Differential equation derivation for Feynman integrals

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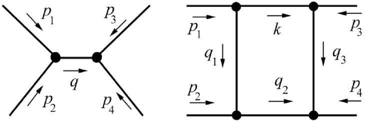
International Conference on Quantum Field Theory, High-Energy Physics, and Cosmology,
July 2022

Feynman Diagrams: Basic Definitions



- Quantum field theory amplitudes are represented as a sum of Feynman Diagrams, graphs for which each line and vertex is represented by a factor in a term of the quantum amplitude.
- Integrating over all unconstrained momenta gives rise to a Feynman Integral, FI. For L loops and n internal lines, and allowing the propagators to be raised to powers ν_j,

$$J(m_1^2, ...m_n^2; p_1^2, ...p_k^2, \alpha_1, ..., \alpha_n) = \int \frac{d^d k_1..d^d k_L}{(i\pi^{d/2})^L} \Pi_{i=1}^n \frac{1}{(q_i^2 - m_i^2)^{\alpha_i}}.$$

- Our final goal is to find a solution of FI as a series in dimensional regularization parameter ε where coefficients are expressed in terms of some special functions with well-established properties.
- Here we discuss properties of differential equation system for FI in most general case, when all masses, external momenta and propagator powers have arbitrary values

IBP Relations, Master Integrals and Differential Equations

- **Integration by parts** leads to a set of recurrence relations among diagrams of a given topology but different powers of the propagators.
  K.G. Chetyrkin, F.V. Tkachov, Nucl. Phys. B 192, 159 (1981)
- The full set of recurrence relations should be solved by finding how the integral with powers of propagators $(j_1 + j_2 + \cdots + j_k)$ reduced to integrals with powers $(j_1 + j_2 + \cdots + j_k - 1)$
- The method involves taking derivatives of each integral with respect to momenta and reducing it to the original integral.
- The relations found permit a **reduction** to a basis set of **master integrals** in terms of which the diagrams of this class may be expressed.
- The differential equation system fo FI is obtained by taking some derivatives of a given master integral with respect to kinematical invariants and masses.
  A.V. Kotikov, Phys. Lett. B 254, 158 (1991)
  Then the result is written in terms of Feynman integrals of the given family and, according to the known reduction, in terms of the master integrals.
- Finally, one obtains a system of differential equations for the master integrals which can be solved with appropriate boundary conditions.

Hypergeometric approach to FI

- An FI could be written in terms of hypergeometric series of Horn type
- The expansion over dimensional regularization parameter (derivatives of Horn type hypergeometric function) could be written in terms of function of the same class
- To express the Feynman integral we need hypergeometric function called generalized Lauricella series:

$$\sum_{m_1,\ldots,m_l}^{\infty} \prod_{i,j} \frac{(a_j)_{\sum_k^l q_k m_k}}{(b_i)_{\sum_k^l q_k m_k}} \prod_{n=1}^{l} \frac{x_n^{m_n}}{m_n!}\,, \quad q_k \in \mathbb{Z}\,,$$

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$$F^{A:B^{(1)};\ldots;B^{(n)}}_{C:D^{(1)};\ldots;D^{(n)}} \left(\begin{array}{c} [(a):\theta^{(1)},\ldots,\theta^{(n)}]:[(b^1):\phi^{(1)}];\ldots;[(b^n):\phi^{(n)}] \\ [(c):\psi^{(1)},\ldots,\psi^{(n)}]:[(d^1):\delta^{(1)}];\ldots;[(d^n):\delta^{(n)}] \end{array} x_1,\ldots,x_n \right)$$

$$= \sum_{s_1,\ldots,s_n=0}^{\infty} \Omega(s_1,\ldots,s_n) \frac{x_1^{s_1}}{s_1!} \frac{x_n^{s_n}}{s_n!}\,,$$

$$\Omega(s_1,\ldots,s_n) = \frac{\prod_{j=1}^{A}(a_j)_{s_1\theta_j^{(1)}+\cdots+s_n\theta_j^{(n)}} \prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{s_1\phi_j^{(1)}} \cdots \prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{s_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{s_1\psi_j^{(1)}+\cdots+s_n\psi_j^{(n)}} \prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{s_1\delta_j^{(1)}} \cdots \prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{s_n\delta_j^{(n)}}}\,,$$

