Basso-Dixon diagrams

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Plan

- Basso-Dixon diagram in $d = 4$ and great determinant representation
  B.Basso, L.Dixon, *Gluing Ladder Feynman Diagrams into Fishnets,* Phys.Rev.Lett. 119 (2017) no.7, 071601.
- Graph building operator and commuting Q-operators
- Construction of eigenfunctions of the Q-operator for any d
  S.D., V.Kazakov, E.Olivucci, *Basso-Dixon Correlators in Two-Dimensional Fishnet CFT,* JHEP 1904 (2019) 032
  S.D., E.Olivucci, *Exactly solvable magnet of conformal spins in four dimensions,* Phys.Rev.Lett. 125 (2020) 3, 031603
  S.D., G.Ferrando, E.Olivucci, *Mirror channel eigenvectors of the d-dimensional fishnets,* e-Print: 2108.12620 [hep-th]

Based on the previous works S.D., G. Korchemsky, A. Manashov, *Noncompact Heisenberg spin magnets from high-energy QCD,* Nucl.Phys. B617 (2001) 375-440
S.D., A.Manashov, *Iterative construction of eigenfunctions of the monodromy matrix for SL(2,C) magnet,* J.Phys. A47 (2014) 305204

The whole story was initiated by the works L.N. Lipatov, *Asymptotic behavior of multicolor QCD at high energies in connection with exactly solvable spin models,* JETP Lett. 59 (1994) 596

L.D. Faddeev and G.P. Korchemsky, *High-energy QCD as a completely integrable model,* Phys. Lett. B 342 (1995) 311

L.N. Lipatov, *Integrability of scattering amplitudes in $N = 4$ SUSY,* J. Phys. A 42 (2009) 304020

Plan

- Heavily based on uniqueness method

  A. N. Vasiliev, Y. M. Pismak, Y. R. Khonkonen, *$1/N$ Expansion: Calculation of the exponent eta in the order $1/N^3$ by the conformal bootstrap method,* Theor. Math. Phys. 50 (1982) 127-134

  D.Kazakov,*The method of uniqueness, a new powerful technique for multiloop calculations,* Phys.Lett.133B,406(1983)

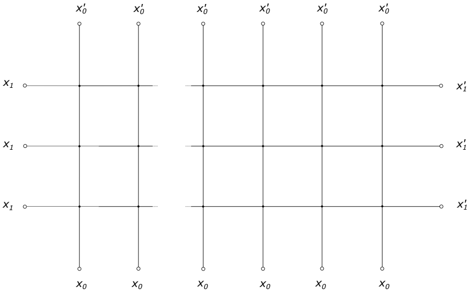
  and in fact the representation of separated variables is relative of the Gegenbauer polynomial technique

  K. Chetyrkin, A. Kataev and F. Tkachov, *New Approach to Evaluation of Multiloop Feynman Integrals: The Gegenbauer Polynomial x Space Technique,* Nucl. Phys. B 174, 345 (1980)

  A.V. Kotikov, *The Gegenbauer polynomial technique: The Evaluation of a class of Feynman diagrams,* Phys. Lett. B 375 (1996) 240

Basso-Dixon diagram

B.Basso, L.Dixon, *Gluing Ladder Feynman Diagrams into Fishnets,* Phys.Rev.Lett. 119 (2017) no.7, 071601.



N horizontal lines ($N = 3$ here), L vertical lines. Solid lines are the scalar propagators $1/(x-y)^2$ where x and y are the two endpoints of each segment. The boundary points are identified into four points (x_0, x_1, x_0', x_1').

$$I_{NL}(x_0, x_1, x_0', x_1') = \frac{(x_0 - x_0')^{-2(N+L)}}{(x_1 - x_0)^{2N}(x_1' - x_0)^{2N}} \left[\frac{(z\bar{z})^{\frac{1}{2}}}{z - \bar{z}}\right]^N I_{NL}(z, \bar{z})$$

$$u = \frac{x_{1'0}^2 x_{10'}^2}{x_{10}^2 x_{1'0'}^2} = z\bar{z} \quad ; \quad v = \frac{x_{11'}^2 x_{00'}^2}{x_{10}^2 x_{1'0'}^2} = (1-z)(1-\bar{z}) \, ,$$

