Invariant Differential Operators : Recent Developments

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Exceptional Lie algebra $\boldsymbol{G_2}$

1 Maxwell equations hierarchy

It is well known that Maxwell equations

$$\partial^\mu F_{\mu\nu} = J_\nu \tag{1a}$$
$$\partial^\mu\, {}^*F_{\mu\nu} = 0 \tag{1b}$$

(where ${}^*F_{\mu\nu} \equiv \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, $\epsilon_{\mu\nu\rho\sigma}$ being totally antisymmetric with $\epsilon_{0123} = 1$), or, equivalently

$$\begin{aligned} \partial_k E_k &= J_0\ (= 4\pi\rho), \quad \partial_0 E_k - \varepsilon_{k\ell m}\partial_\ell H_m = J_k\ (= -4\pi j_k), \\ \partial_k H_k &= 0\,, \quad \partial_0 H_k + \varepsilon_{k\ell m}\partial_\ell E_m = 0\,, \end{aligned} \tag{2}$$

where $E_k \equiv F_{k0}$, $H_k \equiv (1/2)\varepsilon_{k\ell m}F_{\ell m}$, may be rewritten in the following manner:

$$\partial_k F_k^\pm = J_0\,, \quad \partial_0 F_k^\pm \pm i\varepsilon_{k\ell m}\partial_\ell F_m^\pm = J_k\,, \tag{3}$$

where

$$F_k^\pm \equiv E_k \pm iH_k\ . \tag{4}$$

Not so well known is the fact that the eight equations in (3) may be rewritten as two conjugate scalar equations in the following way:

$$I^+\ F^+(z) = J(z,\bar z)\,, \tag{5a}$$
$$I^-\ F^-(\bar z) = J(z,\bar z)\,, \tag{5b}$$

where

$$I^+ = \bar z\partial_+ + \partial_v - \tfrac{1}{2}\Big(\bar z z\partial_+ + z\partial_v + \bar z\partial_{\bar v} + \partial_-\Big)\partial_z\,, \tag{6a}$$
$$I^- = z\partial_+ + \partial_{\bar v} - \tfrac{1}{2}\Big(\bar z z\partial_+ + z\partial_v + \bar z\partial_{\bar v} + \partial_-\Big)\partial_{\bar z}\,, \tag{6b}$$

$$x_\pm \equiv x_0 \pm x_3, \quad v \equiv x_1 - ix_2, \quad \bar v \equiv x_1 + ix_2, \tag{7a}$$
$$\partial_\pm \equiv \partial/\partial x_\pm, \quad \partial_v \equiv \partial/\partial v, \quad \partial_{\bar v} \equiv \partial/\partial \bar v, \tag{7b}$$

$$F^+(z) \equiv z^2(F_1^+ + iF_2^+) - 2zF_3^+ - (F_1^+ - iF_2^+)\,, \tag{8a}$$
$$F^-(\bar z) \equiv \bar z^2(F_1^- - iF_2^-) - 2\bar z F_3^- - (F_1^- + iF_2^-)\,, \tag{8b}$$
$$\begin{aligned} J(z,\bar z) &\equiv \bar z z(J_0 + J_3) + z(J_1 + iJ_2) + \bar z(J_1 - iJ_2) + (J_0 - J_3) = \\ &\equiv \bar z z J_+ + zJ_v + \bar z J_{\bar v} + J_- \end{aligned} \tag{8c}$$

It is easy to recover (3) from (5) - just note that both sides of each equation are first order polynomials in each of the two variables z and $\bar{z}$, then comparing the independent terms in (5) one gets at once (3).

Writing the Maxwell equations in the simple form (5) has also important conceptual meaning. The point is that each of the two scalar operators I^+, I^- is indeed a single object, namely it is an intertwiner of the conformal group, or conformally invariant differential operator, while the individual components in (1) - (3) do not have this interpretation. This is also the simplest way to see that the Maxwell equations are conformally invariant, since this is equivalent to the intertwining property.

Let us be more explicit. The physically relevant representations T^χ of the 4-dimensional conformal algebra $so(4,2) = su(2,2)$ may be labelled by $\chi = [n_1, n_2; d]$, where n_1, n_2 are non-negative integers fixing finite-dimensional irreducible representations of the Lorentz subalgebra, (the dimension being $(n_1+1)(n_2+1)$), and d is the conformal dimension (or energy). (In the literature these Lorentz representations are labelled also by $(j_1, j_2) = (n_1/2, n_2/2)$.) Then the intertwining properties of the operators in (6) are given by:

$$I^+ \ : \ C^+ \longrightarrow C^0 , \quad I^+ \circ T^+ \ = \ T^0 \circ I^+ , \tag{9a}$$
$$I^- \ : \ C^- \longrightarrow C^0 , \quad I^- \circ T^- \ = \ T^0 \circ I^- , \tag{9b}$$

where $T^a = T^{\chi^a}$, $a = 0, +, -$, $C^a = C^{\chi^a}$ are the representation spaces, and the signatures are given explicitly by:

$$\chi^+ = [2, 0; 2] , \quad \chi^- = [0, 2; 2] , \quad \chi^0 = [1, 1; 3] , \tag{10}$$

as anticipated. Indeed, $(n_1, n_2) = (1, 1)$ is the four-dimensional Lorentz representation, (carried by J_μ above), and $(n_1, n_2) = (2, 0), (0, 2)$ are the two conjugate three-dimensional Lorentz representations, (carried by $F^\pm_k$ above), while the conformal dimensions are the canonical dimensions of a current $(d = 3)$, and of the Maxwell field $(d = 2)$. We see that the variables $z, \bar{z}$ are related to the spin properties and we shall call them 'spin variables'.

It is also important that the variables $x_\pm, v, \bar{v}, z, \bar{z}$ have definite group-theoretical meaning, namely, they are six local coordinates on the coset $\mathcal{Y} = SL(4)/B$, where B is the Borel subgroup of $SL(4)$ consisting of all upper diagonal matrices. (Equally well one may take the coset $SL(4)/B^-$, where B^- is the Borel subgroup of lower diagonal matrices.) Under the natural conjugation (cf. also below) this is also a coset of the conformal group $SU(2,2)$.

Now we recollect that closely related to the above fields is the potential A_μ with signature

$$\tilde{\chi}^0 = [1,1;1] \tag{11}$$

so that the analog of (1a) is

$$\partial_\mu A_\nu = F_{\mu\nu} \tag{12}$$

(not forgetting that the RHS is only a subspace). We also recall that there are two more conformal operators involving two scalar fields with signatures:

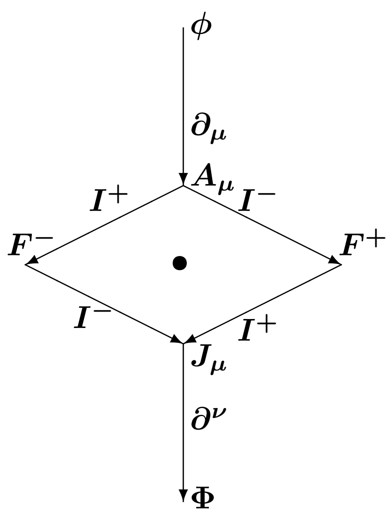
$$\phi = [0,0;0], \quad \Phi = [0,0;4] \tag{13}$$

so that

$$\partial \mu \phi = A_\mu, \quad \partial^\mu J_\mu = \Phi \tag{14}$$

(again the RHSs are subspaces).

Altogether we have the following picture:


ϕ
∂_μ
A_μ
I^+
I^-
F^-
F^+
I^-
I^+
J_μ
∂^ν
Φ

Remark: Note that the $\pm$ pairs are related by integral operators I_{KS}, so-called Knapp-Stein operators, with kernels which are conformal two-point functions. Their action on the signatures is:

$$I_{KS} \ : \ [n_1, n_2; d] \ \longrightarrow \ [n_2, n_1; 4-d] \tag{15}$$

The above picture is the simplest occurrence of conformally invariant differential operators. The general case is given by a 3-parameter generalization given as follows:

