SIGMA MODELS

AS GROSS-NEVEU MODELS

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- the review article

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Spin chains and integrable theories"

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Flag manifold sigma models:
Spin chains and integrable theories

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A B S T R A C T

This review is dedicated to two-dimensional sigma models with flag manifold target spaces, which are generalizations of the familiar $\mathbb{CP}^{n-1}$ and Grassmannian models. They naturally arise in the description of continuum limits of spin chains, and their phase structure is sensitive to the values of the topological angles, which are determined by the representations of spins in the chain. Gapless phases can in certain cases be explained by the presence of discrete 't Hooft anomalies in the continuum theory. We also discuss integrable flag manifold sigma models, which provide a generalization of the theory of integrable models with symmetric target spaces. These models, as well as their deformations, have an alternative equivalent formulation as bosonic Gross–Neveu models, which proves useful for demonstrating that the deformed geometries are solutions of the renormalization group (Ricci flow) equations, as well as for the analysis of anomalies and for describing potential couplings to fermions.

Contents

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SUMMARY OF THE TALK

[Gross-Neveu '1974]

- $\exists$ A broad class of integrable sigma models [Witten '1978] (exactly and explicitly) equivalent to [Andrei-Lowenstein '1979] bosonic (and Bose/Fermi) Gross-Neveu models [Destri-de Vega '1989]

Related: [LeClair, Ludwig, 2000
Saleur, Schomerus -2010]

[NOT BOSONIZATION!]

- One can easily construct deformations (trigonometric, in principle also elliptic)

- Simple proof of renormalizability at one loop
$\mapsto$ generalized Ricci flow

Riemann surface

$$\Sigma_2 \to \mathcal{M}$$

target space

MORE...

- Sigma models = models with polynomial interactions!
- New approach to models with fermions (cancellation of "integrability anomalies")
- New method for constructing SUSY theories (from target space SUSY)
- General class = models with quiver variety phase spaces
- Integrable sigma models on Riemann surfaces?

KEY EXAMPLE: THE $\mathbb{CP}^{n-1}$ MODEL AS A GN-MODEL

Take $U, V \in \mathbb{C}^n$, define $\Psi := \begin{pmatrix} U \\ \bar{V} \end{pmatrix}$ "Dirac boson"

$$\mathcal{L} = \bar{\Psi}_a \not{D} \Psi_a + \varkappa \left(\bar{\Psi}_a \frac{1+\sigma_3}{2} \Psi_a \right) \cdot \left(\bar{\Psi}_b \frac{1-\sigma_3}{2} \Psi_b \right)$$

$U(1)$ ($\mathbb{C}^*$) covariant derivative

Bosonic chiral gauged Gross-Neveu model

In components, $\mathcal{L} = V \cdot \bar{D} U + \bar{U} \cdot D \bar{V} + \varkappa (\bar{U} \cdot U)(\bar{V} \cdot V)$

Eliminating $V, \bar{V}$ $\mapsto$ $\boxed{\mathcal{L} = \frac{1}{\varkappa} \frac{D\bar{U} \cdot \bar{D} U}{\bar{U} \cdot U}}$ = $\mathbb{CP}^{n-1}$ model in the GLSM approach

$\bar{U} \cdot U = 1$ – standard "Hopf" gauge

ROLE OF CHIRAL SYMMETRY

For Euclidean worldsheet signature (Σ_2 = Riemann surface)

$$\boxed{\text{Chiral symmetry} = \text{Complexification of the usual (flavor) symmetry}}$$

[Zumino '1977, Mehta '1990]

In our example $U \mapsto \lambda U$, $V \mapsto \lambda^{-1} V$

so that $\cancel{|U|^4}$ $\cancel{|V|^4}$ $|U|^2 \cdot |V|^2$ ✓

Fermi GN-model: classically integrable model [Neveu, Papanicolaou '1978]
for fermion bilinears (these are "moment maps/currents")

FIRST APPLICATION. QUANTUM MECHANICS.

$$\mathcal{L} = V \cdot \frac{\bar{D}U}{dt} + \bar{U} \cdot \frac{D\bar{V}}{dt} + \varkappa (\bar{U} \cdot U)(\bar{V} \cdot V) \,, \quad \text{where } \frac{\bar{D}U}{dt} = \frac{dU}{dt} - i\bar{A}U$$

Canonical quantization: $[U_i, V_j] = \delta_{ij} \mapsto V_j = -\dfrac{\partial}{\partial U_j}$

$$\boxed{V \cdot U = \bar{U} \cdot \bar{V} = 0 \mapsto \sum_{i=1}^{n} U_i \frac{\partial \psi}{\partial U_i} = \sum_{i=1}^{n} \bar{U}_i \frac{\partial \psi}{\partial \bar{U}_i} = 0}$$

ψ is a homogeneous function of $\{U_i\}$

Hamiltonian = Laplace operator:

$$H\psi = \left(\sum_{i=1}^{n} \bar{U}_i U_i\right) \sum_{j=1}^{n} \frac{\partial^2 \psi}{\partial U_j \partial \bar{U}_j}$$