Basso-Dixon great formula

Step one $\nu_k \in \mathbb{R}\,;\ell_k = 0,1,2,\ldots\,;k = 1,\ldots,N\ \ ;N < L+1$

$$\boldsymbol{\nu} = (\nu_1\,,\nu_2,\ldots,\nu_N)\quad;\quad \boldsymbol{\ell} = (\ell_1\,,\ell_2,\ldots,\ell_N)$$

$$I_{NL}(z,\bar{z}) = \sum_{\ell_1,\ldots,\ell_N}\int_{-\infty}^{+\infty}\frac{d\nu_1\cdots d\nu_N}{(2\pi)^N N!}\,\mu(\boldsymbol{\nu},\boldsymbol{\ell})\prod_{k=1}^{N}(\ell_k+1)\,\frac{z^{i\nu_k+\frac{\ell_k+1}{2}}\,\bar{z}^{i\nu_k-\frac{\ell_k+1}{2}}}{\left(\frac{(\ell_k+1)^2}{4}+\nu_k^2\right)^{N+L}}$$

$$\mu(\boldsymbol{\nu},\boldsymbol{\ell}) = \prod_{1\le i<k\le N}\left((\nu_i-\nu_k)^2+\frac{(\ell_i-\ell_k)^2}{4}\right)\left((\nu_i-\nu_k)^2+\frac{(\ell_i+\ell_k+2)^2}{4}\right)$$

Ladder diagramm $N = 1\,,L = p$

$$I_{1p}(z,\bar{z}) = \sum_{\ell}\int_{-\infty}^{+\infty}\frac{d\nu}{2\pi}\,(\ell+1)\,\frac{z^{i\nu+\frac{\ell+1}{2}}\,\bar{z}^{i\nu-\frac{\ell+1}{2}}}{\left(\frac{(\ell+1)^2}{4}+\nu^2\right)^{p+1}}$$

Basso-Dixon great formula

N. I. Usyukina and A. I. Davydychev, *Exact results for three and four point ladder diagrams with an arbitrary number of rungs,* Phys. Lett. B305, 136 (1993)

$$I_{1,p}(z,\bar{z}) = L_p(z,\bar{z}) = \sum_{j=0}^{p} \frac{(-1)^j (2p-j)!}{p!j!(p-j)!} \ln^j (z\bar{z}) \left(\mathrm{Li}_{2p-j}(z) - \mathrm{Li}_{2p-j}(\bar{z})\right)$$

with $\mathrm{Li}_k(z) = \sum_{\ell=1}^{+\infty} z^\ell/\ell^k$ – the polylogarithm.
Step two

$$I_{NL}(z,\bar{z}) = \frac{\det M}{\prod_{k=1}^{N}(L-N+2k-2)!(L-N+2k-1)!}$$

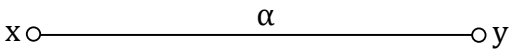
where M is a $N \times N$ Hankel matrix with ij element

$$M_{ij} = (L-N+i+j-2)!(L-N+i+j-1)! \times L_{L-N+i+j-1}(z,\bar{z})$$

B.Basso, L.Dixon, D.Kosower, A.Krajenbrink, De-liang Zhong, *Fishnet four-point integrals: integrable representations and thermodynamic limits,* JHEP 07 (2021) 168

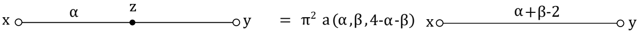
Simple diagrammatic rules

- The function $(x-y)^{-2\alpha} = ((x-y)^{\mu}(x-y)_{\mu})^{-\alpha}$ is represented by the line with index α connecting points x and y



- Chain rule $a(\alpha) = \frac{\Gamma(2-\alpha)}{\Gamma(\alpha)}$ and $a(\alpha, \beta, \cdots, \gamma) = a(\alpha)\, a(\beta) \cdots a(\gamma)$.