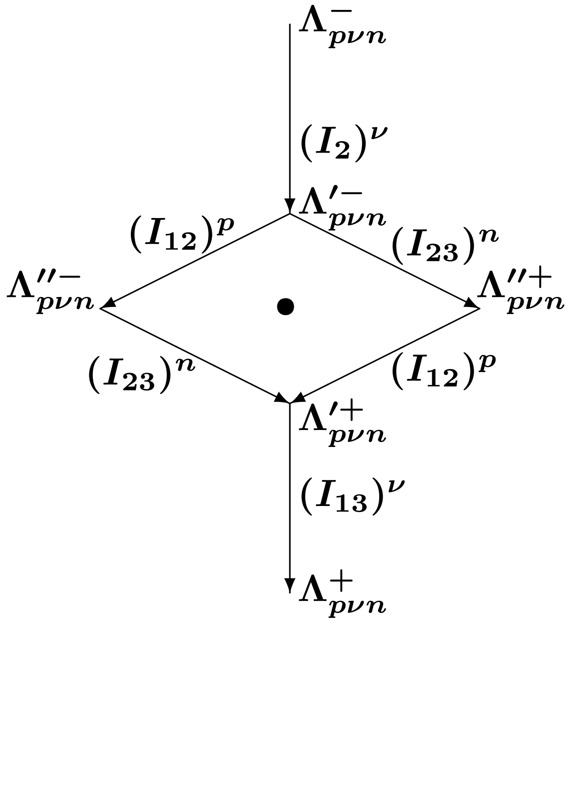
$$\begin{aligned}
\chi^-_{p\nu n} &= [p-1, n-1;\ 2-\nu-\tfrac{1}{2}(p+n)] \quad (\phi) \\
\chi^+_{p\nu n} &= [n-1, p-1;\ 2+\nu+\tfrac{1}{2}(p+n)] \quad (\Phi) \\
\chi'^-_{p\nu n} &= [p+\nu-1, n+\nu-1;\ 2-\tfrac{1}{2}(p+n)] \quad (A_\mu) \\
\chi'^+_{p\nu n} &= [n+\nu-1, p+\nu-1;\ 2+\tfrac{1}{2}(p+n)] \quad (J_\mu) \\
\chi''^-_{p\nu n} &= [\nu-1, p+n+\nu-1;\ 2+\tfrac{1}{2}(p-n)] \quad (F^-) \\
\chi''^+_{p\nu n} &= [p+n+\nu-1, \nu-1;\ 2+\tfrac{1}{2}(n-p)] \quad (F^+)
\end{aligned} \tag{16}$$

where p, ν, n are positive integers which are exactly the Dynkin labels m_1, m_2, m_3 of $sl(4)$ for $\chi^-_{p\nu n}$.

We call "multiplets" such collection of representations related by intertwining differential operators.

The simplest example we considered first is obtained for $p = \nu = n = 1$.

The multiplets (sextets here) are given now in the following figure:


$\Lambda^{-}_{p\nu n}$
$(I_2)^{\nu}$
$\Lambda'^{-}_{p\nu n}$
$(I_{12})^{p}$
$(I_{23})^{n}$
$\Lambda''^{-}_{p\nu n}$
$\Lambda''^{+}_{p\nu n}$
$(I_{23})^{n}$
$(I_{12})^{p}$
$\Lambda'^{+}_{p\nu n}$
$(I_{13})^{\nu}$
$\Lambda^{+}_{p\nu n}$

where the differential operators are given explicitly by:

$$
\begin{aligned}
(I_2)^m &= (\bar z_1 z_1 \partial_+ + z_1 \bar z_2 \partial_v + \bar z_1 z_2 \partial_{\bar v} + \bar z_2 z_2 \partial_-)^m = \\
&= \left((\bar z_1, \bar z_2)\ \sigma^\mu \partial_\mu \ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right)^m , && (17a)\\
(I_{12})^m &= \left((\bar z_1, \bar z_2)\ \sigma^\mu \partial_\mu \, \varepsilon \ \begin{pmatrix} \partial_{z_1} \\ \partial_{z_2} \end{pmatrix} \right)^m , && (17b)\\
(I_{23})^m &= \left((\partial_{\bar z_1}, \partial_{\bar z_2})\ \varepsilon \sigma^\mu \partial_\mu \ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right)^m , && (17c)\\
(I_{13})^m &= \left((\partial_{\bar z_1}, \partial_{\bar z_2})\ \sigma^\mu \partial_\mu \ \begin{pmatrix} \partial_{z_1} \\ \partial_{z_2} \end{pmatrix} \right)^m , && (17d)
\end{aligned}
$$

where σ_μ are the Pauli matrices, $\varepsilon = i\sigma_2$. Note that here for the finite-dimensional irreps of the Lorentz subalgebra we have passed from polynomials in $z, \bar z$ of degrees n_1, n_2, to homogeneous polynomials in z_1, z_2 of degree n_1 and in $\bar z_1, \bar z_2$ of degree n_2. The two realizations are easily related via $z = z_1/z_2$, $\bar z = \bar z_1/\bar z_2$.

The above picture is valid also for the 4-dimensional Euclidean conformal algebra $so(5,1)$, and also for the Lie algebra $so(3,3)$.

Next we recall that the conformal algebra of 2n-dimensional Minkowski space-time is the algebra $so(2n,2)$. Actually we shall consider a more general picture, namely, the Lie algebras $\mathcal{G} = so(p,q)$.

The analogue of the Lorentz subalgebra is:

$$\mathcal{M} = so(p-1, q-1) . \tag{18}$$

The analogue of Minkowski space-time is $\mathcal{N}$ with:

$$\dim \mathcal{N} = p + q - 2 . \tag{19}$$

We label the signature of the representations of $\mathcal{G}$ as follows:

$$
\begin{aligned}
\chi &= \{ n_1 , \dots , n_h ; c \} , && (20)\\
& n_j \in \mathbb{Z}/2 , \quad c = d - \tfrac{p+q-2}{2} , \quad h \equiv [\tfrac{p+q-2}{2}], \\
& |n_1| < n_2 < \cdots < n_h , \quad p+q \ \text{ even} , \\
& 0 < n_1 < n_2 < \cdots < n_h , \quad p+q \ \text{ odd} ,
\end{aligned}
$$

where the last entry of χ labels the characters of $\mathcal{A}$, and the first h entries are labels of the finite-dimensional nonunitary irreps of $\mathcal{M} = so(p-1, q-1)$.

The reason to use the parameter c instead of d will become clear below.

The analogue of the multiplets in (16) here is:

$$\begin{aligned}
\chi_1^{\pm} &= \{\epsilon\, n_1\,, \dots,\, n_h\,;\, \pm n_{h+1}\}\ , \\
&\qquad n_h < n_{h+1}\ , \\
\chi_2^{\pm} &= \{\epsilon\, n_1\,, \dots,\, n_{h-1}\,,\, n_{h+1}\,;\, \pm n_h\} \\
\chi_3^{\pm} &= \{\epsilon\, n_1, \dots, n_{h-2}, n_h, n_{h+1}\,;\, \pm n_{h-1}\} \\
&\dots \\
\chi_{h-1}^{\pm} &= \{\epsilon\, n_1\,, n_2\,, n_4\,, \dots,\, n_h\,,\, n_{h+1}\,;\, \pm n_3\} \\
\chi_h^{\pm} &= \{\epsilon\, n_1\,, n_3\,, \dots,\, n_h\,,\, n_{h+1}\,;\, \pm n_2\} \\
\chi_{h+1}^{\pm} &= \{\epsilon\, n_2\,, n_3\,, \dots,\, n_h\,,\, n_{h+1}\,;\, \pm n_1\} \\
\epsilon &= \begin{cases} \pm\,, & p+q \ \ even \\ 1, & p+q \ \ odd \end{cases}
\end{aligned} \tag{21}$$

where $\epsilon = \pm$ is correlated with $\chi^{\pm}$, the last entry is the value of c. Clearly, the multiplets correspond 1-to-1 to the finite-dimensional irreps of $so(p+q,\mathbb{C})$ with signature $\{n_1,\dots,n_h,n_{h+1}\}$ and we are able to use previous results due to so called "parabolic relation" between the $so(p,q)$ algebras for $p+q$ -fixed.

Note that the number of representations in the corresponding multiplets is equal to $2[\frac{p+q}{2}] = 2(h+1)$.

Further, we denote by $C_i^{\pm}$ the representation space with signature $\chi_i^{\pm}$.

Below we give the multiplets pictorially first for $p+q$ even, then for $p+q$ odd.

$$
\begin{tikzcd}
& C_1^- \arrow[d, "d_1"] & \\
& C_2^- & \\
& \vdots & \\
& C_{h-1}^- \arrow[d, "d_{h-1}"] & \\
& C_h^- \arrow[dl, "d_h"'] \arrow[dr, "d'_h"] & \\
C_{h+1}^- \arrow[dr, "d'_h"'] & \bullet & C_{h+1}^+ \arrow[dl, "d_h"] \\
& C_h^+ \arrow[d, "d'_{h-1}"] & \\
& C_{h-1}^+ & \\
& \vdots & \\
& C_2^+ \arrow[d, "d'_1"] & \\
& C_1^+ &
\end{tikzcd}
$$

Invariant differential operators for $so(p,q)$ for $p+q$ even,
$h = \frac{1}{2}(p+q-2)$

$$\begin{tikzcd}
C_1^- \arrow[d, "d_1"] \\
C_2^- \\
\vdots \\
C_{h-1}^- \arrow[d, "d_{h-1}"] \\
C_h^- \arrow[d, "d_h"] \\
C_{h+1}^- \arrow[d, "d_{h+1}"] \\
C_{h+1}^+ \arrow[d, "d'_h"] \\
C_h^+ \arrow[d, "d'_{h-1}"] \\
C_{h-1}^+ \\
\vdots \\
C_2^+ \arrow[d, "d'_1"] \\
C_1^+
\end{tikzcd}$$

Invariant differential operators for $so(p,q)$ for $p+q$ odd,
$h = \frac{1}{2}(p+q-1)$

The degrees of the operators in the two pictures are:

$$\begin{aligned}
&\deg d_i = \deg d'_i = n_{h+2-i} - n_{h+1-i}, \qquad i = 1, \dots, h, \\
&\deg d'_h = n_2 + n_1, \quad (p+q) - \text{even} \ , \\
&\deg d_{h+1} = 2n_1, \quad (p+q) - \text{odd}
\end{aligned} \tag{22}$$

where d'_h is omitted from the first line for $(p+q)$ even.