The Mellin-Barnes Representation

- Balanced Mellin-Barnes representation of FI:

$$J_{bal}(\{s\},\{m\},\{\alpha\}) = C\int_{-i\infty}^{+i\infty}\prod_{j;l=1}^{l=n} \mathrm{d}u_l \, \frac{\Gamma(\sum_i a_{ij}u_i + b_j)}{\Gamma(\sum_i c_{ij}u_i + d_j)}\Gamma(-u_l) z_l^{u_l+c_l}(-1)^{g_l u_l}$$

- we could construct n ratio of two polynomials over variables u_i (treat all continuous variables as independent):

$$\frac{P_i(\vec{u})}{Q_i(\vec{u})} = \frac{\Phi(\vec{u}+\vec{e}_i)}{\Phi(\vec{u})}$$

  At the process of ratio construction one are allowed to cancel only factors with Γ functions, and leave similar polynomial factors in $\vec{u}$ untouched in numerator and denominator.

- With Euler differential operator $\theta_i = z_i\frac{d}{dz_i}$ we find the following n linear system of homogeneous differential equations:

$$(-1)^{g_j} Q_j(\vec{u})|_{u_i\to\theta_i}\frac{1}{z_i}J_{bal}(\{s\},\{m\},\{\alpha\}) = P_j(\vec{u})|_{u_i\to\theta_i}J_{bal}(\{s\},\{m\},\{\alpha\})$$

  Generally, it is a system of n differential equations, with n variables and χ maximum power of Euler differential operator.

Differential Contiguous Relations

- step-up $L^+_{b_j}$ and step-down $L^-_{d_j}$ operators shift indices b_j, d_j by a unit:

$$H(\mathbf{a},\vec{b}+\vec{e}_j,\mathbf{c},\vec{d};\ \vec{z}) = L^+_{b_j} H(\mathbf{a},\vec{b},\mathbf{c},\vec{d};\ \vec{z}) = \left(\sum_i a_{ij}\theta_i + b_j\right) H(\mathbf{a},\vec{b},\mathbf{c},\vec{d};\ \vec{z})$$

$$H(\mathbf{a},\vec{b},\mathbf{c},\vec{d}-\vec{e}_j;\ \vec{z}) = L^-_{d_j} H(\mathbf{a},\vec{b},\mathbf{c},\vec{d};\ \vec{z}) = \left(\sum c_{ij}\theta_i + d_j\right) H(\mathbf{a},\vec{b},\mathbf{c},\vec{d};\ \vec{z})$$

- The inverse operators $L^-_{b_j}$, $L^+_{d_j}$ can not be directly constructed form Mellins-Barnes representation
- Together operators $L^-_{b_j}$, $L^+_{d_j}$, $L^+_{b_j}$, $L^-_{d_j}$ helps one to change parameters of FI (Horn hypergeometric function) on integer number and find relation between the number of non-trivial master integrals found from IBP (which are not expressible in terms of Gamma functions) and the maximal power of derivatives generated by the $L^-_{b_j}$, $L^+_{d_j}$
- The inverse operators $L^-_{b_j}$, $L^+_{d_j}$ inherit information about simplification of hypergeometric function (lowering its order)

$$\left(\prod_{j\in m_k^+,\, n_j\in\{0,a_{kj}\}} L^+_{b_j+n_j} - \frac{1}{z_k}\theta_k \prod_{j\in m_k^-,\, n_j\in\{0,c_{kj}-1\}} L^-_{d_j-n_j}\right) H(\mathbf{a},\vec{b},\mathbf{c},\vec{d};\vec{z}) = 0$$

Differential Equation System Derivation

- n differential equation system:

$$\left(\prod_{j\in m_k^+, n_j\in\{0,a_{kj}\}} L^+_{b_j+n_j} - \frac{1}{z_k}\theta_k \prod_{j\in m_k^-, n_j\in\{0,c_{kj}-1\}} L^-_{d_j-n_j} \right) H(\mathbf{a},\vec{b},\mathbf{c},\vec{d};\vec{z}) = 0$$

- This system has several flaws that should be eliminated before resolving it. First, it is incomplete in sense that there exist more linearly independent differential equations. Second, we have to do multivariable specialization to remove auxiliary variables that we introduce in balanced MB representation
- we imply that our system have only finite number of solutions in fundamental system
- we consider the multivariable specialization of initial differential equation:

$$z_j = y_j(\vec{x}), \qquad j = 1, \ldots, n$$
$$\vec{x} = (x_1, \ldots, x_k), \qquad k < n\,.$$