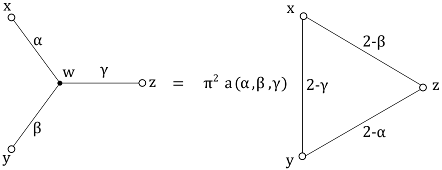
$$\int d^4 z \frac{1}{(x-z)^{2\alpha}(z-y)^{2\beta}} = \pi^2 a(\alpha, \beta, 4-\alpha-\beta) \frac{1}{(x-y)^{2(\alpha+\beta-2)}} ,$$



- Star-triangle relation $\alpha + \beta + \gamma = 4$

$$\int d^4 w \frac{1}{(x-w)^{2\alpha}(y-w)^{2\beta}(z-w)^{2\gamma}} = \frac{\pi^2 a(\alpha, \beta, \gamma)}{(y-z)^{2(2-\alpha)}(z-x)^{2(2-\beta)}(x-y)^{2(2-\gamma)}}$$

where $a(\alpha) = \frac{\Gamma(2-\alpha)}{\Gamma(\alpha)}$ and $a(\alpha, \beta, \cdots, \gamma) = a(\alpha)\, a(\beta) \cdots a(\gamma)$.



Operator V_N

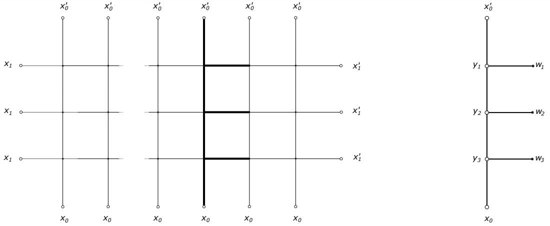
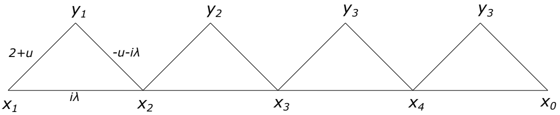
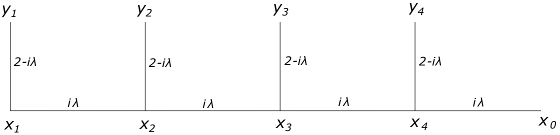
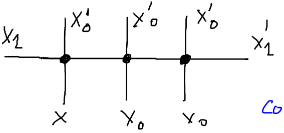


Diagram $\leftrightarrow V_N^{L+1}$



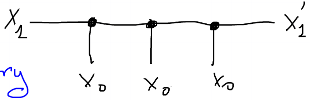


- The diagram for the Q-operator $Q_N(u)$: one-parametric family of commuting operators $[Q_N(u), Q_N(v)] = 0$.
- Reduction of the diagram: $u \to -i\lambda$ and $\lambda \to -i$ gives $Q_N(u) \to V_N$.



$x_0' \to \infty$

conformal symmetry



Operator reformulation

$$[V\varphi](x) = \int d^4y \, \frac{\varphi(y)}{(x-x_0)^2 (x-y)^2}$$

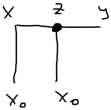
Integral kernel

$$V(x,y) = \frac{1}{(x-x_0)^2 (x-y)^2} =$$

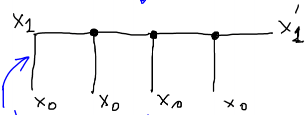
Kernel for operator V

convolution

$$V^2(x,y) = \int d^4z \, V(x,z) V(z,y) =$$



$V^4(x_1, x_1')$



amputation → Ladder

Spectral representation for operator V

Eigenfunctions $\Psi_{\nu,\ell}(x) = \dfrac{(x-x_0)^{\mu_1\ldots\mu_\ell}}{(x-x_0)^{2\left(1+i\nu+\frac{\ell}{2}\right)}}$

$x^{\mu_1\ldots\mu_\ell} = x^{\mu_1} x^{\mu_2} \ldots x^{\mu_\ell}$ – traces

$$\frac{1}{(x-x_0)^2}\int d^4y \, \frac{1}{(x-y)^2} \cdot \frac{(y-x_0)^{\mu_1\ldots\mu_\ell}}{(y-x_0)^{2\left(1+i\nu+\frac{\ell}{2}\right)}} = \lambda(\nu,\ell)\,\frac{(x-x_0)^{\mu_1\ldots\mu_\ell}}{(x-x_0)^{2\left(1+i\nu+\frac{\ell}{2}\right)}}$$

$$\lambda(\nu,\ell) = \frac{\pi^2}{\left(1-i\nu+\frac{\ell}{2}\right)\left(i\nu+\frac{\ell}{2}\right)}$$