Eigenfunctions: $|M\rangle := \dfrac{1}{|U|^{2M}} \sum \psi_{i_1 \ldots i_M | j_1 \ldots j_M} U_{i_1} \ldots U_{i_M} \bar{U}_{j_1} \ldots \bar{U}_{j_M}$

Traceless

$$H|M\rangle = -M(n+M-1)|M\rangle$$

DEFORMATIONS

Vast literature on integrable deformations:
[Cherednik '1981] [Klimčík '2002]
[Fateev '1993+] [Delduc-Magro-Vicedo '2013]
[Hoare-Tseytlin] [Alfimov-Feigin-Litvinov]

We replace the GN-Lagrangian:

$$\mathcal{L} = \overline{\Psi}_a \not{D} \Psi_a + (r_s)^{cd}_{ab} \left(\overline{\Psi}_a \frac{1+\gamma_3}{2} \Psi_c \right) \cdot \left(\overline{\Psi}_d \frac{1-\gamma_3}{2} \Psi_b \right)$$

classical r-matrix
(s = deformation parameter)

[Belavin Drinfeld '1980]

Noether current $= J\,dz + \overline{J}\,d\overline{z}$, then $\partial \overline{J} = [\overline{J}, r_s(J)]$

Flatness of the connection
$A_u := r_u(J)\,dz - r_{us^{-1}}(\overline{J})\,d\overline{z}$

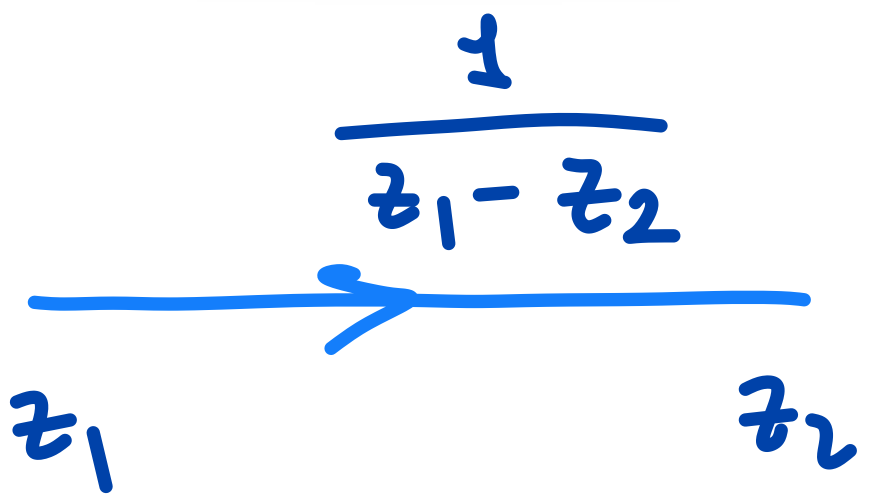
$\longleftrightarrow$

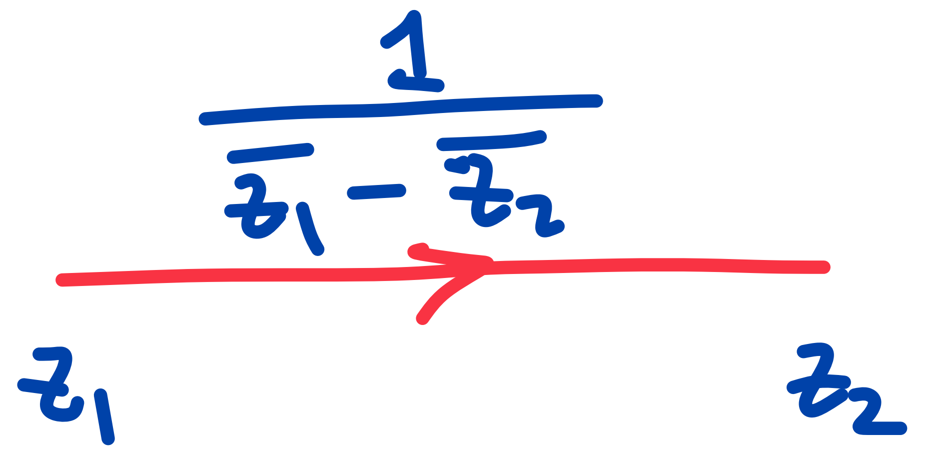
[Costello Yamazaki '2019]