Again the $\pm$ pairs are related by KS operators with obvious action on the signatures. There is a peculiarity, namely, that for $p+q$ odd, for the pair $C^{\pm}_{h+1}$ the KS operator acting from C^{-}_{h+1} to C^{+}_{h+1} has degenerated (due to regularization of the kernel) to the differential operator d_{h+1} .

2 Intertwining differential operators related to Hermitean symmetric spaces

Since the study and description of detailed classification should be done group by group we had to decide which groups to study first. A natural choice would be non-compact groups that have discrete series representations. By the Harish-Chandra criterion these are groups where holds:

$$\mathrm{rank}\, G = \mathrm{rank}\, K, \tag{23}$$

where K is the maximal compact subgroup of the non-compact group G. Another formulation is to say that the Lie algebra $\mathcal{G}$ of G has a compact Cartan subalgebra.

Example: The groups $SO(p,q)$ have discrete series, except when both p,q are odd numbers. $\diamondsuit$

This class is still rather big, thus, we decided to start with a subclass, namely, the class of Hermitian symmetric spaces. The practical criterion is that in these cases, the maximal compact subalgebra $\mathcal{K}$ is of the form:

$$\mathcal{K} \ = \ so(2) \oplus \mathcal{K}' \ . \tag{24}$$

The Lie algebras from this class are:

$$so(n,2), \quad sp(n,R), \quad su(m,n), \quad so^*(2n), \quad E_{6(-14)}\,, \quad E_{7(-25)} \tag{25}$$

These groups/algebras have highest/lowest weight representations, and relatedly holomorphic discrete series representations.

The most widely used of these algebras are the conformal algebras $so(n,2)$ in n-dimensional Minkowski space-time which we already considered and use now to introduce some more notions. In that case, there is a maximal Bruhat decomposition:

$$\begin{aligned} so(n,2) \ &= \ \mathcal{P} \oplus \tilde{\mathcal{N}} \ = \ \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N} \oplus \tilde{\mathcal{N}}\ , \\ \mathcal{M} \ &= \ so(n-1,1)\ , \quad \dim \mathcal{A} = 1, \quad \dim \mathcal{N} = \dim \tilde{\mathcal{N}} = n \end{aligned} \tag{26}$$

that has direct physical meaning, namely, $so(n-1,1)$ is the Lorentz algebra of n-dimensional Minkowski space-time, the subalgebra $\mathcal{A} = so(1,1)$ represents the dilatations, the conjugated subalgebras $\mathcal{N}$, $\tilde{\mathcal{N}}$ are the algebras of translations, and special conformal transformations, both being isomorphic to n-dimensional Minkowski space-time.

The subalgebra $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ ($\cong \mathcal{M} \oplus \mathcal{A} \oplus \tilde{\mathcal{N}}$) is a maximal parabolic subalgebra.

It is also important that the complexification of the maximal compact subalgebra is isomorphic to the complexification of the first two factors of the Bruhat decomposition:

$$\mathcal{K}^{\mathbb{C}} \;=\; so(n,\mathbb{C}) \oplus so(2,\mathbb{C}) \;\cong\; so(n-1,1)^{\mathbb{C}} \oplus so(1,1)^{\mathbb{C}} = \mathcal{M}^{\mathbb{C}} \oplus \mathcal{A}^{\mathbb{C}}\,. \tag{27}$$

In particular, the coincidence of the complexification of the subalgebras:

$$\mathcal{K}'^{\mathbb{C}} \;=\; \mathcal{M}^{\mathbb{C}} \tag{28}$$

means that the sets of finite-dimensional (nonunitary) representations of $\mathcal{M}$ are in 1-to-1 correspondence with the finite-dimensional (unitary) representations of $\mathcal{K}'$.

It turns out that some of the hermitian-symmetric algebras share the above-mentioned special properties of $so(n,2)$. This subclass consists of:

$$so(n,2), \quad sp(n,\mathbb{R}), \quad su(n,n), \quad so^*(4n), \quad E_{7(-25)} \tag{29}$$

the corresponding analogs of Minkowski space-time V being:

$$\mathbb{R}^{n-1,1}, \quad \mathrm{Sym}(n,\mathbb{R}), \quad \mathrm{Herm}(n,\mathbb{C}), \quad \mathrm{Herm}(n,\mathbb{Q}), \quad \mathrm{Herm}(3,\mathbb{O}) \tag{30}$$

where we use standard notation $\mathbb{R},\mathbb{C},\mathbb{Q},\mathbb{O}$ for the four division algebras: real, complex, quaternion, octonion.

In view of applications to physics, we proposed to call these algebras 'conformal Lie algebras', (or groups).

We have started the study of the above class in the framework of the present approach in some cases and we expose these below.

Before passing to the examples we mention also an useful notion:

Definition: Let $\mathcal{G},\mathcal{G}'$ be two non-compact semisimple Lie algebras with the same complexification $\mathcal{G}^{\mathbb{C}} \cong \mathcal{G}'^{\mathbb{C}}$. We call them parabolically related if they have parabolic subalgebras $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$, $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$, such that: $\mathcal{M}^{\mathbb{C}} \cong \mathcal{M}'^{\mathbb{C}}$ ($\Rightarrow \mathcal{P}^{\mathbb{C}} \cong \mathcal{P}'^{\mathbb{C}}$). $\diamondsuit$

Certainly, there may be several such parabolic relationships for any given algebra $\mathcal{G}$.

3 The Lie algebras su(n,n)

Let $\mathcal{G} = su(n,n)$, $n \geq 2$ (though the case $su(2,2)$ was already treated). The maximal compact subgroup is $\mathcal{K} \cong u(1) \oplus su(n) \oplus su(n)$.

We choose a maximal parabolic $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ such that $\mathcal{A} \cong so(1,1)$, $\mathcal{M} = sl(n,\mathbb{C})_{\mathbb{R}}$.

We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\chi = \{ n_1, \ldots, n_{n-1}, n_{n+1}, \ldots, n_{2n-1}; c \}, \ n_j \in \mathbb{Z}_+ , \ c = d - \tfrac{1}{2}n^2 \tag{31}$$

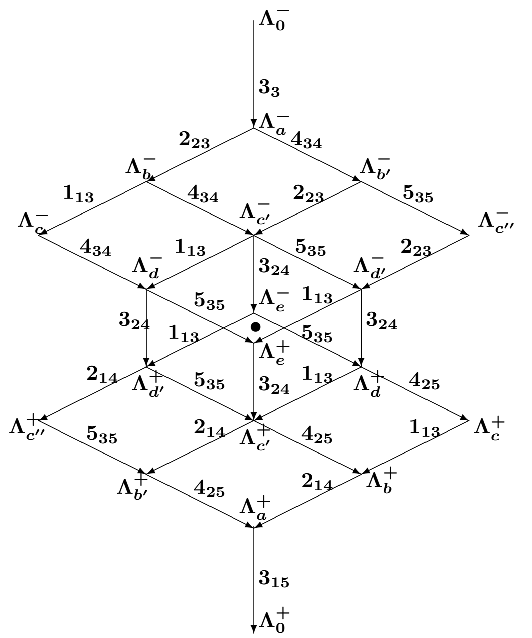
where the last entry of χ labels the characters of $\mathcal{A}$, and the first $2n-2$ entries are labels of the finite-dimensional nonunitary irreps of $\mathcal{M}$.