Orthogonality of eigenfunctions

$$\int d^4x \, \frac{(x-x_0)^{\alpha_1\ldots\alpha_\ell}}{(x-x_0)^{2\left(1+i\nu+\frac{\ell}{2}\right)}} \, \frac{(x-x_0)^{\beta_1\ldots\beta_{\ell'}}}{(x-x_0)^{2\left(1-i\nu'+\frac{\ell'}{2}\right)}} = \frac{2^{1-\ell}\pi^3}{\ell+1}\,\delta_{\ell\ell'}\,\delta(\nu-\nu')\,\Pi^{\alpha_1\ldots\alpha_\ell}_{\beta_1\ldots\beta_\ell}$$

Completeness of eigenfunctions

$$\sum_{\ell\geq 0} \frac{\ell+1}{2^{1-\ell}\pi^3}\int_{-\infty}^{+\infty} d\nu \, \frac{(x-x_0)^{\mu_1\ldots\mu_\ell}}{(x-x_0)^{2\left(1+i\nu+\frac{\ell}{2}\right)}} \, \frac{(y-x_0)^{\mu_1\ldots\mu_\ell}}{(y-x_0)^{2\left(1-i\nu+\frac{\ell}{2}\right)}} = \delta^4(x-y)$$

Spectral decomposition in bra-ket notations

$|n\rangle$ – eigenvector $\quad V|n\rangle = \lambda(n)\,|n\rangle$

completeness $\longrightarrow$ resolution of unity $\quad \mathbb{1} = \sum_n |n\rangle\langle n|$

$$V = V\cdot\mathbb{1} = V\sum_n |n\rangle\langle n| = \sum_n \lambda_n |n\rangle\langle n|$$

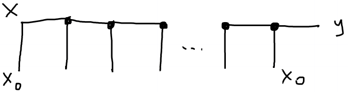
$$V^L = \sum_n \lambda_n^L\,|n\rangle\langle n|$$

$$\langle x|n\rangle \longleftrightarrow \frac{(x-x_0)^{\mu_1-\mu_\ell}}{(x-x_0)^{2(1+i\nu+\frac{\ell}{2})}}$$

$$\langle x|V^L|y\rangle = \sum_n \lambda_n^L\,\langle x|n\rangle\langle n|y\rangle$$

$$\langle n|y\rangle \longleftrightarrow \frac{(y-x_0)^{\mu_1-\mu_\ell}}{(y-x_0)^{2(1-i\nu+\frac{\ell}{2})}}$$

$$V^L(x,y) = \sum_{\ell\geq 0}\frac{\ell+1}{2^{1-\ell}\pi^3}\int_{-\infty}^{+\infty}d\nu\left(\frac{\pi^2}{(1-i\nu+\frac{\ell}{2})(i\nu+\frac{\ell}{2})}\right)^L \frac{(x-x_0)^{\mu_r\cdot\mu_n}\,(y-x_0)^{\mu_r\cdot\mu_n}}{(x-x_0)^{2(1+i\nu+\frac{\ell}{2})}\,(y-x_0)^{2(1-i\nu+\frac{\ell}{2})}}$$



Diagonalization of the operator $V_N \leftrightarrow$ Transition to the representation of separated variables

E. K. Sklyanin, *Separation of variables - new trends,* Prog.Theor.Phys.Suppl. 118 (1995) 35-60

- Spectrum of operator V_N is continuous
- Eigenfunctions $\nu_k \in \mathbb{R}\,, \ell_k\,, m_k \in \mathbb{Z}_+$

$$\Psi_{\nu_1,\ell_1,m_1\,,\dots\,,\nu_N,\ell_N,m_N}(x_1\,,\dots\,,x_N) = |\nu_1,\ell_1,m_1\,,\dots\,,\nu_N,\ell_N,m_N\rangle = |\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}\rangle$$

$$V_N\,|\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}\rangle = \lambda(\nu_1,\ell_1)\cdots\lambda(\nu_N,\ell_N)\,|\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}\rangle$$