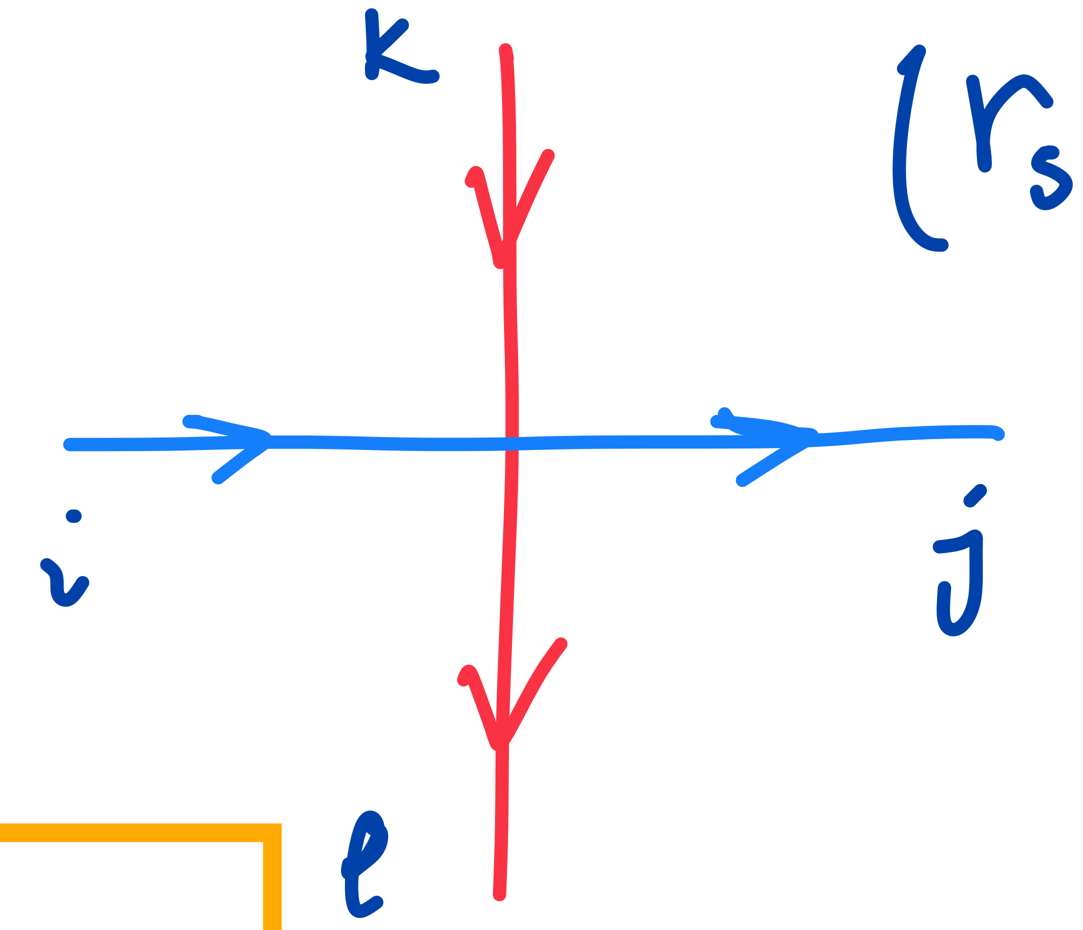
Classical Yang-Baxter equation

8

ONE-LOOP β-FUNCTION





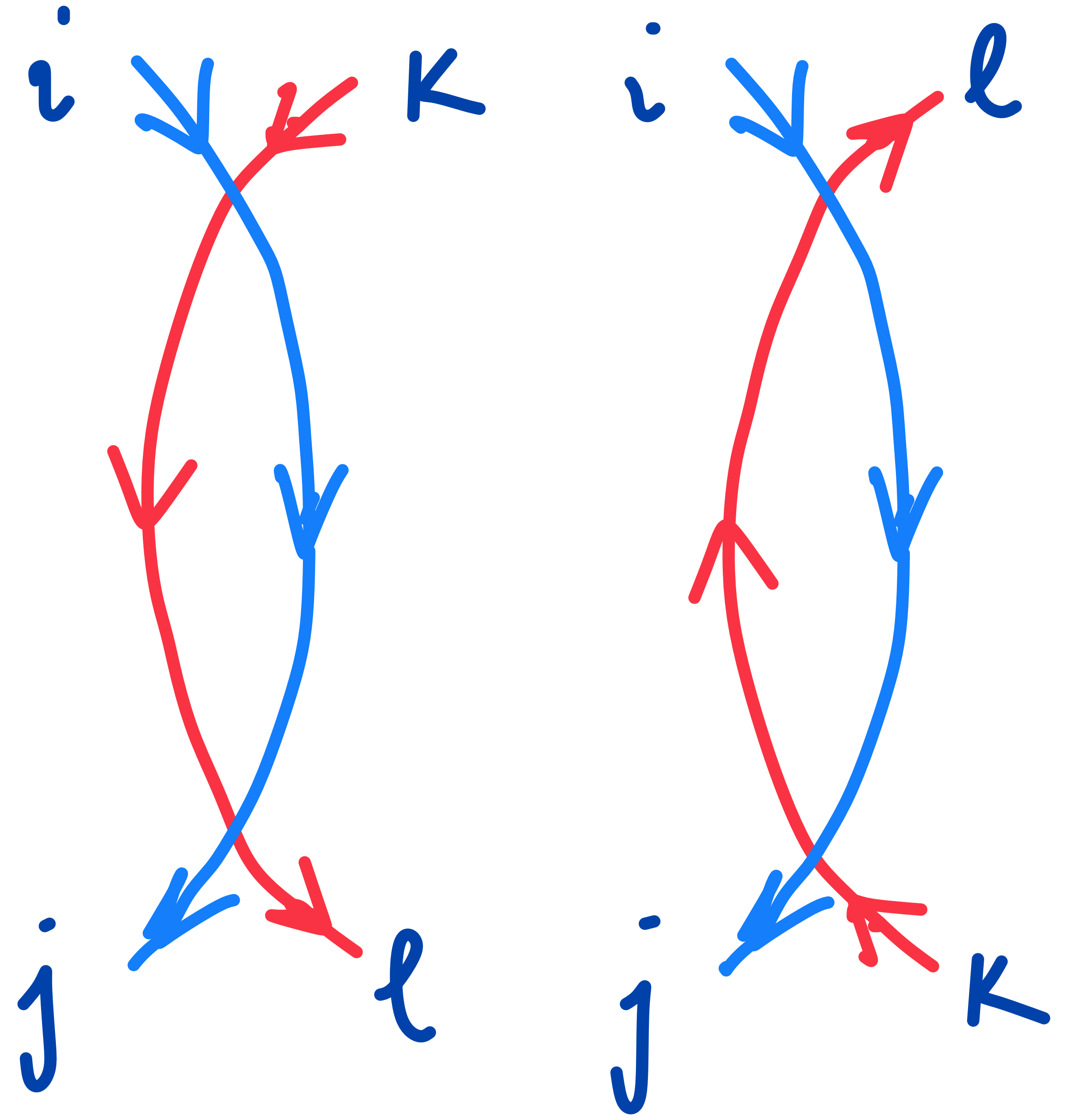


$$(r_s)^{kl}_{ij}$$

Diagrams for the one-loop β-function:

(No background field method necessary!)

$$\beta^{kl}_{ij} = \sum_{p,q=1}^{n} \left((r_s)^{kq}_{ip} (r_s)^{ql}_{pj} - (r_s)^{ql}_{ip} (r_s)^{kq}_{pj} \right)$$



trigonometric deformation parameter

$\mathbb{CP}^{n-1}$

RG time

The RG-flow eq-n $\frac{d}{d\tau} r^{kl}_{ij} = \beta^{kl}_{ij}$ has the solution $s = e^{n\tau}$

EXAMPLE : THE "SAUSAGE" MODEL

$\mathbb{CP}^1$: $n=2 \Rightarrow s=e^{2\tau}$

Deformed metric: $ds^2 = (s^{-1}-s)\dfrac{|dW|^2}{(s+|W|^2)(s^{-1}+|W|^2)}$, $0<s<1$

Ricci flow: $-\dfrac{d}{d\tau} g_{w\bar{w}} = R_{w\bar{w}}$

$$\text{Length} \sim |\log(s)| = n|\tau|$$

[Fateev
Onofri '1994
Zamolodchikov]

$\mathbb{CP}^{n-1}$: $(\mathbb{C}^*)^{n-1}$ in the UV-limit $\xleftrightarrow{\text{RG flow}}$ Homogeneous $\mathbb{CP}^{n-1}$ metric in the IR-limit

$s \mapsto 0$ $\qquad\qquad$ $s \mapsto 1$

GENERALIZED RICCI FLOW

One recovers the geometric form of the model by integrating over $V, \bar{V}$.
