International Conference on Quantum Field Theory, High Energy Physics and Cosmology

# UV Divergences of Scattering Amplitudes in D-dimensional Yang-Mills Theories

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# Introduction

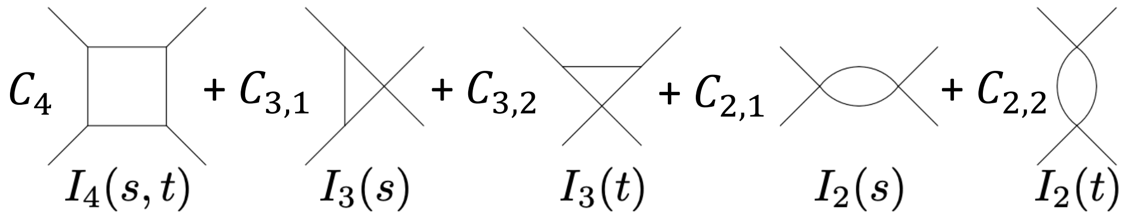
- The object: one-loop four gluon on-shell amplitude for a different helicity

- The process:

Feynman Rules $\Longrightarrow$ $\sum \mathrm{Numerators} \times \mathrm{Diagrams}$ $\xrightarrow{\text{PVReduction}}$ $\sum_i C_{i,j} MI_j$

- The result:

$$A_4^{(1)}(helicity, dimension) =$$



# Preliminaries

Massless momenta in the center-of-mass frame:

$$
\begin{aligned}
p_1 &= (p, p, 0, 0, 0, \ldots 0), \\
p_2 &= (p, -p, 0, 0, 0, \ldots 0), \\
p_3 &= (-p, -p\ \cos\theta, -p\ \sin\theta, 0, 0, \ldots 0), \\
p_4 &= (-p, p\ \cos\theta, p\ \sin\theta, 0, 0, \ldots 0),
\end{aligned}
$$

The Mandelstam variables

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = 4\,p^2, \\
t &= (p_2 + p_3)^2 = -2\,p^2(1 + \cos\theta), \\
u &= (p_1 + p_3)^2 = -2\,p^2(1 - \cos\theta).
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
p &= \sqrt{s/4}, \\
\cos\theta &= -\frac{s + 2t}{s}, \\
\sin\theta &= \frac{2}{s}\sqrt{tu}.
\end{aligned}
$$

# Polarization vectors in D=4

- Transversality

$$p_\mu \epsilon^\mu_\pm(p) = 0$$

- Normalization

$$\epsilon_\lambda(p) \cdot \epsilon^*_{\lambda'}(p) = -\delta_{\lambda,\lambda'}$$

- Completeness

$$\sum_{\lambda=\pm} \epsilon^\mu_\lambda(p) \epsilon^{\nu *}_\lambda(p) = -\eta^{\mu\nu} + n^\mu \overline{n}^\nu + \overline{n}^\mu n^\nu$$

# Polarization vectors in D=4

$$\epsilon_+(p_1) = \frac{1}{\sqrt{2}}(0,0,i,-1), \qquad \epsilon_-(p_1) = \frac{1}{\sqrt{2}}(0,0,-i,-1),$$

$$\epsilon_+(p_2) = \frac{1}{\sqrt{2}}(0,0,i,1), \qquad \epsilon_-(p_2) = \frac{1}{\sqrt{2}}(0,0,-i,1),$$

$$\epsilon_+(p_3) = \frac{1}{\sqrt{2}}(0,-i\sin\theta, i\cos\theta,-1), \qquad \epsilon_-(p_3) = \frac{1}{\sqrt{2}}(0,i\sin\theta,-i\cos\theta,-1),$$

$$\epsilon_+(p_4) = \frac{1}{\sqrt{2}}(0,-i\sin\theta, i\cos\theta,1), \qquad \epsilon_-(p_4) = \frac{1}{\sqrt{2}}(0,i\sin\theta,-i\cos\theta,1).$$

# Polarization vectors in D=6

- Normalization

$$\epsilon^{\mu}_{a\dot{a}}\epsilon_{\mu b\dot{b}} = \varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}}$$