One point example

$$[V_1\,\Phi]\,(x) \leftrightarrow (x-x_0)^{-2}\int d^4y\,(x-y)^{-2}\,\Phi(y)\,.$$

$$\Psi_{\nu,\ell,m}(x) = \frac{(x-x_0)^{\mu_1\cdots\mu_\ell}}{(x-x_0)^{2(1+i\nu+\frac{\ell}{2})}} \quad ; \quad \lambda(\nu,\ell) = \frac{\pi^2}{\left(1-i\nu+\frac{\ell}{2}\right)\left(i\nu+\frac{\ell}{2}\right)}$$

$$\int d^4x\,\frac{x^{\alpha_1\dots\alpha_\ell}}{x^{2(1+\ell/2+i\nu)}}\,\frac{x^{\beta_1\dots\beta_{\ell'}}}{x^{2(1+\ell'/2-i\nu')}} = c_\ell\delta_{\ell\ell'}\delta\left(\nu-\nu'\right)P^{\alpha_1\dots\alpha_\ell}_{\beta_1\dots\beta_\ell}$$

$$\sum_{\ell\geq 0}\frac{1}{c_\ell}\int_{\mathbb{R}} d\nu\,\frac{x^{\mu_1\dots\mu_\ell}}{x^{2(1+\ell/2+i\nu)}}\,\frac{y^{\mu_1\dots\mu_\ell}}{y^{2(1+\ell/2-i\nu)}} = \delta^{(4)}\,(x-y) \quad ; \quad c_\ell = \frac{\pi^3}{2^{\ell-1}\,(\ell+1)},$$

Eigenfunctions

$$V_N \, |\boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m}\rangle = \lambda(\nu_1, \ell_1) \cdots \lambda(\nu_N, \ell_N) \, |\boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m}\rangle$$

- Iterative construction

$$\Psi_{\nu_1,\ell_1,m_1\,,\ldots\,,\nu_N,\ell_N,m_N}(x_1\,,\,\ldots\,,x_N) = \Lambda^{(N)}_{\nu_N,\ell_N,m_N} \cdots \Lambda^{(2)}_{\nu_2,\ell_2,m_2} \Psi_{\nu_1,\ell_1,m_1}$$

- Main commutation relation

$$V_N \, \Lambda^{(N)}_{\nu_N,\ell_N,m_N} = \lambda(\nu_N, \ell_N) \, \Lambda^{(N)}_{\nu_N,\ell_N,m_N} \, V_{N-1}$$

$$V_N \, \Lambda^{(N)} \Lambda^{(N-1)} \cdots \Lambda^{(2)} \Psi_{\nu_1,\ell_1,m_1} = \lambda(\nu_N, \ell_N) \, \Lambda^{(N)} \, V_{N-1} \, \Lambda^{(N-1)} \cdots \Lambda^{(2)} \Psi_{\nu_1,\ell_1,m_1} =$$

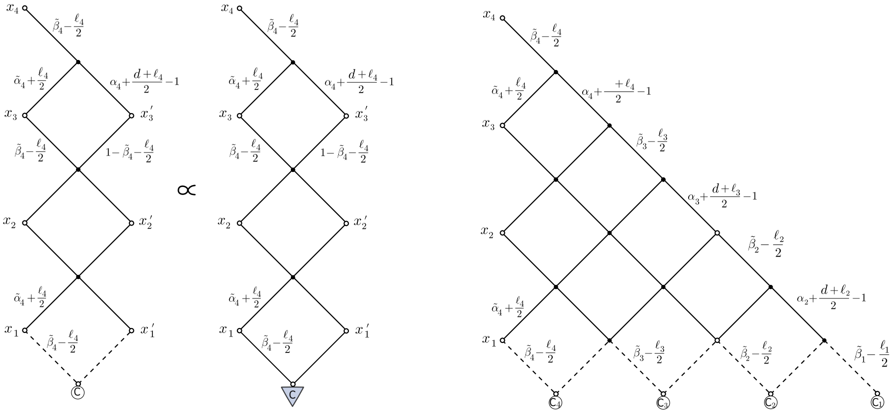
$$\lambda(\nu_N, \ell_N) \, \lambda(\nu_{N-1}, \ell_{N-1}) \, \Lambda^{(N)} \Lambda^{(N-1)} \, V_{N-1} \, \cdots \Lambda^{(2)}_{\nu_2,\ell_2,m_2} \Psi_{\nu_1,\ell_1,m_1} = \ldots$$