The number of ERs in the main multiplets is equal to:

$$|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| \, / \, |W(\mathcal{M}^{\mathbb{C}}, \mathcal{H}_m^{\mathbb{C}})| = \binom{2n}{n}$$

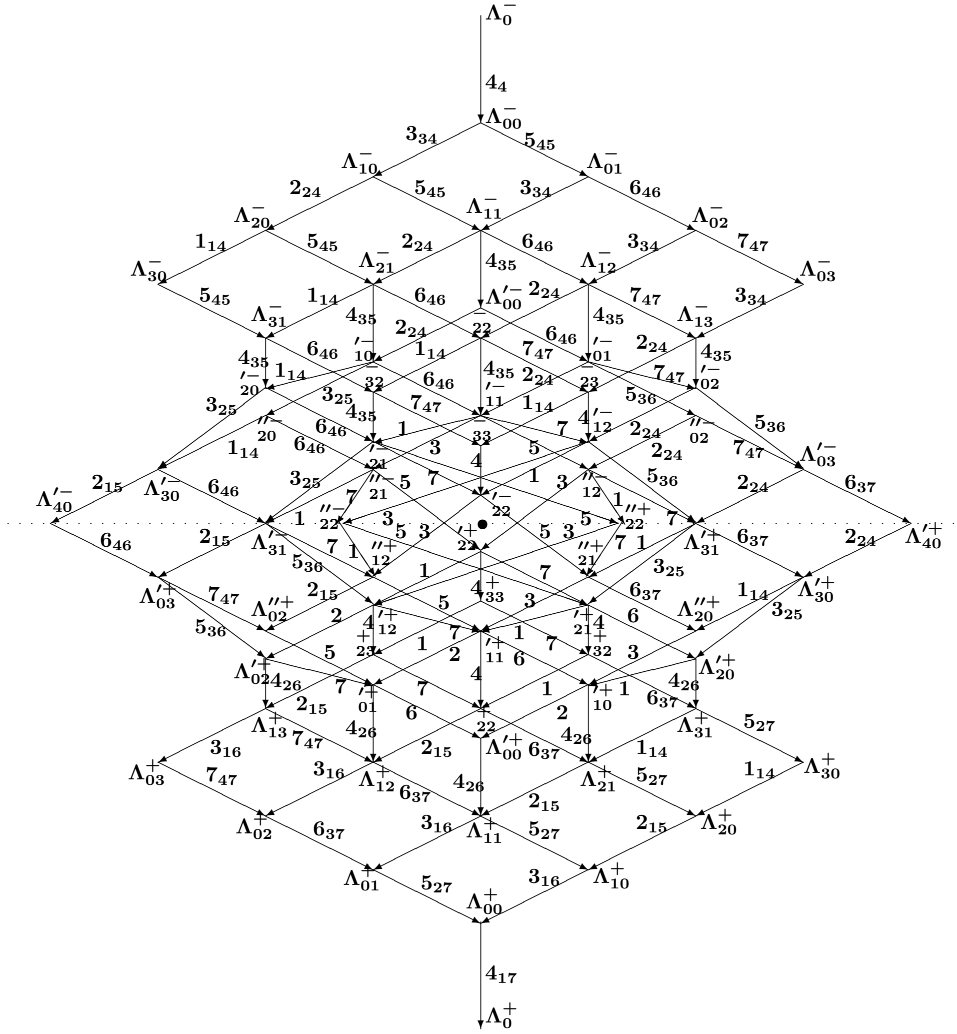
Below we give the multiplets for $su(n,n)$ for $n = 3,4$. They are valid also for $sl(2n,\mathbb{R})$ with $\mathcal{M}$-factor $sl(n,\mathbb{R}) \oplus sl(n,\mathbb{R})$, and when $n = 4$ these are multiplets also for the Lie algebra $su^*(8)$ with $\mathcal{M}$-factor $su^*(4) \oplus su^*(4)$. We present the results only pictorially while the details may be found in [VKD1][1]

[1]Vladimir K. Dobrev, **Invariant Differential Operators, Volume 1: Noncompact Semisimple Lie Algebras and Groups**, De Gruyter Studies in Mathematical Physics vol. 35 (De Gruyter, Berlin, Boston, 2016, ISBN 978-3-11-042764-6), 408 + xii pages.



Main multiplets for $su(3,3)$ **and** $sl(6,\mathbb{R})$

with parabolic $\mathcal{M}$**-factors** $sl(3,\mathbb{C})_{\mathbb{R}}$, $sl(3,\mathbb{R}) \oplus sl(3,\mathbb{R})$**, resp.**



Main multiplets for $su(4,4)$, $sl(8,\mathbb{R})$, $su^*(8)$

with parabolic $\mathcal{M}$-factors $sl(4,\mathbb{C})_{\mathbb{R}}$, $sl(4,\mathbb{R}) \oplus sl(4,\mathbb{R})$, $su^*(4) \oplus su^*(4)$, resp.

4 Cases sp(n,ℝ) and sp(r,r)

Let $\mathcal{G} = sp(n,\mathbb{R})$, the split real form of $sp(n,\mathbb{C}) = \mathcal{G}^{\mathbb{C}}$. The maximal compact subgroup of $\mathcal{G}$ is $\mathcal{K} \cong u(1) \oplus su(n)$.

We choose a maximal parabolic $\mathcal{P} = \mathcal{M}\mathcal{A}\mathcal{N}$ such that $\mathcal{A} \cong so(1,1)$, while $\mathcal{M} = sl(n,\mathbb{R})$.

We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\chi = \{ n_1 , \ldots , n_{n-1} ; c \} , \qquad n_j \in \mathbb{N} , \quad c = d - (n+1)/2 \quad (32)$$

where the last entry of χ labels the characters of $\mathcal{A}$, and the first $n-1$ entries are labels of the finite-dimensional nonunitary irreps of $\mathcal{M}$, (or of the finite-dimensional unitary irreps of $su(n)$).

For $n = 2r$ the algebra $sp(2r,\mathbb{R})$ with $\mathcal{M}$-factor $sl(2r,\mathbb{R})$ is parabolically related to $\mathcal{G} = sp(r,r)$ with $\mathcal{M}$-factor $su^*(2r)$, noting that $(su^*(2r))^{\mathbb{C}} = sl(2r,\mathbb{C})$. The algebra $sp(r,r)$ has maximal compact subalgebra $\mathcal{K} = sp(r) \oplus sp(r)$ and has discrete series representations but no highest/lowest weight representations.

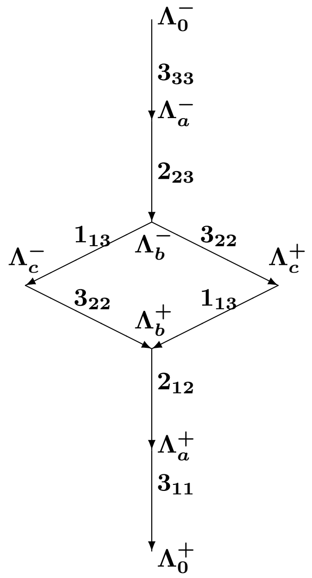
There are several types of multiplets. The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of $sp(n,\mathbb{R})$, i.e., they will be labelled by the n positive Dynkin labels $m_i \in \mathbb{N}$. The number of ERs in the main multiplets is:

$$\frac{|W(\mathcal{G}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}})|}{|W(\mathcal{M}^{\mathbb{C}},\mathcal{H}_m^{\mathbb{C}})|} = \frac{|W(sp(n,\mathbb{C}))|}{|W(sl(n,\mathbb{C}))|} = \frac{2^n(n)!}{((n)!)} = 2^n \qquad (33)$$

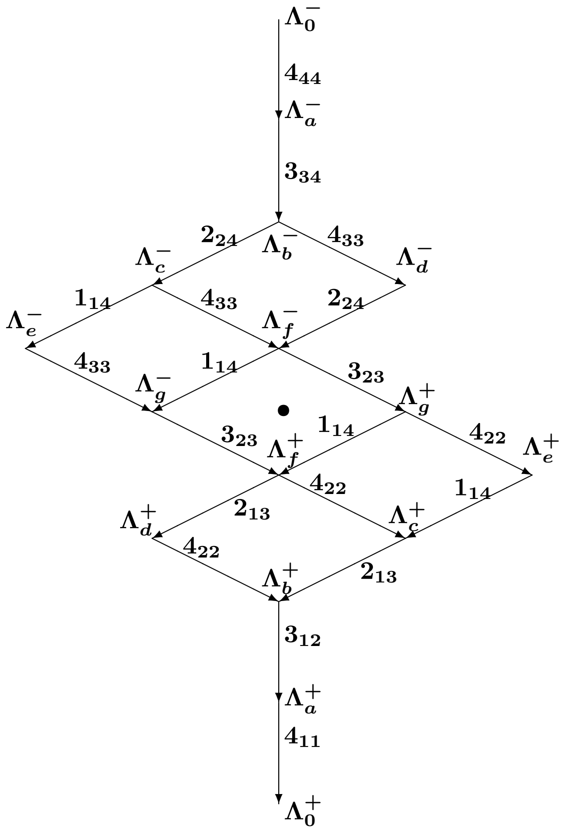
It is difficult to give explicitly the multiplets for general n. Thus, we present the cases $3 \leq n \leq 6$ and only pictorially.

Note that the cases $n = 1,2$ were already considered recalling that $sp(1,\mathbb{R}) \cong sl(2,\mathbb{R})$, $sp(2,\mathbb{R}) \cong so(3,2)$). Also the case $sp(1,1)$ was considered recalling that $sp(1,1) \cong so(4,1)$.

