varepsilon) & 2+\varepsilon & 3+\frac{\varepsilon}{3}\\ 3 & 3-\varepsilon & 3+\varepsilon & \frac{5}{2}+\varepsilon & 2+\frac{\varepsilon}{3} & \end{matrix};-s\right]
$$

$$
-\frac{\pi\varepsilon(3\varepsilon+10)\csc(\pi\varepsilon)\Gamma(\varepsilon+2)}{4\left(2\varepsilon^2+7\varepsilon+6\right)\Gamma(3-\varepsilon)}\,{}_7F_6\left[\begin{matrix}1 & 2 & 2 & 2(1+\varepsilon) & 2+\varepsilon & 2+\varepsilon & 3+\frac{3\varepsilon}{5}\\ 3 & 3 & 3-\varepsilon & 3+\varepsilon & \frac{5}{2}+\varepsilon & 2+\frac{3\varepsilon}{5} & \end{matrix};s\right]
$$

$$
-\frac{\pi^{3/2}2^{-2(\varepsilon+2)}\csc(\pi\varepsilon)\Gamma(2\varepsilon+3)}{\Gamma(2-\varepsilon)\Gamma\left(\varepsilon+\frac{5}{2}\right)}\,{}_3F_2\left[\begin{matrix}1 & 2 & 2(1+\varepsilon)\\ \frac{5}{2} & \frac{5}{2}+\varepsilon & \end{matrix};-\frac{s}{4}\right]
$$

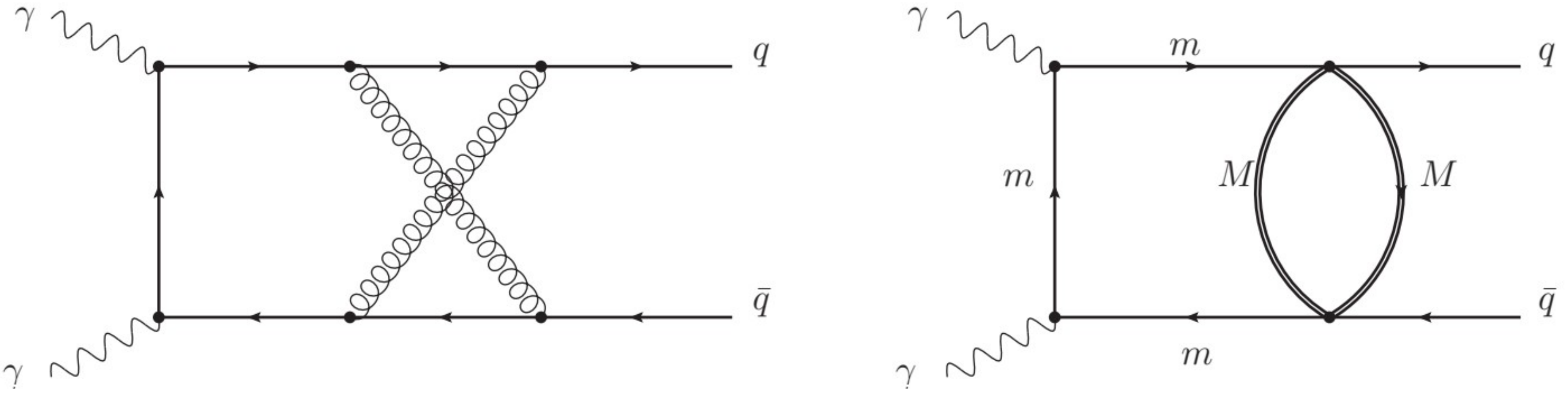
M. Bezuglov and A. Onishchenko arXiv:2112.05096

Generalized Kampé de Fériet function

$$
F^{p:q:k}_{l:m:n}\left[\begin{matrix}(a_p)\\ (\alpha_l)\end{matrix}\,\middle|\,\begin{matrix}(b_q)\\ (\beta_m)\end{matrix}\,\middle|\,\begin{matrix}(c_k)\\ (\gamma_n)\end{matrix};x;y\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}
$$