The "elementary" solution $S = e^{n\tau}$ is a solution to the complicated RG-flow eqns

One-loop determinant arising from the $V, \bar{V}$-integration

$$-\dot{g}_{ij} = R_{ij} + \frac{1}{4} H_{imn} H_{jm'n'} g^{mm'} g^{nn'} + 2 \nabla_i \nabla_j \Phi$$

$$-\dot{B}_{ij} = -\frac{1}{2} \nabla^k H_{kij} + \nabla^k \Phi \cdot H_{kij}$$

[Curci-Paffuti '1987]

$$-\dot{\Phi} = \text{const.} - \frac{1}{2} \nabla^k \nabla_k \Phi + \nabla^k \Phi \cdot \nabla_k \Phi + \frac{1}{24} H_{kmn} H^{kmn}$$

MODELS WITH FERMIONS

Purely bosonic models are affected by chiral anomalies

[Schwinger '1962]

$$\mathcal{L} = \overline{\Psi_a} \not{D} \Psi_a \longmapsto S_{eff} \sim n \int d^2z \, F_{z\bar{z}} \frac{1}{\Delta} F_{z\bar{z}}$$

not invariant w.r.t. complexified gauge transformations

$$A \to A + \partial \alpha, \quad \bar{A} \to \bar{A} + \bar{\partial} \bar{\alpha}$$

[Alvarez-Gaume-Moore-Vafa '1986]

Elementary way of cancelling the anomaly:

$$\tilde{\mathcal{L}} = \overline{\Psi_a} \not{D} \Psi_a + \overline{\Theta} \not{D} \Theta$$

Bosons

Fermions

The determinants simply cancel!

THE GENERAL SETUP. 1.

There is a more conceptual way.