C. Cheung, D. O'Connel, JHEP 0907 (2009) 075, arXiv:0902.0981 [hep-th].

- Polarization vectors

$$\epsilon_{1\dot{1}}(p_1) = \frac{1}{\sqrt{2}}(0,0,i,-1,0,0),$$
$$\epsilon_{2\dot{2}}(p_1) = \frac{1}{\sqrt{2}}(0,0,-i,-1,0,0),$$
$$\epsilon_{1\dot{2}}(p_1) = \frac{1}{\sqrt{2}}(0,0,0,0,i,-1),$$
$$\epsilon_{2\dot{1}}(p_1) = \frac{1}{\sqrt{2}}(0,0,0,0,i,1).$$

# Polarization vectors in D

$$\epsilon_+(p_1) = \frac{1}{\sqrt{2}}(0,0,i,-1,\underbrace{0,...,0}_{D-4}), \qquad \epsilon_-(p_1) = \frac{1}{\sqrt{2}}(0,0,-i,-1,\underbrace{0,...,0}_{D-4}),$$

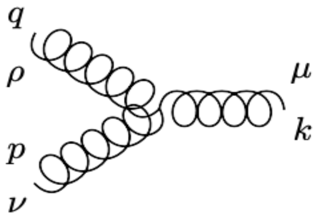
$$\epsilon_+(p_2) = \frac{1}{\sqrt{2}}(0,0,i,1,\underbrace{0,...,0}_{D-4}), \qquad \epsilon_-(p_2) = \frac{1}{\sqrt{2}}(0,0,-i,1,\underbrace{0,...,0}_{D-4}),$$

$$\epsilon_+(p_3) = \frac{1}{\sqrt{2}}(0,-i\sin\theta,i\cos\theta,-1,\underbrace{0,...,0}_{D-4}), \qquad \epsilon_-(p_3) = \frac{1}{\sqrt{2}}(0,i\sin\theta,-i\cos\theta,-1,\underbrace{0,...,0}_{D-4}),$$

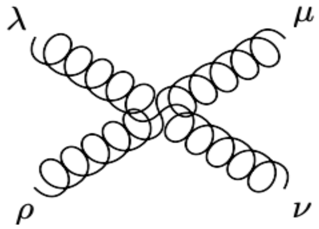
$$\epsilon_+(p_4) = \frac{1}{\sqrt{2}}(0,-i\sin\theta,i\cos\theta,1,\underbrace{0,...,0}_{D-4}), \qquad \epsilon_-(p_4) = \frac{1}{\sqrt{2}}(0,i\sin\theta,-i\cos\theta,1,\underbrace{0,...,0}_{D-4}).$$

# Colour-ordered Feynman rules

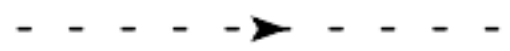

$= -i\frac{\eta_{\mu\nu}}{k^2}$



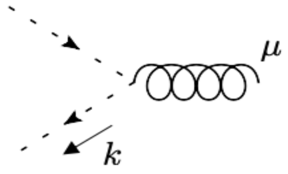

$= \frac{i}{\sqrt{2}}\left(\eta_{\mu\nu}(k-p)_\rho + \eta_{\nu\rho}(p-q)_\nu + \eta_{\rho\mu}(q-k)_\nu\right)$




$= \frac{i}{2}\left(2\eta_{\mu\rho}\eta_{\nu\lambda} - (\eta_{\mu\nu}\eta_{\rho\lambda} + \eta_{\mu\lambda}\eta_{\nu\rho})\right)$