System of integrals describing two-loop corrections to processes in nonrelativistic QCD

$$\gamma\gamma/gg \longrightarrow q\bar{q}$$

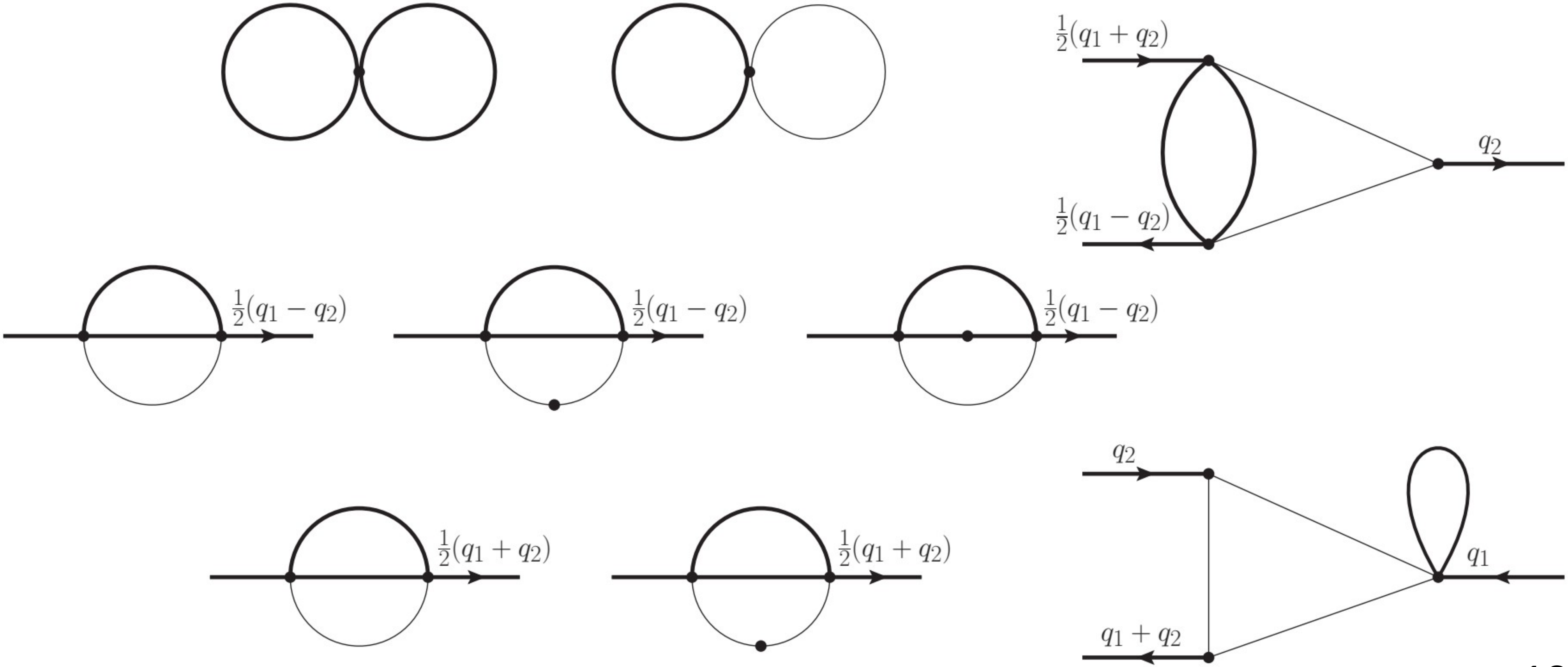


$$x = \frac{m^2}{M^2}$$

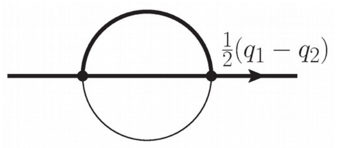
$$\int\int \frac{d^dkd^dl}{((l+q_1)^2-x)^{a_1}((l-q_2)^2-x)^{a_2}(l^2-x)^{a_3}((k-l)^2-1)^{a_4}((k+q_1/2-q_2/2)^2-1)^{a_5}}$$

Kniehl, B. A., Kotikov, A. V., Onishchenko, A. I., & Veretin, O. L. Nuclear Physics B, 948, 114780. (2019).