Recall the $\mathbb{CP}^{n-1}$ model:

$$\mathcal{L} = \underbrace{V\cdot\bar{D}U + \bar{U}\cdot D\bar{V}}_{\text{Poincaré–Liouville one-form ("pdq")}} + \underbrace{\varkappa\,(\bar{U}\cdot U)\,(\bar{V}\cdot V)}_{\text{"Hamiltonian"}}$$

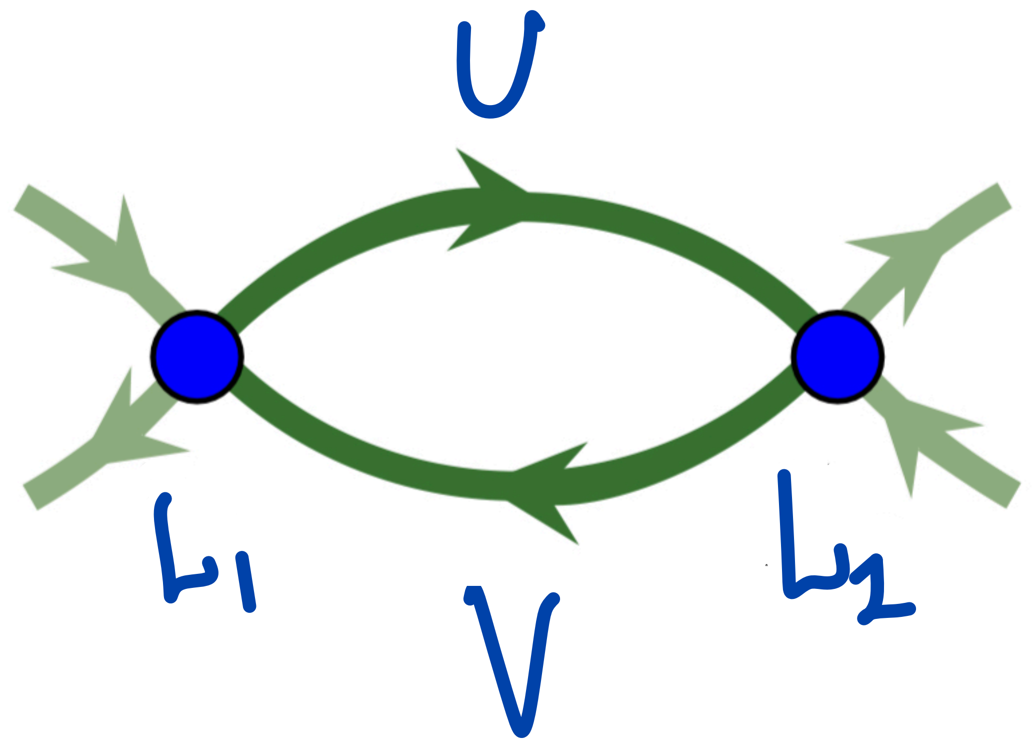
Poincaré–Liouville one-form ("pdq"), "Hamiltonian"
for the complex symplectic form $\sum_{i=1} dV_i \wedge dU_i$

$\beta\gamma$-system [Costello, Yamazaki '2019] + GLSM-representation [DB '2017]

$$\boxed{\text{Gauge field} \longleftrightarrow \text{Complex symplectic reduction!}}$$

THE GENERAL SETUP. 2.

- Complex symplectic quiver supervariety Φ (phase space)



Vector spaces L_1, L_2

$U \in \mathrm{Hom}(L_1, L_2)$ $\quad V \in \mathrm{Hom}(L_2, L_1)$

- Matter fields $U \oplus V$ in representation $W \oplus W^\vee$ of complex gauge supergroup G_{gauge}
- Complex global symmetry group G_{global}, with corresponding complex moment map J

THE GENERAL SETUP. 3.

- Lagrangian has the form

$$\boxed{\mathcal{L} = V\cdot \bar{D}\, U + \bar{U}\cdot D\bar{V} + \varkappa\, \mathrm{STr}(J\bar{J})}$$

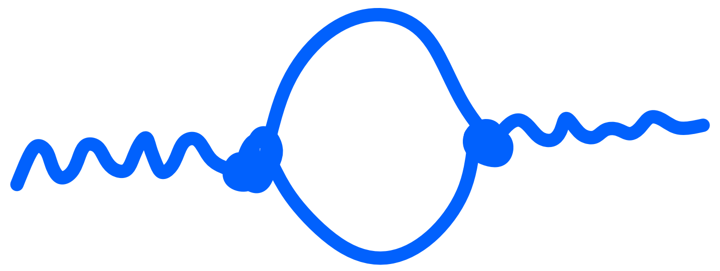
in the $\mathbb{CP}^{n-1}$-example $J = V\otimes V$

coupling constant

- Chiral anomaly cancellation condition (WZNW-type) [Schwinger '1962] [Polyakov '1983, Wiegmann]

$$\mathrm{STr}_W(T_a T_b) = 0$$

$\mathfrak{g}_{gauge}$



- Conjecture: this also ensures the cancellation of "integrability" (Yangian/Lüscher non-local charge) anomalies [Polyakov '1977, Goldschmidt-Witten '1980, Abdalla et.al. '1981-84]

EXAMPLES. $\mathbb{CP}^{n-1}$ MODELS WITH FERMIONS

Choosing the super-phase space Φ and G_{global}, one can obtain all known integrable $\mathbb{CP}^{n-1}$ models

$$\Phi = T^* \underbrace{\mathbb{CP}^{n-1|n}}_{=\text{fermionic "conifold" bundle } \Pi(\mathcal{O}(1)\oplus\cdots\oplus\mathcal{O}(1)) \text{ over } \mathbb{CP}^{n-1}} \qquad G_{gauge} = \mathbb{C}^*$$