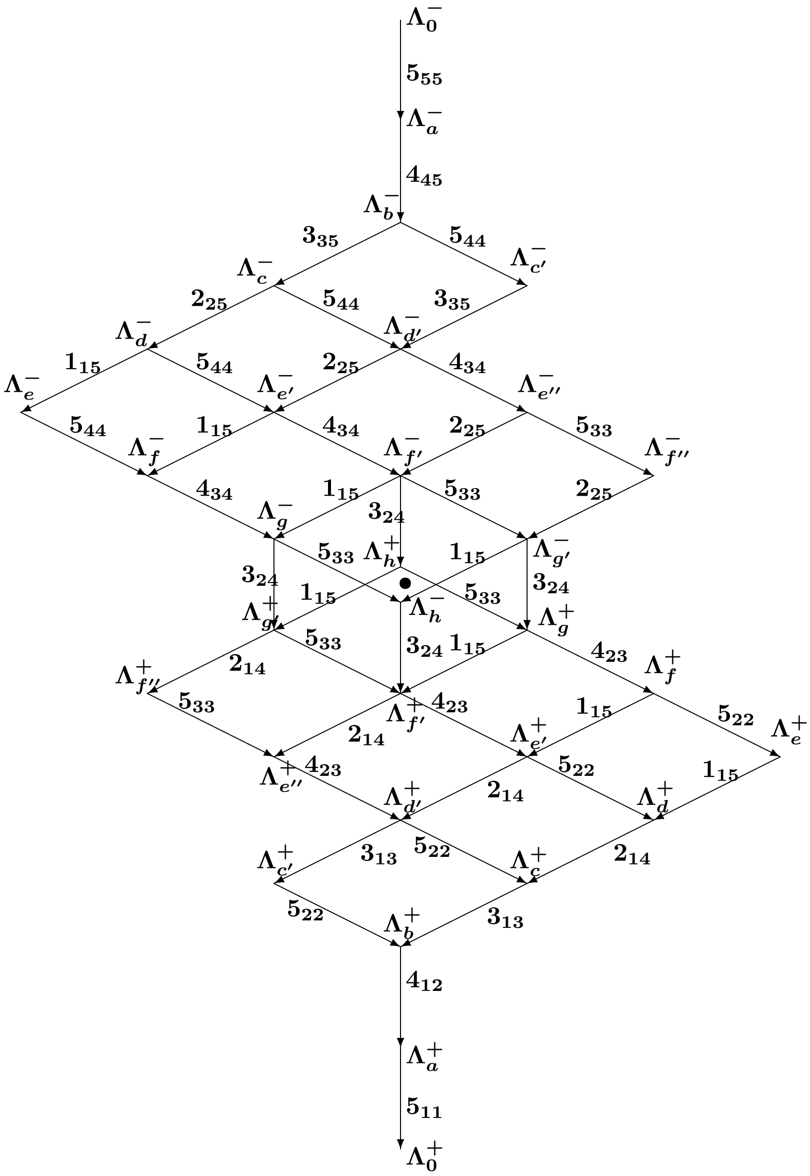
Also note that the diagram for $sp(3,\mathbb{R})$ looks similar to the one for $so(6,2)$, however the parametrizations are obviously different as the ranks are different.



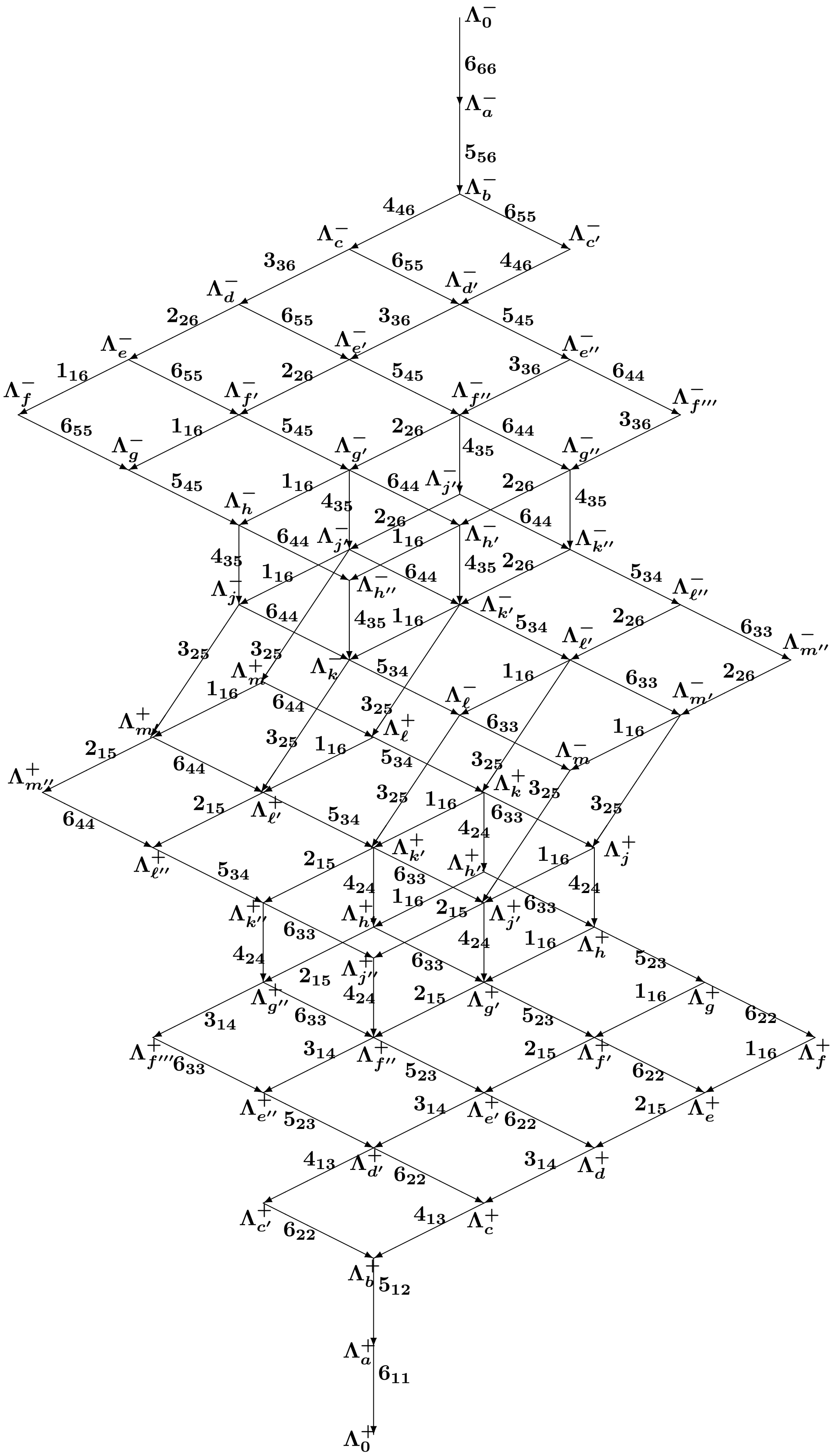
Main multiplets for $Sp(3, I\!R)$



Main multiplets for $sp(4, I\!\!R)$ **and** $sp(2,2)$



Main multiplets for $sp(5, I\!\!R)$



Main multiplets for $sp(6,\mathbb{R})$ and $sp(3,3)$

5 SO*(4n) case

Let $\mathcal{G} = so^*(4n)$. We choose a maximal parabolic $\mathcal{P} = \mathcal{MAN}$ such that $\mathcal{A} \cong so(1,1)$, $\mathcal{M} = su^*(2n)$. Since the algebras $so^*(4n)$ belong to the class called 'conformal Lie algebras' we have:

$$\mathcal{K}^{\mathbb{C}} \cong u(1)^{\mathbb{C}} \oplus sl(2n,\mathbb{C}) \cong \mathcal{A}^{\mathbb{C}} \oplus \mathcal{M}^{\mathbb{C}} \tag{34}$$

Here we have the series of algebras: $so^*(4)$, $so^*(8)$, $so^*(12)$, ... However the first two cases are reduced to well known conformal algebras due to the coincidences: $so^*(4) \cong so(3) \oplus so(2,1)$, $so^*(8) \cong so(6,2)$.

Thus, we shall study the algebra $\mathcal{G}_6 \equiv so^*(12)$.