- Symmetry $\leftrightarrow$ Faddeev-Zamolodchikov algebra

$$\Lambda^{(N)}_{\nu,\ell,m} \, \Lambda^{(N-1)}_{\nu',\ell',m'} = R^{nn'}_{mm'}(\nu - \nu') \, \Lambda^{(N)}_{\nu',\ell',n'} \, \Lambda^{(N-1)}_{\nu,\ell,n}$$

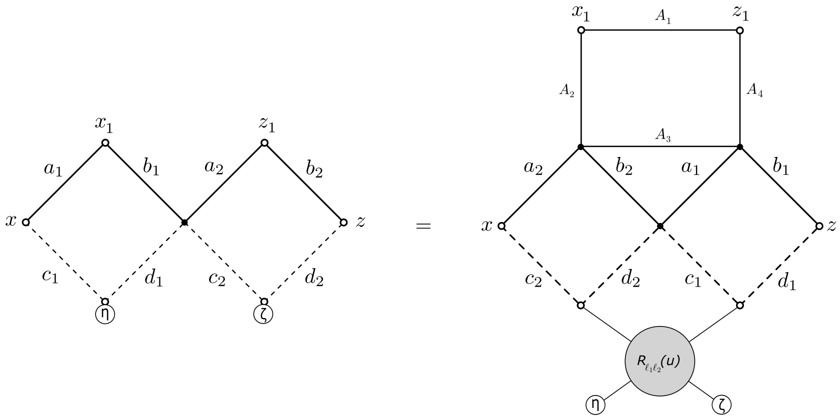
where $R(\nu - \nu')$ – R-matrix, solution of the Yang-Baxter equation

Eigenfunctions



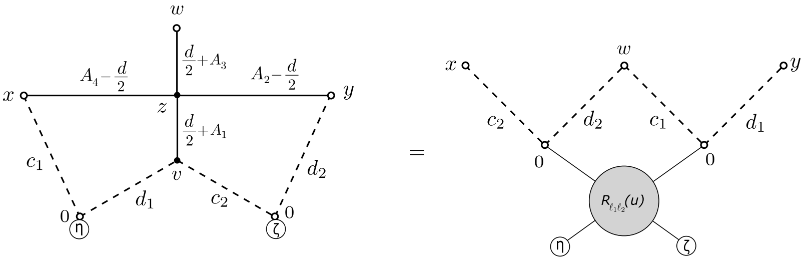
Layer operators and construction of eigenfunction.

Eigenfunctions



Interchange relation $\leftrightarrow$ symmetry of eigenfunctions $\leftrightarrow$ Faddeev-Zamolodchikov algebra

Eigenfunctions



For simplicity $x_0 \to 0$

$$\int \mathrm{d}^d z \mathrm{d}^d v \frac{(z-w)^{2\left(iu+\frac{\ell_1+\ell_2}{2}-1\right)} \left[\zeta \cdot \left(\frac{y}{y^2}-\frac{v}{v^2}\right)\right]^{\ell_1} \left[\eta \cdot \left(\frac{x}{x^2}-\frac{v}{v^2}\right)\right]^{\ell_2}}{(z-x)^{2\left(iu+\frac{\ell_{21}}{2}\right)} (z-y)^{2\left(iu+\frac{\ell_{12}}{2}\right)} (z-v)^{2\left(d-1+\frac{\ell_1+\ell_2}{2}-iu\right)} v^{2\left(1-\frac{\ell_1+\ell_2}{2}+iu\right)}}$$

$$\leftrightarrow \frac{w^{2\left(iu+\frac{\ell_1+\ell_2}{2}-1\right)}}{x^{2\left(iu+\frac{\ell_{21}}{2}\right)} y^{2\left(iu+\frac{\ell_{12}}{2}\right)}} \left[R_{\ell_1,\ell_2}(u)\zeta^{\otimes \ell_1} \otimes \eta^{\otimes \ell_2}\right] \cdot \left[\left(\frac{x}{x^2}-\frac{w}{w^2}\right)^{\otimes \ell_1} \otimes \left(\frac{y}{y^2}-\frac{w}{w^2}\right)^{\otimes \ell_2}\right]$$