Differential Equation System Derivation II

$$\left[\begin{array}{c|c} M_1 & \\ \hline B_1 & M_2 \end{array}\right] \left[\begin{array}{c} \frac{\partial^\chi H(\vec{z})}{\partial z_{j_1}\ldots\partial z_{j_\chi}} \\ \vdots \\ \frac{\partial^i H(\vec{z})}{\partial z_{j_1}\ldots\partial z_{j_i}} \\ \vdots \\ \frac{\partial H(\vec{z})}{\partial z_j} \\ \hline \frac{\partial^l H(\vec{x})}{\partial x_{j_1}\ldots\partial x_{j_l}} \\ \vdots \\ H(\vec{x}) \end{array}\right] = 0$$

- Applying the chain rule χ times we construct system of differential equation where various derivatives of order less or equal χ w.r.t. new variables $\vec{x}$ are expressed in terms of derivatives w.r.t. old $\vec{z}$ and derivatives of $y_i(\vec{x})$ functions.

Full differential equation system and its specialization

- For the case of several variables we get the system:

$$M_2 \begin{bmatrix} \frac{\partial^l H(\vec{x})}{\partial x_{j_1} \ldots \partial x_{j_l}} \\ \vdots \\ H(\vec{x}) \end{bmatrix} = 0$$

- for one variable we get equation of high order:

$$\sum_{i=0}^{\chi} Q_{\chi - i} \frac{\partial^i H(x_1)}{\partial^i x_1} = 0$$

- If the multivariable specialization falls into a singular locus of the initial PDE system, the rank of the new PDE system will be lower than the initial one, for any combination of the parameters.
- The singular loci of the new PDEs are composed of ones from old PDEs and locus of multivariable specialization. For some particular combinations of parameters and variables, the loci of the new PDEs could be diminished.
- we may observe simplifications in the class of functions that satisfy the new PDE system after the application of projective or more general pull-back transformations of variables.

Full differential equation system and Pfaff form:

- By introducing the derivations of initial system we could set it in full differentials of dependent and independent variables:

$$d\omega_i(\vec{z}) = \Omega_{ij}(\vec{z})\omega_j(\vec{z})dz_k$$

- Setting the values of independent variables we could make a specialization to one variable:

$$\omega_i'(x) = \tilde{\Omega}_{ij}(x, parameters)\omega_j(x)$$

- The system for MI is full differential equation system and could be also written down on form of one differential equation of higher order

Special value parameters and factorization

- Although the system of differential equations is full, sometimes IBP reduction could find additional independent relation(s) between MI
- additional linear relations are of the form

$$\sum_{i=0}^{\chi} Q_{\chi-i}\frac{\partial^i H(x)}{\partial^i x} = \sum_k c_k x_k^{\lambda}$$

- There is one-to-one correspondence between factorization of differential equation and block-diagonal form of differential equation system
- Hyperexponential solutions are responsible for factorization
- factorization manifest itself in system form as diagonal blocks
- left and right term factorization are equal only in the case of full factorization in the field of rational functions.

$$L(x) = L_1(x)L_2(x)\ldots L_n(x)$$

- for some special cases this relation for MI could be inferred from IBP procedure
- factorization works also for multiscale case, where all parameters except one treated as constant.

One-Loop Two-point Diagram, different masses

- one-loop two-point diagram with different masses and arbitrary powers of propagators:

$$J(\alpha_1, \alpha_2, m_1, m_2) = \int \frac{\mathrm{d}^n k}{(k^2 - m_1^2)^{\alpha_1} \left((k-p)^2 - m_2^2\right)^{\alpha_2}} .$$

- By constructing step-up and step down operators we obtain the system of partial differential equations of second order with two variables for $J(\alpha, \beta, m_1, m_2)$:

$$\theta_1 \left(-\alpha_1 + \frac{n}{2} + \theta_1\right) - \frac{(2\alpha_1 + 2\alpha_2 - n - 2\theta_1 - 2\theta_2)(\alpha_1 + \alpha_2 - n - \theta_1 - \theta_2 + 1)}{2z_1} = 0 ,$$

$$\theta_2 \left(-\alpha_2 + \frac{n}{2} + \theta_2\right) - \frac{(2\alpha_1 + 2\alpha_2 - n - 2\theta_1 - 2\theta_2)(\alpha_1 + \alpha_2 - n - \theta_1 - \theta_2 + 1)}{2z_2} = 0 .$$