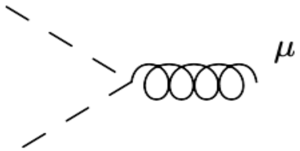
$= \frac{i}{k^2}$



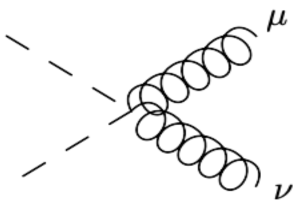

$= -\frac{i}{\sqrt{2}}k_\mu$



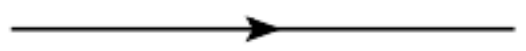
$= \frac{i}{k^2}$



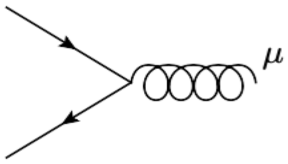

$= \frac{i}{\sqrt{2}}(p-q)_\mu$




$= -\frac{i}{2}\eta_{\mu\nu}$

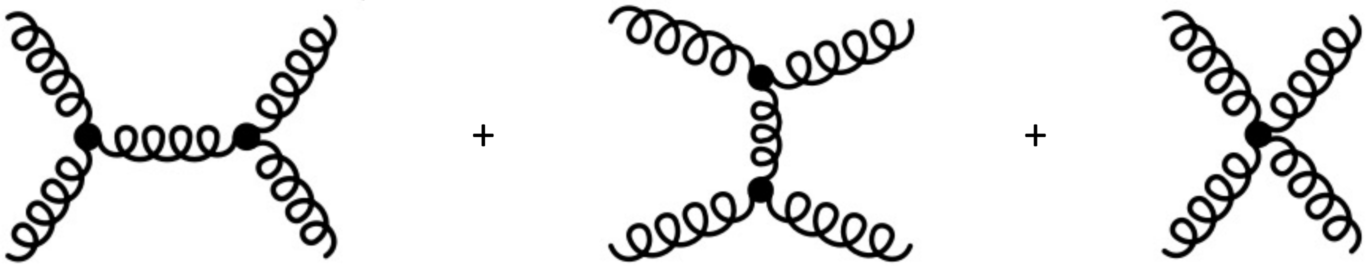


$= -i\frac{\not{k}}{k^2}$




$= -\frac{i}{\sqrt{2}}\gamma_\mu$

# Tree Level



$$A_4^{(0)}(1^+, 2^+, 3^+, 4^+) = 0,$$

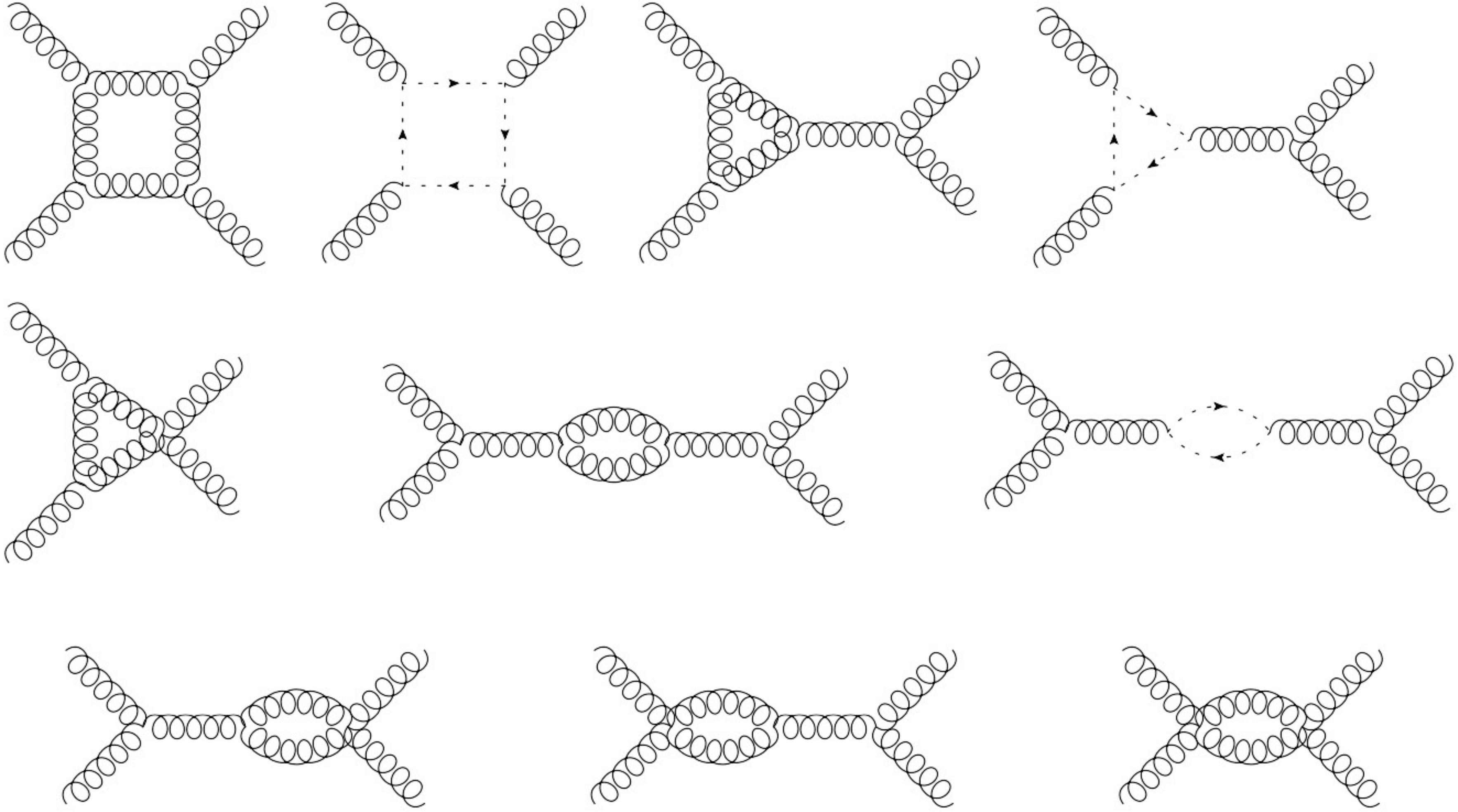
$$A_4^{(0)}(1^+, 2^+, 3^+, 4^-) = 0,$$

$$A_4^{(0)}(1^+, 2^+, 3^-, 4^-) = -ig^2 \frac{s}{t},$$

$$A_4^{(0)}(1^+, 2^-, 3^-, 4^+) = -ig^2 \frac{t}{s},$$

$$A_4^{(0)}(1^+, 2^-, 3^+, 4^-) = -ig^2 \frac{(s+t)^2}{st}.$$

# One-Loop Level



# One-Loop Level

$$\mathcal{A}_4(1,2,3,4) = \mathcal{A}_4^{(0)}(1,2,3,4) M_4(s,t)$$

- Adjacent helicity case

$$M_4^{(1)}(1^+,2^+,3^-,4^-) =$$
$$-\frac{st(16(3-4D+D^2)s^2+16(2-3D+D^2)st-(2(16-14D+D^2)-d(8-6D+D^2))t^2)}{16(-3+D)(-1+D)(s+t)^2} I_4(s,t)$$