Representation of separated variables

- Orthogonality

$$\langle \boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m} | \boldsymbol{\nu}', \boldsymbol{\ell}', \boldsymbol{m}' \rangle = \mu^{-1}(\boldsymbol{\nu}, \boldsymbol{\ell})\, \delta(\boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m} | \boldsymbol{\nu}', \boldsymbol{\ell}', \boldsymbol{m}')$$

$$\Lambda^{(N)\dagger}_{\nu',\ell',m'}\, \Lambda^{(N)}_{\nu,\ell,m} \quad = \quad \frac{R^{nm'}_{mn'}(\nu-\nu')\, \Lambda^{(N-1)}_{\nu,\ell,n}\, \Lambda^{(N-1)\dagger}_{\nu',\ell',n'}}{\left[(\nu-\nu')^2 + \frac{(\ell-\ell')^2}{4}\right]\left[(\nu-\nu')^2 + \frac{(d-2+\ell+\ell')^2}{4}\right]}$$

- Completeness

$$\sum_{\boldsymbol{\ell},\boldsymbol{m}} \int d^N\boldsymbol{\nu}\, \mu(\boldsymbol{\nu}, \boldsymbol{\ell})\, |\boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m}\rangle\langle \boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m}| = \mathbb{1}$$

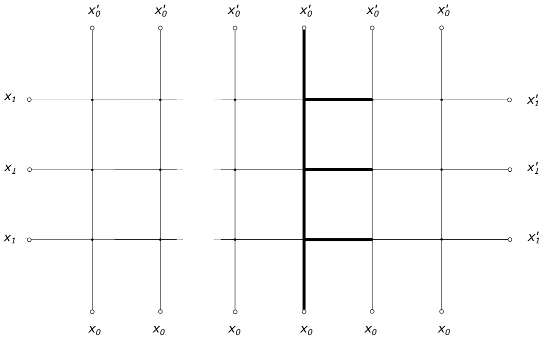
$$\mu(\boldsymbol{\nu}, \boldsymbol{\ell}) \;=\; \prod_{1\le i<k\le N} \left((\nu_i-\nu_k)^2 + \frac{(\ell_i-\ell_k)^2}{4}\right)\left((\nu_i-\nu_k)^2 + \frac{(\ell_i+\ell_k+d-2)^2}{4}\right)$$

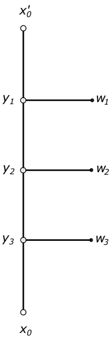
- Spectral decomposition

$$V^{L+1} = \sum_{\boldsymbol{\ell},\boldsymbol{m}} \int d^N\boldsymbol{\nu}\, \mu(\boldsymbol{\nu}, \boldsymbol{\ell})\, \lambda^{L+1}(\nu_1, \ell_1)\cdots\lambda^{L+1}(\nu_N, \ell_N)\, |\boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m}\rangle\langle \boldsymbol{\nu}, \boldsymbol{\ell}, \boldsymbol{m}|$$

Representation of separated variables

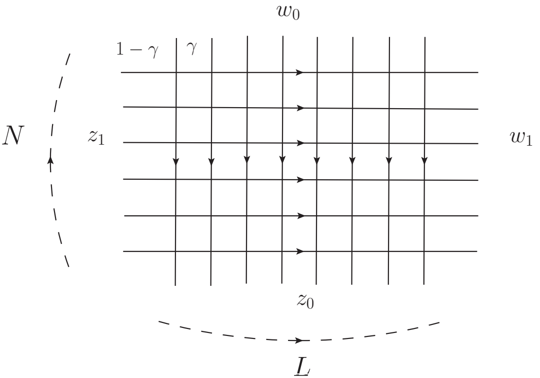
$$V^{L+1}\left(x_1\,,\dots\,,x_N\,,x'_1\,,\dots\,,x'_N\right)=\sum_{\boldsymbol{\ell},\boldsymbol{m}}\int d^N\boldsymbol{\nu}\,\mu(\boldsymbol{\nu},\boldsymbol{\ell})\,\lambda^{L+1}(\nu_1,\ell_1)\cdots\lambda^{L+1}(\nu_N,\ell_N)$$
$$\overline{\Psi}_{\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}}\left(x_1\,,\dots\,,x_N\right)\Psi_{\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}}\left(x'_1\,,\dots\,,x'_N\right)$$