$G_{global} \subset$ Symplectomorphism group $PSL(n|n)$

- $G_{global} = SL(n) \times \mathbb{1} \subset PSL(n|n)$ — [Abdalla et.al. '1981–84]

  "Minimally coupled fermions"

- $G_{global} = PSL(n|n)$ — [Read-Saleur '2001, Witten '2003, Schomerus et.al. '2010]

  $\mathbb{CP}^{n-1|n}$ sigma model

THE SUSY $\mathbb{CP}^{n-1}$ MODEL.

fermionic tangent bundle

Here we take $\Phi = T^*E$, where $E = \Pi T(\mathbb{CP}^{n-1})$

We can write $\Phi = T^*\mathbb{C}^{n|n} /\!/ G_{gauge}$

Supersymplectic reduction!

$$G_{gauge} = \left\{ \begin{pmatrix} \lambda & 0 \\ \xi & \lambda \end{pmatrix} \in SL(1|1) \subset SL(n|n) \right\}$$

bosonic parameter $\simeq \mathbb{C}^*$

fermionic parameter $\simeq \mathbb{C}$

$T^*\mathbb{C}^{n|n}$: coordinates on $\mathbb{C}^{n|n}$: $\mathcal{U} = \begin{pmatrix} U \\ C \end{pmatrix}$

coordinates in the fiber: $\mathcal{V} = (V \; B)$

MORE...

Start with the "free" Lagrangian

$$\mathcal{L}_{free} = V \cdot \bar{\mathcal{D}}\, U + \bar{U} \cdot \mathcal{D} \bar{V} \,, \quad \text{where } \bar{\mathcal{D}} = \bar{\partial} + i \bar{\mathcal{A}}_{super}, \quad \bar{\mathcal{A}}_{super} = \begin{pmatrix} \bar{A} & 0 \\ \bar{W} & \bar{A} \end{pmatrix}$$

$\mapsto$ invariant w.r.t. SUSY transformations

$$\delta U = \varepsilon_1 C \,, \quad \delta B = -\varepsilon_1 V \,, \quad \delta C = -\varepsilon_2 D U \,, \quad \delta V = \varepsilon_2 D B$$

$$\{ Q_1, Q_2 \} = D$$

[Kapustin '2005]
[Grassi–Policastro–Scheidegger '2007]

Interactions: take $G_{global} = SL(n) \underset{\text{diagonal}}{\subset} SL(n|n)$

$$J = U \otimes V - C \otimes B$$

YET MORE...

The full Lagrangian is

$$\mathcal{L} = V \cdot \bar{D} U + \bar{U} \cdot D \bar{V} + \varkappa \, \mathrm{Tr}(J \bar{J})$$

SUSY-inv.!

- Gauge: $\bar{U} \circ U = 1$, $\bar{U} \circ C = 0$
- Eliminate $V, \bar{V}$

$$\mathcal{L} = \frac{1}{\varkappa} |\bar{D} U|^2 + \bar{\Theta} \not{D} \Theta - \varkappa \left(\bar{\Theta} \frac{1+\sigma_3}{2} \Theta \right) \left(\bar{\Theta} \frac{1-\sigma_3}{2} \Theta \right)$$

$\Theta = \begin{pmatrix} C \\ B \end{pmatrix}$

chiral Gross-Neveu model

$$\bar{U} \circ \Theta = \begin{pmatrix} \bar{U} \circ C \\ \bar{U} \circ B \end{pmatrix} = 0$$

gauge fixing

moment map = 0

[D'Adda, Di Vecchia, Lüscher '1979]

The bosonic part is also a chiral GN-model, albeit a bosonic one!

ALL-LOOP β-FUNCTIONS?

The above systems are examples of

$$S = S_{CFT} + \int d^2z \, d_{a\bar{b}} J^a \bar{J}^b$$

coupling constants

CFT with Kac-Moody symmetry

∃ an all-loop β-function proposal

Homogeneous case ($d_{a\bar{b}} = \varkappa \, \delta_{a\bar{b}}$):

$$\boxed{\beta_\varkappa = \frac{1}{2} \frac{C_2 \varkappa^2}{\left(1 + \frac{1}{2} K \varkappa\right)^2}}$$

level of Kac-Moody algebra

[Kutasov '1989
Gerganov, LeClair, Moriconi '2001]

SUSY model: $K = 0$ ⇒ β-function is one-loop exact!