$$-\frac{(-4+D)st(16(-1+D)s+(-20-d(-2+D)+18D)t)}{8(-3+D)(-1+D)(s+t)^2} I_3(s)$$
$$+\frac{(-4+D)t^2(-16(-1+D)s+(20+d(-2+D)-18D)t)}{8(-3+D)(-1+D)(s+t)^2} I_3(t)$$
$$-\frac{(-2+d)(12-7D+D^2)t}{4(-3+D)(-1+D)(s+t)} I_2(s) + \frac{(2(6+d-8D)s+(20+d(-2+D)-18D)t)}{4(-1+D)(s+t)} I_2(t).$$

# One-Loop Level

- Non-adjacent helicity case

$$M_4^{(1)}(1^+,2^-,3^+,4^-) =$$
$$-\frac{1}{16(-1+D)(s+t)^4}\Big(\frac{1}{-3+D}st(16(3-4D+D^2)s^4+16(10-13D+3D^2)s^3t$$
$$+(224-2(146-d)D+(62+d)D^2)s^2t^2+16(10-13D+3D^2)st^3+16(3-4D+D^2)t^4)I_4(s,t)$$
$$+\frac{1}{D-3}2s^2t(16(2-3D+D^2)s^2+(64-2(46+d)D+(34-d)D^2)st+16(2-3D+D^2)t^2)I_3(s)$$
$$+\frac{1}{D-3}2st^2(16(2-3D+D^2)s^2+(64-2(46+d)D+(34-d)D^2)st+16(2-3D+D^2)t^2)I_3(t)$$
$$+\frac{1}{-2+D}4t(s+t)(8(2+d-6D+2D^2)s^2-(-2+D)(20-34D+d(6+D))st$$
$$-2(6+d-8D)(-2+D)t^2)I_2(s)+\frac{1}{-2+D}4s(s+t)(-2(6+d-8D)(-2+D)s^2$$
$$-(-2+D)(20-34D+d(6+D))st+8(2+d-6D+2D^2)t^2)I_2(t)\Big).$$

# d=4, D=4

- Adjacent

$$C_4 = -st, \quad C_{3,1} = 0, \quad C_{3,2} = 0, \quad C_{2,1} = 0, C_{2,2} = -\frac{11}{3}$$