- it is equivalent to the equation of Appell hypergeometric function $F_4(a, b, c_1, c_2, z_1, z_2)$ and has 4 different solutions. The singular locus on $\mathbf{P}^2$ is $z_1 = 0$, $z_2 = 0$, the line at infinity, $z_1^2 + z_2^2 + 1 = 2z_1 z_2 + 2z_1 + 2z_2$

One-Loop Two-point Diagram, different masses

- we choose $\chi = 4$, the number of derivatives w.r.t. x. In this case we need $\chi - \eta = 2$ differentiations w.r.t. sets of variables z_1, z_2 and obtain an Fuchsian differential equation over one variable

  $$L_4(x) J(\alpha_1, \alpha_2, m_1, m_2) = 0\,,$$

  where $L_4(x)$ is the differential operator of the fourth order.
- it has 4 singular points inherited from initial differential system, so the final answer could not be expressed in terms of hypergeometric function of one variable
- monodromy is reduced, is defined by $\{a, b, c_1 - a, c_1 - b, c_2 - a, c_2 - b, c_1 + c_2 - a, c_1 + c_2 - b\} \in Z$, and in our case we have $-b + c_1 + c_2 = 3$, so one solution of system degenerates to Puiseux-type and one-variable equation for F_4 must factorize by first-order differential operator
- $$L_1(x) L_3(x) J(\alpha_1, \alpha_2, m_1, m_2) = 0$$

  By defining arbitrary constants, we could see that final answer for FI with two different masses and arbitrary powers of propagators has only two F_4 terms for the variables $z_1 = p^2/m_2^2$, $z_2 = m_1^2/m_2^2$ and three terms in variables $z_1 = m_2^2/p^2$, $z_2 = m_2^2/p^2$

One-Loop Two-point Diagram, equal masses

- J is equivalent to a two-point FI with equal masses $m_1 = m_2$: $z_1 = y_1(x) = x$ and $z_2 = y_2(x) = x$.
- we could find an differential equation for the case of equal masses by two different ways: consider the case $z_1 = z_2 = x$ and $z_1 = x, z_2 = const = x$.
- this univariate specialization does not belong to singular locus, the rank of new differential system should be the same.

$$\tilde{L}_4(x) F_4(x, x) = 0$$

- we have three distinct poles at points $0, 1/4, \infty$. Compare the singular points and local exponents with differential equation for hypergeometric function ${}_4F_3$, we came to the well-known result for univariate specialization of F_4

$$F_4\left(\begin{array}{c} a, b \\ c_1, c_2 \end{array} \middle| x, x \right) = {}_4F_3\left(\begin{array}{c} a, b, \frac{c_1+c_2}{2}, \frac{c_1+c_2-1}{2} \\ c_1, c_2, c_1+c_2-1 \end{array} \middle| 4x \right).$$

One-Loop Two-point Diagram, equal masses II

- The monodromy group of initial differential equation is reduced due to equation for the parameters $-b + c_1 + c_2 = 3$, so we find the factorization of $\tilde{L}_4(x)$ if we substitute parameters:

$$L_1(x)L_3(x)J(\alpha_1, \alpha_2, m, m) = 0,$$

$$L_1(x) = \frac{\mathrm{d}}{dx} + \frac{((x-4)(-\alpha_1 - \alpha_2 + n) + 3x - 8)}{x(x-4)},$$

$$L_3(x) = \frac{\mathrm{d}^3}{dx^3} + \frac{-(x-8)(\alpha_1 + \alpha_2 - n - 3) + 2n + 18}{(x-4)x} \frac{\mathrm{d}^2}{dx^2}$$

$$- \frac{4\left((\alpha_1+\alpha_2)(5(\alpha_1+\alpha_2) - 8n + 1) + 3n^2\right) + x(2\alpha_1 - n - 2)(-2\alpha_2 + n + 2)}{4(x-4)x^2} \frac{\mathrm{d}}{dx}$$

$$+ \frac{(\alpha_1+\alpha_2 - n + 1)(\alpha_1 + \alpha_2 - n + 2)(2(\alpha_1+\alpha_2) - n)}{2(x-4)x^3},$$

- The final answer for two-point FI with equal masses could be expressed through hypergeometric function ${}_3F_2$ and polynomial expression.

Thank you for your Attention

thank you for your attention