[Morozov
Perelomov '1984
Shifman]

To compare:

4-loop β-function of the pure GN-model:

$$\begin{aligned}\beta(g) = &(d-2)g - (N-1)\frac{g^2}{\pi} + (N-1)\frac{g^3}{2\pi^2} \\ &+ (N-1)(2N-7)\frac{g^4}{16\pi^3} + (N-1)[-2N^2 - 19N \\ &+ 24 - 6\zeta_3(11N-17)]\frac{g^5}{48\pi^4} + \mathcal{O}(g^6) \qquad (4.9)\end{aligned}$$

[Gracey
Luthe
Schröder
'2016]

MORE ON ANOMALIES / RIEMANN SURFACES

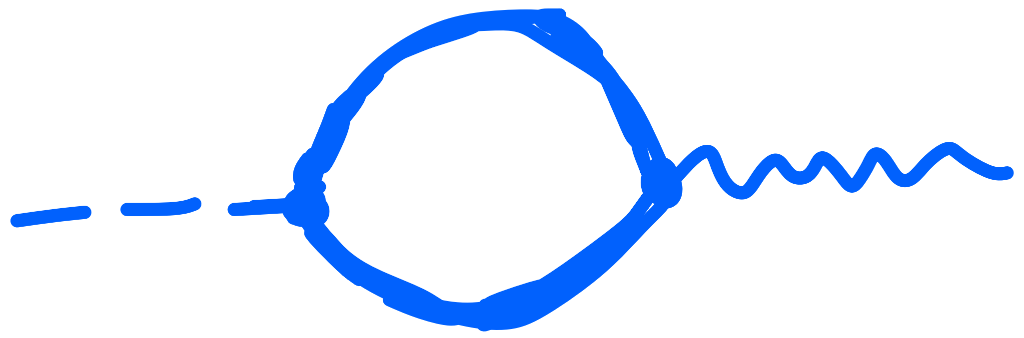
GN models can be considered on Riemann surfaces Σ_g:

U = section of line bundle L, V = section of $K \otimes L^{-1}$

Absence of mixed gauge/gravitational anomaly:

$$\boxed{\mathrm{STr}_W(\mathcal{J}\, T_a) = 0}$$

matrix of spins



In most models $\mathrm{STr}_W(T_a) = 0$ (same gauge charges for Bose/Fermi fields)

$\Rightarrow \mathcal{J} \propto \mathbb{1}$: A-type top. twist [Witten '1988-91]

MORE ON ANOMALIES / RIEMANN SURFACES...

Amusingly, in gauged GN models the best gauge is (on $\mathbb{R}^2$)

$\boxed{A = \bar{A} = 0}$ (Gauge transfos: $\bar{A} \mapsto \bar{A} - i\bar{\partial}\chi = 0 \Rightarrow \chi(z,\bar{z}) = \frac{i}{\pi}\int \frac{d^2w}{z-w}\bar{A}(w,\bar{w})$)

↑ Gauge field analogue of conformal gauge in string theory!

On Σ_g, and for rank-'k' bundle V, one can choose a 'flat' gauge

$$dA - i\, A \wedge A = \frac{p}{k}\,\mathrm{vol}_\Sigma\, \mathbb{1}_k$$

$c_1(V)$

Solutions give rise to moduli

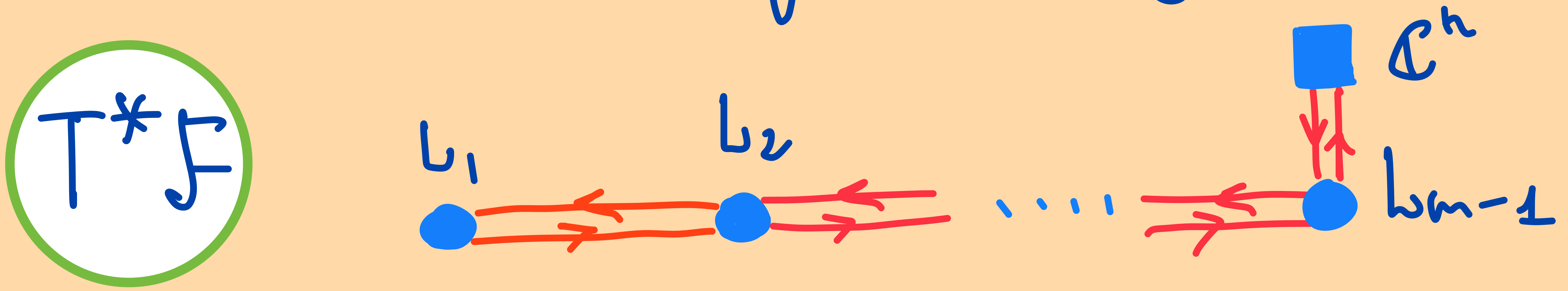
$$\left\{ \prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} = e^{\frac{2\pi i p}{k}} \right\} \Big/ U(k)$$

[Narasimhan–Seshadri '1965]
[Atiyah–Bott 1983]
[Hori '1996]

SUMMARY & OUTLOOK

- Sigma models = chiral Gross-Neveu models
- Integrable sigma models related to quiver supervarieties

Ex.: classically integrable flag manifold sigma models [DB'2015+]



[Nakajima '1994]

- Polynomial interactions
- All-loop β-functions?
- Ashteker variables
- Ultralocal Lax pairs [Bytsko '1994]
- Sigma models on Riemann surfaces
- Full quantum theory?

POLYNOMIAL INTERACTIONS.

The GN-model has quartic interactions

One can completely get rid of the gauge fields $A, \bar{A}$.