We label the signature of the ERs of $\mathcal{G}_6$ as follows:

$$\chi = \{ n_1 , n_2 , n_3 , n_4 , n_5 ; c \} , \qquad n_j \in \mathbb{Z}_+ , \quad c = d - \tfrac{15}{2} \tag{35}$$

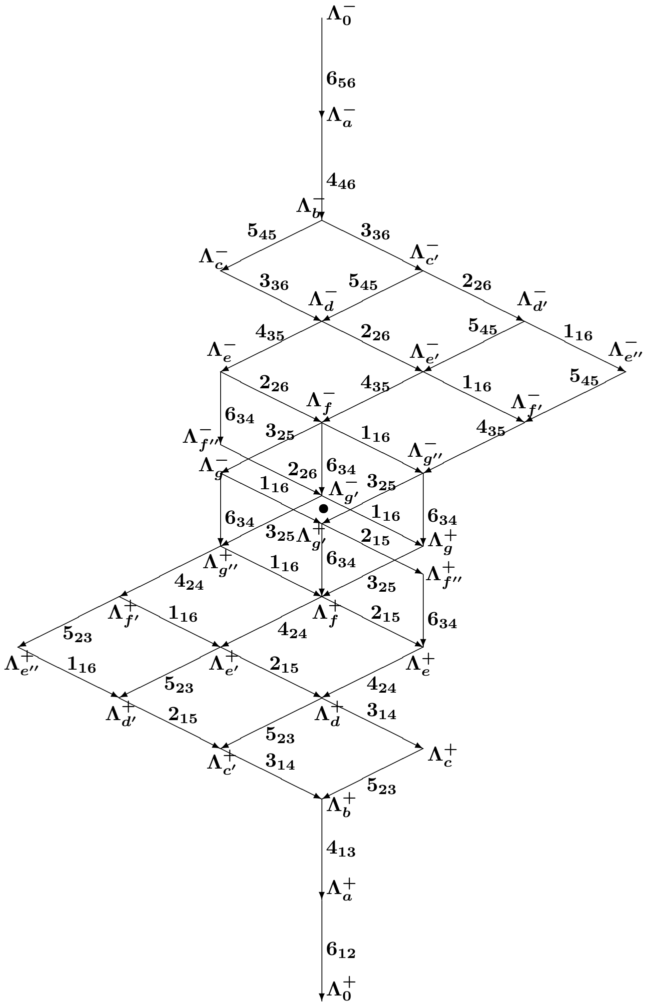
where the last entry of χ labels the characters of $\mathcal{A}$, and the first five entries are labels of the finite-dimensional nonunitary irreps of $\mathcal{M}_6 = su^*(6)$.

Finally, we remind that the above considerations are applicable also for the parabolically related algebra $so(6,6)$ with parabolic $\mathcal{M}$-factor $sl(6,\mathbb{R})$. It has discrete series representations but no highest/lowest weight representations.

The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of $so^*(12)$, i.e., they are labelled by the six positive Dynkin labels $m_i \in \mathbb{N}$. The number of ERs/GVMs in the main multiplets is:

$$|W(\mathcal{G}_6^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| \,/\, |W(\mathcal{M}_6^{\mathbb{C}}, \mathcal{H}_m^{\mathbb{C}})| = |W(so(12,\mathbb{C}))| \,/\, |W(sl(6,\mathbb{C}))| = 32 \tag{36}$$

where $\mathcal{H}^{\mathbb{C}}, \mathcal{H}_m^{\mathbb{C}}$ are Cartan subalgebras of $\mathcal{G}_6^{\mathbb{C}}, \mathcal{M}_6^{\mathbb{C}}$, resp.



Main multiplets for $so^*(12)$ and $so(6,6)$

with parabolic factor $\mathcal{M}^{\mathbb{C}} = sl(6,\mathbb{C})$

6 Exceptional Lie algebras $E_{7(-25)}$ and $E_{7(7)}$

Let $\mathcal{G} = E_{7(-25)}$. The maximal compact subgroup is $\mathcal{K} \cong e_6 \oplus so(2)$, while the completion space has $\dim \mathcal{Q} = 54$. We work with maximal parabolic $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ with $\mathcal{M} \cong E_{6(-26)}$.

We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\chi = \{ n_1 , \ldots , n_6 \, ; c \} , \qquad n_j \in \mathbb{N} , \quad c = d - 9 \tag{37}$$

where the last entry of χ labels the characters of $\mathcal{A}$, and the first 6 entries are labels of the finite-dimensional nonunitary irreps of $\mathcal{M}$, (or of the finite-dimensional unitary irreps of the compact e_6). The signatures expressed through the Dynkin labels:

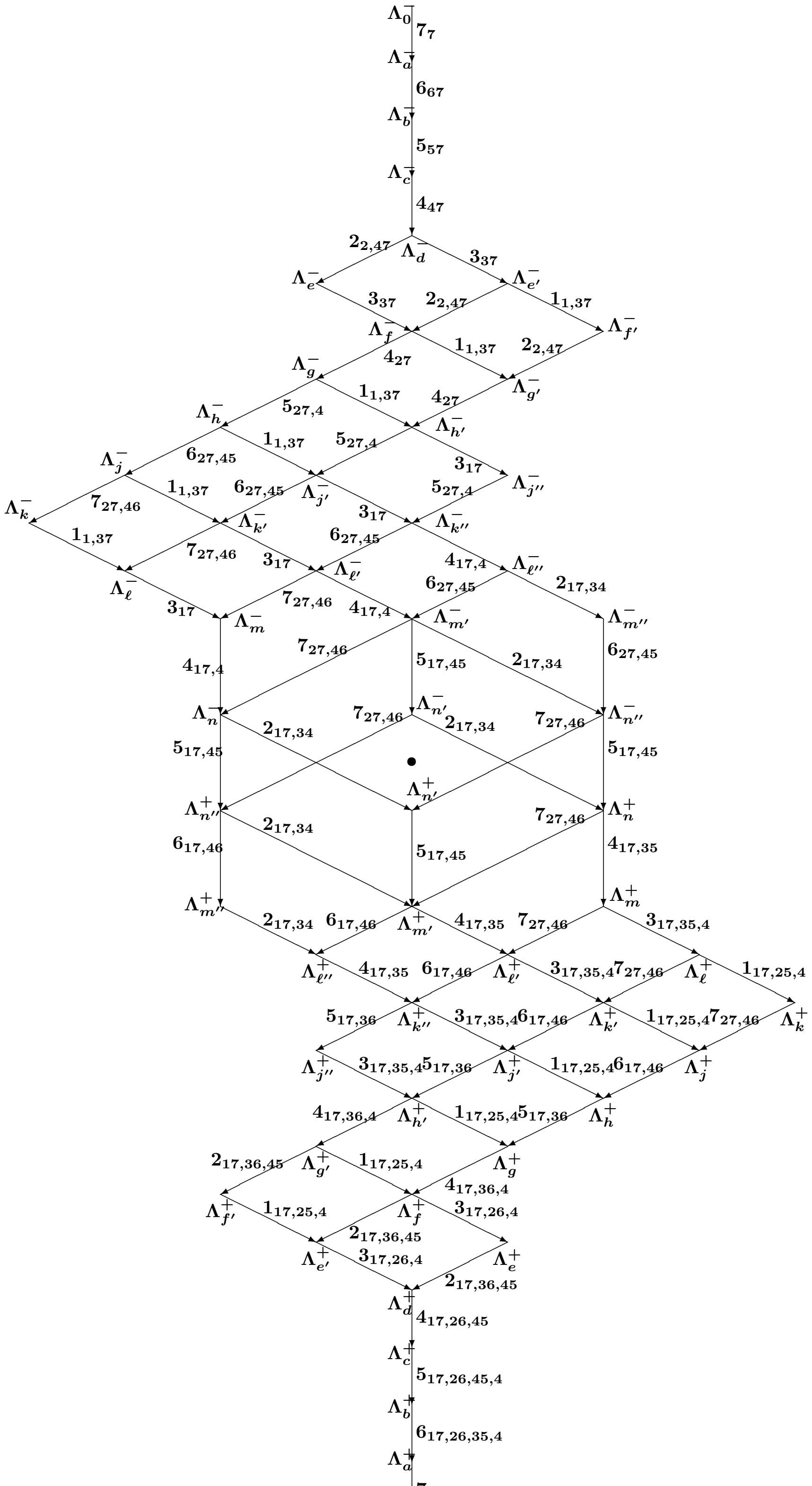
$$\begin{aligned} & n_i = m_i \, , \quad c = -\tfrac{1}{2}(m_{\tilde{\alpha}} + m_7) = \\ & = -\tfrac{1}{2}(2m_1 + 2m_2 + 3m_3 + 4m_4 + 3m_5 + 2m_6 + 2m_7) \end{aligned} \tag{38}$$

The same holds for the parabolically related exceptional Lie algebra $E_{7(7)}$ (with $\mathcal{M}$-factor $E_{6(6)}$). Its maximal compact subgroup is $\mathcal{K} \cong su(8)$, while the completion space has $\dim \mathcal{Q} = 70$. This algebra has discrete series representations (as $\mathrm{rank}\mathcal{G} = \mathrm{rank}\mathcal{K}$), but no highest/lowest weight representations.