System of integrals describing two-loop corrections to processes in nonrelativistic QCD

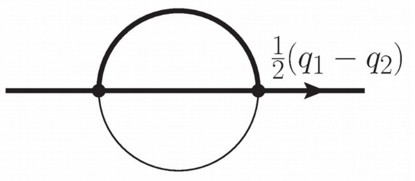


Elliptic sunset

$$J_3 = (1+3\varepsilon)$$ 

$$(1+2\varepsilon)(\varepsilon-2\varepsilon^2+2x^2(1+\varepsilon^2))J_3-(3-4(\varepsilon-1)\varepsilon+2x^2(1+\varepsilon)(5+4\varepsilon))x\frac{dJ_3}{dx}$$
$$-(4(3+\varepsilon)+x^2(7+5\varepsilon))x^2\frac{d^2J_3}{dx^2}-(4+x^2)x^3\frac{d^3J_3}{dx^3}+\cdots=0$$

We will look for solutions in the form: $J_3=\sum_{n=0}^{\infty}\left(c_n+d_nx^{-\varepsilon}\right)x^{2n+1}+\sum_{n=0}^{\infty}\left(f_n+g_nx^{-\varepsilon}\right)x^{2n}$



$$=\frac{\pi\csc(\pi\varepsilon)\varepsilon\Gamma(\varepsilon)}{x^{\varepsilon}\Gamma(1-\varepsilon)}\,{}_4F_3\left[\begin{matrix}\frac{1}{2} & 1 & \frac{1}{2}+\frac{\varepsilon}{2} & 1+\frac{\varepsilon}{2}\\ \frac{3}{4} & \frac{5}{4} & 1-\varepsilon & \end{matrix};-\frac{x^2}{4}\right]-\frac{\sqrt{\pi}\varepsilon\Gamma(\varepsilon)\Gamma(2\varepsilon)}{4^{\varepsilon}\Gamma\left(\varepsilon+\frac{3}{2}\right)}\,{}_4F_3\left[\begin{matrix}\frac{1}{2}+\varepsilon & 1 & \frac{1}{2}+\frac{\varepsilon}{2} & 1+\varepsilon\\ 1-\frac{\varepsilon}{2} & \frac{3}{4}+\frac{\varepsilon}{2} & \frac{5}{4}+\frac{\varepsilon}{2} & \end{matrix};-\frac{x^2}{4}\right]$$
$$-\frac{\sqrt{\pi}2^{-2(\varepsilon+1)}x\Gamma(\varepsilon+1)\Gamma(2\varepsilon+2)}{(\varepsilon-1)\Gamma\left(\varepsilon+\frac{5}{2}\right)}\,{}_4F_3\left[\begin{matrix}1 & 1+\varepsilon & \frac{3}{2}+\varepsilon & 1+\frac{\varepsilon}{2}\\ \frac{3}{2}-\frac{\varepsilon}{2} & \frac{5}{4}+\frac{\varepsilon}{2} & \frac{7}{4}+\frac{\varepsilon}{2} & \end{matrix};-\frac{x^2}{4}\right]$$

Exact solutions for all master integrals can be expressed in terms of generalized hypergeometric functions

These results are consistent with those previously obtained by other methods.

M.Y. Kalmykov and B.A. Kniehl, Nucl. Phys. B 809(2009) 365

Conclusions

- A new method was developed for obtaining an exact solution of elliptic Feynman integrals, in terms of the dimensional regularization parameter, based on the solution of differential equations for the complete system of master integrals by the Frobenius method.

- The use of this method made it possible to obtain exact solutions for a nonplanar elliptic vertex, as well as for a system of master integrals describing two-loop corrections to processes in nonrelativistic QCD.

- Solutions are expressed in terms of well convergent sums which can be easily computed numerically with arbitrary precision.

Future plans

- Generalize the developed technique to the case of "more complicated" elliptic integrals

Thank you for your attention!