- Non-adjacent

$$C_4 = -\frac{st(s^2+st+t^2)^2}{(s+t)^4},$$
$$C_{3,1} = -\frac{2s^2t(2s^2+3st+2t^2)}{(s+t)^4}, \quad C_{3,2} = -\frac{2st^2(2s^2+3st+2t^2)}{(s+t)^4},$$
$$C_{2,1} = -\frac{t(14s^3+33s^2t+30st^2+11t^3)}{3(s+t)^4}, C_{2,2} = -\frac{s(11s^3+30s^2t+33st^2+14t^3)}{3(s+t)^4}$$

# d=4, D=4

UV-divergent parts of the master integrals in D = 4 - 2ϵ

$$Sing\ I_4(s,t) = 0,\ Sing\ I_3(s) = 0,\ Sing\ I_3(t) = 0,\ Sing I_2(s) = \frac{1}{\epsilon},\ Sing\ I_2(t) = \frac{1}{\epsilon}$$

The amplitudes in both helicity cases

$$M_4^{(1)}(1^+,2^+,3^-,4^-)|_{d=4,D=4} = -\frac{11}{3}\frac{1}{\epsilon}$$
$$M_4^{(1)}(1^+,2^-,3^+,4^-)|_{d=4,D=4} = -\frac{11}{3}\frac{1}{\epsilon}$$

# d=6, D=4 and d-arbitrary

- d=6, D=4

$$M_4^{(1)}(1^+,2^+,3^-,4^-)|_{d=6,D=4} = -\frac{10}{3}\frac{1}{\epsilon}$$
$$M_4^{(1)}(1^+,2^-,3^+,4^-)|_{d=6,D=4} = -\frac{10}{3}\frac{1}{\epsilon}$$

S. Davies, Phys. Rev. D 84 (2011), 094016 doi:10.1103/PhysRevD.84.094016 [arXiv:1108.0398 [hep-ph]]

- d - arbitrary, D=4

$$M_4^{(1)}(1^+,2^+,3^-,4^-)|_{d,D=4} = \frac{d-26}{6}\frac{1}{\epsilon}$$
$$M_4^{(1)}(1^+,2^-,3^+,4^-)|_{d,D=4} = \frac{d-26}{6}\frac{1}{\epsilon}$$

D. I. Kazakov, JHEP 03 (2003), 020 doi:10.1088/1126-6708/2003/03/020 [arXiv:hep-th/0209100 [hep-th]]

# d=4, D=4, scalar in-the-loop

- Adjacent

$$C_4 = 0, \quad C_{3,1} = 0, \quad C_{3,2} = 0, \quad C_{2,1} = 0, \quad C_{2,2} = \frac{1}{6}$$

- Non-adjacent

$$C_4 = -\frac{s^3 t^3}{2(s+t)^4}, \quad C_{3,1} = \frac{s^3 t^2}{(s+t)^4}, \quad C_{3,2} = -\frac{s^2 t^3}{(s+t)^4},$$
$$C_{2,1} = \frac{t(-2s^3 + 3s^2 t + 6st^2 + t^3)}{6(s+t)^4}, \; C_{2,2} = \frac{s(s^3 + 6s^2 t + 3st^2 - 2t^3)}{6(s+t)^4}$$

# d=4, D=4, scalar in-the-loop

- Scalar in-the-loop for any helicity

$$M^{(1)}_{4,scalar}|_{D=4} = \frac{1}{6\epsilon}$$

- Scalar trick

$$M_4^{(1)}(1^+,2^+,3^-,4^-)|_{d=4,D=4} + (d-4)M^{(1)}_{4,scalar}(1^+,2^+,3^-,4^-)|_{D=4} = \frac{d-26}{6}\frac{1}{\epsilon}$$

$$M_4^{(1)}(1^+,2^-,3^+,4^-)|_{d=4,D=4} + (d-4)M^{(1)}_{4,scalar}(1^+,2^-,3^+,4^-)|_{D=4} = \frac{d-26}{6}\frac{1}{\epsilon}$$

# d=D=6

UV-divergent parts of the master integrals in D = 6 - 2ϵ

$$Sing\ I_4(s,t) = 0,\ Sing\ I_3(s) = -\frac{1}{2\epsilon},\ Sing\ I_3(t) = -\frac{1}{2\epsilon},$$
$$Sing I_2(s) = \frac{s}{6\epsilon},\ Sing\ I_2(t) = \frac{t}{6\epsilon}.$$