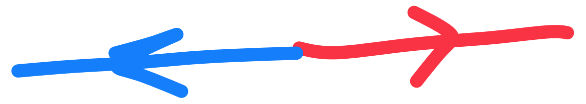
if anomaly-free

Variation w.r.t. $A, \bar{A}$: $V \cdot U = \bar{U} \cdot \bar{V} = 0$

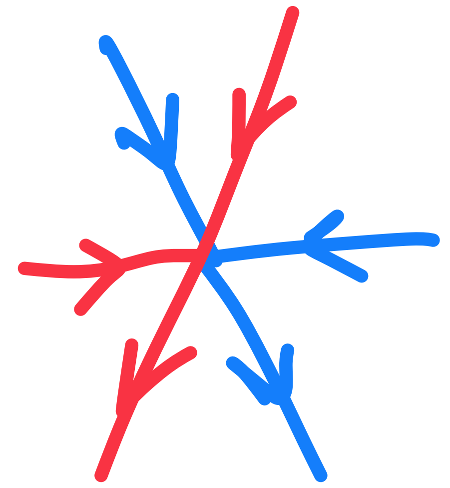
Gauge invariance $U \mapsto \lambda U$, $V \mapsto \lambda^{-1} V$ $\xrightarrow[\text{fixing}]{\text{gauge}}$ $U_n = 1$

Solve the constraints: $V_n = -\sum_{i=1}^{n-1} V_i U_i$

modified Green's functions



sextic vertices



$$\mathcal{L} = \sum_{k=1}^{n-1} V_k \bar{\partial} U_k + \bar{U}_k \partial \bar{V}_k + \varkappa |V_k|^2 +$$

$$+ \varkappa \sum_{i=1}^{n-1} |V_i|^2 \times \sum_{j=1}^{n-1} |V_j|^2 + \varkappa \left| \sum_{j=1}^{n-1} U_j V_j \right|^2 + \varkappa \sum_{i=1}^{n-1} |V_i|^2 \times \left| \sum_{j=1}^{n-1} U_j V_j \right|^2$$

quartic vertices

DIMENSIONAL REDUCTIONS OF 4D GRAVITY

dim. red.

4D gravity + matter [Breitenlohner–Maison–Gibbons '1988] [Breitenlohner–Maison '2000]

2D sigma model with a Hermitian symmetric target space [Ehlers '1959] [Geroch '1971] [Belinskii–Zakharov '1978]

such as $\dfrac{U(1,n-1)}{U(1)\times U(n-1)}$ $\quad n=2 \Rightarrow$ pure gravity $\dfrac{SL(2,\mathbb{R})}{SO(2)}$

To get this replace $\mathrm{Tr}(J\bar{J}) \mapsto \mathrm{Tr}(\Lambda J \Lambda \bar{J})$, $\Lambda = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$

Noether current: $J \sim A \otimes E$, we saw that $J \sim U \otimes V$
[Brodbeck–Zagermann '2000]

Ashtekar canonical variables

Conjecture: (U, V) are the dim. red. of Ashtekar variables

Explains both ultralocality and polynomiality of interactions!

POLYNOMIAL OSCILLATOR REPRESENTATIONS

$$\mathcal{L} = \underbrace{V \cdot \frac{D\bar{U}}{dt} + \bar{U} \cdot \frac{D\bar{V}}{dt}}_{\text{"pdq"-type term}} + \underbrace{2\left(aA + \bar{a}\bar{A}\right)}_{\text{Complex FI-term}}$$

Noether charge = moment map = classical spin

$$SL(2,\mathbb{C}): \; J = U \otimes V - \frac{(V \circ U)}{2} 1_2$$

$$\boxed{J^2 = a^2 \mathbb{1}_2 \quad \text{(complex adjoint orbit)}}$$

Set $U_2 = 1$ and quantize: $\quad [U_1, V_1] = 1$

$$\boxed{J^+ = -\frac{\partial}{\partial U_1}, \; J^- = U_1\left(2a + U_1 \frac{\partial}{\partial U_1}\right), \; J^3 = a + U_1 \frac{\partial}{\partial U_1}}$$

[Dyson Maleev ≈ 1955]

ULTRALOCAL LAX PAIRS

[Faddeev Takhtajan $\simeq$1985]

$$\{L(z,\zeta_1) \underset{,}{\otimes} L(w,\zeta_2)\} = [r(z-w), L(z,\zeta_1)\otimes 1 + 1 \otimes L(w,\zeta_2)]\ \delta(\zeta_1-\zeta_2)$$

↑ component of A along ζ

classical r-matrix (as in mechanics)

Some sigma models do admit ultralocal Lax pairs:

- $\mathbb{C}P^1$ [Bytsko '1994], "Sausage" [Bazhanov-Kotousov-Lukyanov '2017]
- Hermitian symmetric spaces [Brodbeck-Zagermann '1999]
- Complex homogeneous target spaces [DB '2016] [Delduc-Kameyama-Lacroix-Magro-Vicedo 2019]
- Faddeev-Reshetikhin type models [Caudrelier-Stoppato-Vicedo '2020]
- Gross-Neveu models are manifestly ultralocal