The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of E_7, i.e., they will be labelled by the seven positive Dynkin labels $m_i \in \mathbb{N}$. The number of ERs in these main multiplets is:

$$\frac{|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})|}{|W(\mathcal{M}^{\mathbb{C}}, \mathcal{H}_m^{\mathbb{C}})|} = \frac{|W(E_7)|}{|W(E_6)|} = \frac{2^{10}\, 3^4\, 5.7}{2^7\, 3^4\, 5} = 56 \tag{39}$$

The multiplets are depicted in the Figure below:



Main Type for $E_{7(-25)}$ and $E_{7(7)}$

7 Exceptional Lie algebras $E_{6(-14)}$, $E_{6(6)}$, $E_{6(2)}$

Let $\mathcal{G} = E_{6(-14)}$. It has discrete series representations and highest/lowest weight representations. The split rank is equal to 2, while $\mathcal{M}_0 \cong so(6) \oplus so(2)$. The maximal compact subalgebra is $\mathcal{K} = so(10) \oplus so(2)$, while the completion space has $\dim \mathcal{Q} = 32$.

We work with the maximal cuspidal parabolic subalgebra suitable for the class of conformal Lie algebras:

$$\mathcal{M} = su(5,1) , \qquad \dim \mathcal{N}^{\pm} = 21 , \qquad \dim \mathcal{A} = 1 \tag{40}$$

We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\chi = \{ n_1 , n_3 , n_4 , n_5 , n_6 ; c \} , \quad c = d - \tfrac{11}{2} , \tag{41}$$

where the last entry of χ labels the characters of $\mathcal{A}$, and the first five entries are labels of the discrete series of $\mathcal{M}$, then $n_j \in \mathbb{N}$, or of limits of discrete series, when some of n_j are zero.

We consider along with $E_{6(-14)}$ two algebras parabolically related to it, namely, $E_{6(6)}$ and $E_{6(2)}$, with parabolic $\mathcal{M}$-factors: $sl(6,\mathbb{R})$, $su(3,3)$, resp. They have $\mathcal{K} = sp(4)$, $\mathcal{K} = su(6) \oplus su(2)$, resp. $\dim \mathcal{Q} = 42, 40$, resp.

The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of $\mathcal{G}$, i.e., they will be labelled by the six positive Dynkin labels $m_i \in \mathbb{N}$. It turns out that each such multiplet contains 70 ERs/GVMs

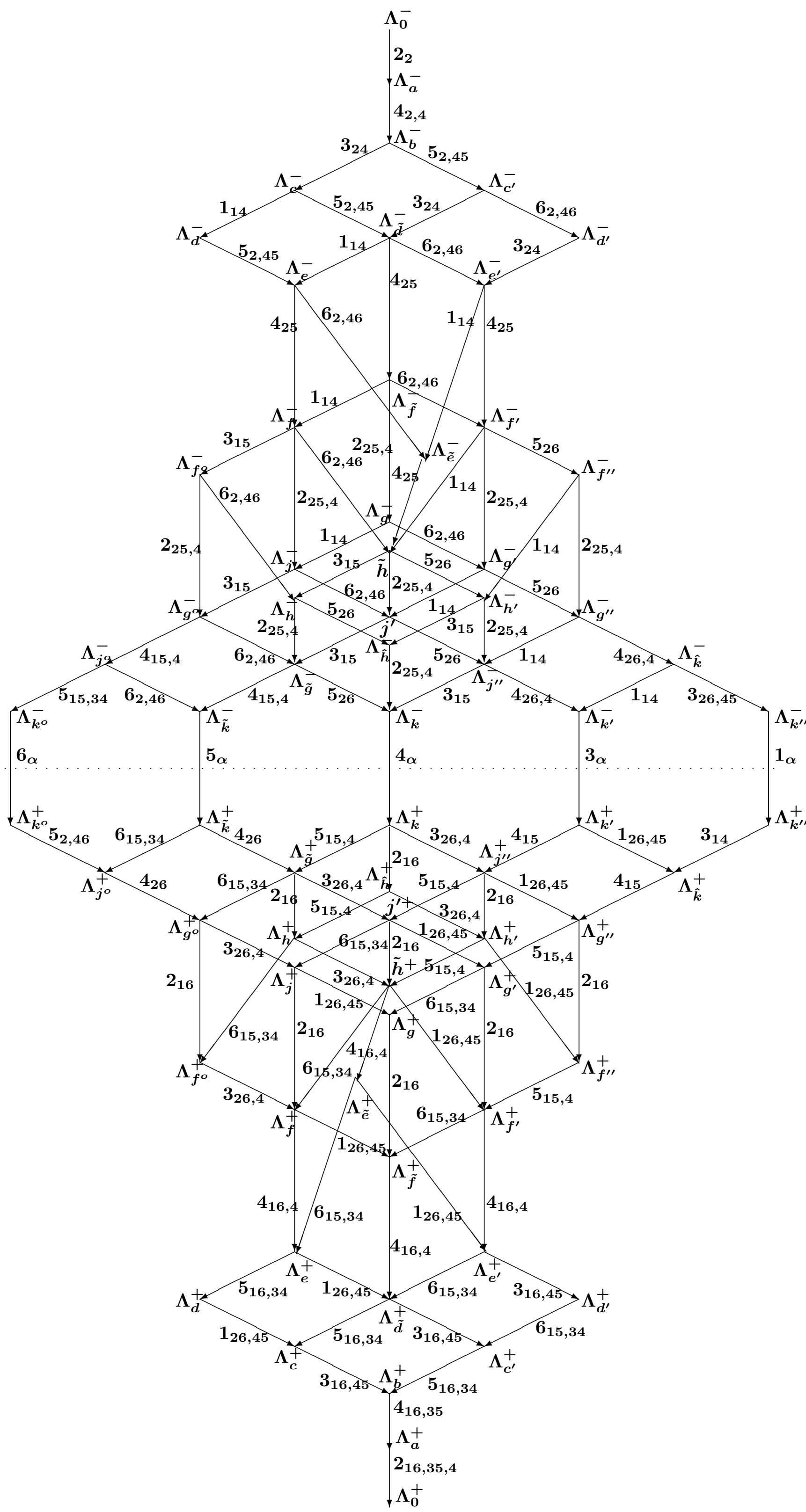
The ERs in the multiplet are related by intertwining integral and differential operators. The Knapp-Stein integral operators intertwining the pairs will be denoted by:

$$G^{\pm} : \mathcal{C}_{\chi^{\mp}} \longrightarrow \mathcal{C}_{\chi^{\pm}} \tag{42}$$

As in all previous cases matters are arranged so that in every multiplet only the ER with signature χ_0^- contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace $\mathcal{E}$. And in every multiplet only the ER χ_0^+ contains the anti/holomorphic discrete series representation. The conformal weight of the holomorphic case has the restriction $d = \frac{1}{2}(11 + m_{\tilde{\alpha}}) \geq 11$.

The multiplets are given explicitly in the Figure below.

The KS operators relate the ERs which are disposed summetrically w.r.t. the dashed line.



Main Type for $E_{6(-14)}$, $E_{6(6)}$, $E_{6(2)}$

8 Exceptional Lie algebra G_2

Let $\mathcal{G}^{\mathbb{C}} = G_2$, with Cartan matrix: $(a_{ij}) = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$, simple roots α_1, α_2 with products: $(\alpha_2, \alpha_2) = 3(\alpha_1, \alpha_1) = -2(\alpha_2, \alpha_1)$. We choose $(\alpha_1, \alpha_1) = 2$, then $(\alpha_2, \alpha_2) = 6$, $(\alpha_2, \alpha_1) = -3$. As we know G_2 is 14–dimensional. The positive roots may be chosen as:

$$\Delta^+ = \{\alpha_1, \quad \alpha_2, \quad \alpha_1 + \alpha_2, \quad \alpha_2 + 2\alpha_1, \quad \alpha_2 + 3\alpha_1, \quad 2\alpha_2 + 3\alpha_1\} \tag{43}$$

The Weyl group $W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})$ of G_2 is the dihedral group of order 12.

The complex Lie algebra G_2 has one non-compact real form: $\mathcal{G} = G_{2(2)}$ which is naturally split. Its maximal compact subalgebra is $\mathcal{K} = su(2) \oplus su(2)$, also written as $\mathcal{K} = su(2)_S \oplus su(2)_L$ to emphasize the relation to the root system (after complexification the first factor contains a short root, the second - a long root). We remind that $\mathcal{G} = G_{2(2)}$ has discrete series representations. Actually, it is quaternionic discrete series since $\mathcal{K}$ contains as direct summand (at least one) $su(2)$ subalgebra. The number of discrete series is equal to the ratio $|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})|/|W(\mathcal{K}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})|$, where $\mathcal{H}$ is a compact Cartan subalgebra of both $\mathcal{G}$ and $\mathcal{K}$, W are the relevant Weyl groups. Thus, the number of discrete series in our setting is three. One case will be explicitly identified below.

The compact Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$ will be chosen to coincide with the Cartan subalgebra of $\mathcal{K}$ and we may write: $\mathcal{H} = u(1)_S \oplus u(1)_L$.