The amplitudes in both helicity cases

$$M_4^{(1)}(1^+,2^+,3^-,4^-)|_{d=6,D=6} = 0,$$

$$M_4^{(1)}(1^+,2^-,3^+,4^-)|_{d=6,D=6} = 0.$$

# d=D=8

UV-divergent parts of the master integrals in D = 8 - 2ϵ

$$Sing\ I_4(s,t) = \frac{1}{6\epsilon},\ Sing\ I_3(s) = -\frac{s}{24\epsilon},\ Sing\ I_3(t) = -\frac{t}{24\epsilon},$$
$$Sing I_2(s) = \frac{s^2}{60\epsilon},\ Sing\ I_2(t) = \frac{t^2}{60\epsilon}.$$

The amplitudes in both helicity cases

$$M_4^{(1)}(1^+,2^+,3^-,4^-)|_{d=8,D=8} = -\frac{4}{35\epsilon}st$$
$$M_4^{(1)}(1^+,2^-,3^+,4^-)|_{d=8,D=8} = -\frac{29}{210\epsilon}st.$$

# d=D=10

UV-divergent parts of the master integrals in D = 10 - 2ϵ

$$Sing\ I_4(s,t) = \frac{s+t}{120\epsilon},\ Sing\ I_3(s) = -\frac{s^2}{360\epsilon},\ Sing\ I_3(t) = -\frac{t^2}{360\epsilon},$$
$$Sing I_2(s) = \frac{s^3}{840\epsilon},\ Sing\ I_2(t) = \frac{t^3}{840\epsilon}.$$

The amplitudes in both helicity cases

$$M_4^{(1)}(1^+,2^+,3^-,4^-)|_{d=10,D=10} = -\frac{st(39s+49t)}{7560\epsilon}$$
$$M_4^{(1)}(1^+,2^-,3^+,4^-)|_{d=10,D=10} = -\frac{53st(s+t)}{7560\epsilon}.$$

# d - arbitrary

- D = 6

$$M_4^{(1)}(1^+, 2^+, 3^-, 4^-)|_{d,D=6} = 0,$$

$$M_4^{(1)}(1^+, 2^-, 3^+, 4^-)|_{d,D=6} = 0.$$

- D = 8, 10

$$M_4^{(1)}(1^+, 2^+, 3^-, 4^-)|_{d,D=8} = -\frac{d+40}{420\epsilon}st,$$

$$M_4^{(1)}(1^+, 2^-, 3^+, 4^-)|_{d,D=8} = -\frac{d+166}{1260\epsilon}st,$$

$$M_4^{(1)}(1^+, 2^+, 3^-, 4^-)|_{d,D=10} = -\frac{st(3(d+16)s+(d+88)t)}{15120\epsilon},$$

$$M_4^{(1)}(1^+, 2^-, 3^+, 4^-)|_{d,D=10} = -\frac{(d+202)st(s+t)}{30420\epsilon}.$$

# Supersymmetry check

- Particle content of various MSYM theories

| D | $\mathcal{N}$ | $n_g$ | $n_f$ | $n_s$ |
|---|---|---|---|---|
| 4 | 4 | 1 | 4 | 6 |
| 6 | 2 | 1 | 2 | 4 |
| 8 | 1 | 1 | 1 | 2 |
| 10 | 1 | 1 | 1 | 0 |

- Result

$$M_4^{(1)}|_{D=4,6,8,10} = -stI_4(s,t)$$

# Conclusions

- The obtained formulas for the UV divergent contributions for one-loop planar 4-gluon scattering amplitudes have a universal form and can be applied in any dimension. We distinguished the Lorentz algebra dimension d and the loop momenta dimension D mostly for the purpose of verification of the validity of our results and comparison with the known ones. To explore the purely D-dimensional case, one should put d=D in our formulas.

- One-loop amplitude in D=6 has no UV divergences for both gauge and matter fields, while in D=8, 10, etc the signs of all the contributions are the same. It is basically negative but contains a polynomial dependence on the Mandelstam variables and can have different signs depending on the kinematics.

- The UV divergent terms $\sim 1/\varepsilon$ are in one-to-one correspondence with the leading logarithms and one can sum them up using the generalized RG equations. This can be done for several QFT models like SYM and the procedure seems to be realized for the D-dimensional pure Yang-Mills theory.

Thank you for your attention!