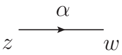


- For the functions $\overline{\Psi}_{\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}}\left(x_1\,,\dots\,,x_N\right)\leftrightarrow$ amputation of the left vertical lines and then reduction $x_k\to x_1$
- For the functions $\Psi_{\boldsymbol{\nu},\boldsymbol{\ell},\boldsymbol{m}}\left(x'_1\,,\dots\,,x'_N\right)\leftrightarrow$ reduction $x'_k\to x'_1$

Two-dimensional Basso-Dixon diagram



$$\eta = \frac{z_0 - w_1}{z_0 - z_1} \frac{w_0 - z_1}{w_0 - w_1}$$

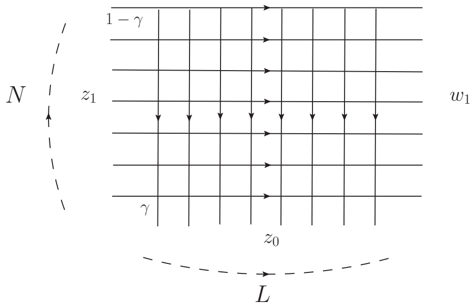


$$= \frac{1}{[w-z]^\alpha}$$

The propagator in $d = 2$ is given by the following expression $(\alpha - \bar{\alpha} \in Z)$

$$\frac{1}{[z-w]^\alpha} \equiv \frac{1}{(z-w)^\alpha (\overline{z} - \overline{w})^{\bar{\alpha}}} = \frac{(\overline{z} - \overline{w})^{\alpha - \bar{\alpha}}}{|z-w|^{2\alpha}}$$

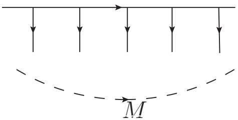
Two-dimensional Basso-Dixon diagram



$$\eta = \frac{z_0 - w_1}{z_0 - z_1}\frac{w_0 - z_1}{w_0 - w_1} \rightarrow \eta = \frac{z_0 - w_1}{z_0 - z_1}$$

$$w_0 \rightarrow \infty$$

$$I_{M+1}(\eta, \bar{\eta})$$



$$I_{L,N}(\eta, \bar{\eta}) \leftrightarrow \underset{1 \le j,k \le N}{Det} \left[(\eta \partial_\eta)^{i-1} (\bar{\eta} \partial_{\bar{\eta}})^{k-1} I_{N+L}(\eta, \bar{\eta}) \right]$$

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Step one

$$I_{L,N}(\eta,\bar\eta) \leftrightarrow \int \mathcal{D}x_1 \cdots \mathcal{D}x_N \prod_{k<j}[x_k - x_j] \prod_{k=1}^{N} \left([\eta]^{ix_k} \lambda^{N+L}(x_k)\right)$$

where

$$ix = \frac{n}{2} + i\nu \ , \ \int \mathcal{D}x = \sum_{n\in Z} \int_{-\infty}^{+\infty} d\nu$$

Step two – Determinant representation

$$\int \mathcal{D}x_1 \cdots \mathcal{D}x_N \prod_{k<j}[x_k - x_j] \prod_{k=1}^{N} [\eta]^{ix_k} \lambda^{N+L}(x_k) = N!\, Det\, M$$

$$M_{ik} = \int \mathcal{D}x\ x^{i-1}\bar{x}^{j-1}[\eta]^{ix} \lambda^{N+L}(x) = (\eta\partial_\eta)^{i-1}(\bar\eta\partial_{\bar\eta})^{k-1} \int \mathcal{D}x\,[\eta]^{ix} \lambda^{N+L}(x)$$
$$i,k = 1,\dots,N$$

Conclusions

- Calculation of diagrams $\leftrightarrow$ Sklyanin SOV method from the theory of integrable spin chains
- Simpler proof of the zig-zag conjecture

  D.J. Broadhurst, D. Kreimer, *Knots and numbers in ϕ^4 theory to 7 loops and beyond,* Int. J. Mod. Phys. C 6, 519 (1995)

  F.C.S. Brown, O. Schnetz, *Single-valued multiple polylogarithms and a proof of the zig-zag conjecture,* Jour.of Numb. Theory 148, 478-506 (2015)

  S. Derkachov, A.P. Isaev, L. Shumilov, *Conformal triangles and zig-zag diagrams,* Phys.Lett.B 830 (2022) 137150
- Clarification of connections between the methods of the above works. At first glance, everything looks completely different.