The minimal parabolic of $\mathcal{G}$ is:

$$\mathcal{P}_0 = \mathcal{M}_0 \oplus \mathcal{A}_0 \oplus \mathcal{N}_0 = \mathcal{A}_0 \oplus \mathcal{N}_0 \tag{44}$$

There are two isomorphic maximal cuspidal parabolic subalgebras of $\mathcal{G}$ which are of Heisenberg type:

$$\begin{aligned} \mathcal{P}_k &= \mathcal{M}_k \oplus \mathcal{A}_k \oplus \mathcal{N}_k, \quad k = 1, 2; \\ \mathcal{M}_k &= sl(2, \mathbb{R})_k, \quad \dim \mathcal{A}_k = 1, \quad \dim \mathcal{N}_k = 5 \end{aligned} \tag{45}$$

Let us denote by $\mathcal{T}_k$ the compact Cartan subalgebra of $\mathcal{M}_k$. Then $\mathcal{H}_k = \mathcal{T}_k \oplus \mathcal{A}_k$ is a non-compact Cartan subalgebra of $\mathcal{G}$. We choose $\mathcal{T}_1$ to be generated by the short $\mathcal{K}$-compact root $\alpha_1 + \alpha_2$

and $\mathcal{A}_1$ to be generated by the long root α_2, while $\mathcal{T}_2$ to be generated by the long $\mathcal{K}$-compact root $\alpha_2 + 3\alpha_1$ and $\mathcal{A}_2$ to be generated by the short root α_1.

Equivalently, the $\mathcal{M}_1$-compact root of $\mathcal{G}^{\mathbb{C}}$ is $\alpha_1 + \alpha_2$, while the $\mathcal{M}_2$-compact root is $\alpha_2 + 3\alpha_1$. In each case the remaining five positive roots of $\mathcal{G}^{\mathbb{C}}$ are $\mathcal{M}_k$-noncompact.

To characterize the Verma modules we shall use first the Dynkin labels:

$$m_i \ \equiv \ (\Lambda + \rho, \alpha_i^\vee), \quad i = 1, 2, \tag{46}$$

where ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$. Thus, we shall use :

$$\chi_\Lambda \ = \ \{m_1, m_2\} \tag{47}$$

Note that when both $m_i \in \mathbb{N}$ then χ_Λ characterizes the finite-dimensional irreps of $\mathcal{G}^{\mathbb{C}}$ and its real forms, in particular, $\mathcal{G}$. Furthermore, $m_k \in \mathbb{N}$ characterizes the finite-dimensional irreps of the $\mathcal{M}_k$ subalgebra.

We shall use also the Harish-Chandra parameters:

$$m_\beta \ = \ (\Lambda + \rho, \beta^\vee) \ , \tag{48}$$

for any positive root β, and explicitly in terms of the Dynkin labels:

$$\begin{aligned} \chi_{HC} \ = \ \{ \ & m_1, \quad m_3 = 3m_2 + m_1, \quad m_4 = 3m_2 + 2m_1 && \text{(49a)} \\ & m_2, \quad m_5 = m_2 + m_1, \quad m_6 = 2m_2 + m_1, \} && \text{(49b)} \end{aligned}$$

8.1 Induction from minimal parabolic

The main multiplets are in 1-to-1 correspondence with the finite-dimensional irreps of G_2, i.e., they are labelled by the two positive Dynkin labels $m_i \in \mathbb{N}$.

Using this labelling the signatures may be given in the following pair-wise manner:

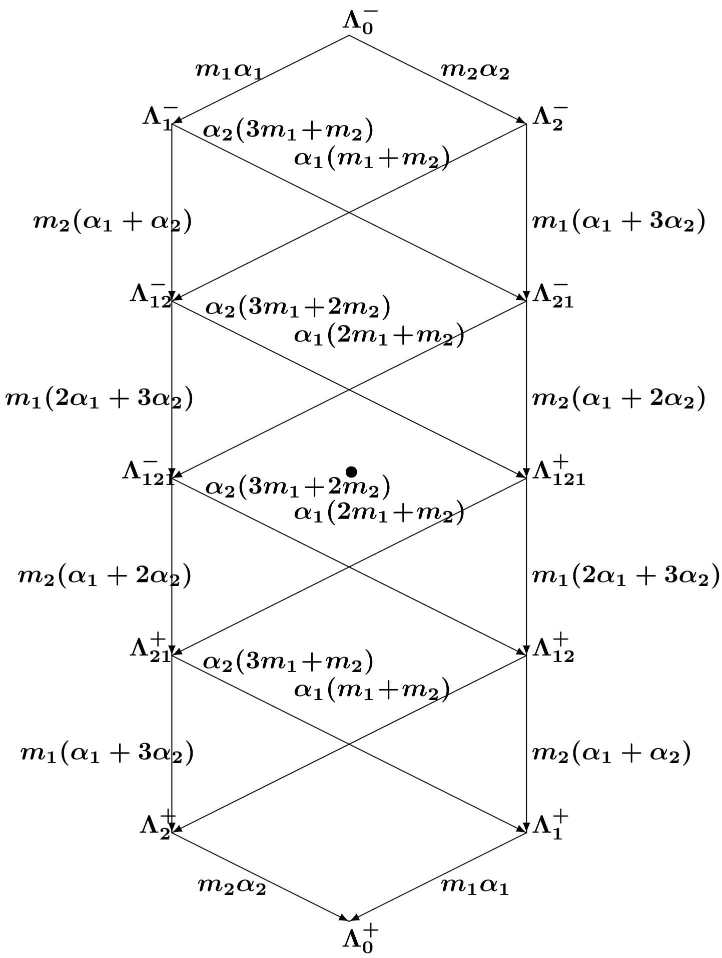
$$\begin{aligned}
\chi_0^\pm &= \{\mp m_1, \mp m_2;\ \ \pm\tfrac{1}{2}(2m_2+m_1)\} \\
\chi_2^\pm &= \{\mp(3m_2+m_1), \pm m_2;\ \ \pm\tfrac{1}{2}(m_2+m_1)\}, \\
\chi_1^\pm &= \{\pm m_1, \mp(m_2+m_1);\ \ \pm\tfrac{1}{2}(2m_2+m_1)\}, \\
\chi_{12}^\pm &= \{\mp(3m_2+2m_1), \pm(m_2+m_1);\ \ \pm\tfrac{1}{2}m_2\} \\
\chi_{21}^\pm &= \{\pm(3m_2+m_1), \mp(2m_2+m_1);\ \ \pm\tfrac{1}{2}(m_2+m_1)\} \\
\chi_{121}^\pm &= \{\mp(3m_2+2m_1), \pm(2m_2+m_1);\ \ \mp\tfrac{1}{2}m_2\},
\end{aligned} \tag{50}$$

We have included as third entry also the parameter $c = -\frac{1}{2}(2m_2+m_1)$, related to the Harish-Chandra parameter of the highest root (recalling that $m_{\alpha_6} = 2m_2+m_1$). It is also related to the conformal weight $d = \frac{3}{2} + c$.

The ERs in the multiplet are related also by intertwining integral Knapp-Stein operators. These operators are defined for any ER, the general action in our situation being:

$$\begin{aligned}
& G_{KS} \ :\ \mathcal{C}_\chi \ \longrightarrow\ \mathcal{C}_{\chi'}\ , \\
& \chi\ =\ [\,n_1, n_2\,;\,c\,]\ , \qquad \chi'\ =\ [\,-n_1, -n_2\,;\,-c\,].
\end{aligned} \tag{51}$$

The main multiplets are given explicitly in the next figure:



Main multiplets for $G_{2(2)}$
using induction from the minimal parabolic

The pairs $\chi^{\pm}$ are symmetric w.r.t. the bullet in the middle of the picture - this symbolizes the Weyl symmetry realized by the Knapp-Stein operators (51): $G^{\pm} : \mathcal{C}_{\chi^{\mp}} \longrightarrow \mathcal{C}_{\chi^{\pm}}$.

Some comments are in order.

Matters are arranged so that in every multiplet only the ER with signature χ_0^- contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional irrep of $G_{2(2)}$ with signature $[m_1, m_2]$. The subspace $\mathcal{E}$ is annihilated by the operators G^+, $\mathcal{D}_{\alpha_1}^{m_1}$, $\mathcal{D}_{\alpha_2}^{m_2}$ and is the image of the operator G^- .
When both $m_i = 1$ then $\dim \mathcal{E} = 1$, and in that case $\mathcal{E}$ is also the trivial one-dimensional UIR of the whole algebra $\mathcal{G}$. Furthermore in that case the conformal weight is zero: $d = \frac{3}{2} + c = \frac{3}{2} - \frac{1}{2}(2m_2 + m_1)_{|m_i=1} = 0$.

In the conjugate ER χ_0^+ there is a unitary discrete series representation (according to the Harish-Chandra criterion) in an infinite-dimensional subspace $\tilde{\mathcal{D}}_0$ with conformal weight $d = \frac{3}{2} + c = \frac{3}{2} + \frac{1}{2}(2m_2 + m_1) = 3, \frac{7}{2}, 4, \ldots$. It is annihilated by the operator G^-, and is in the intersection of the images of the operators G^+ (acting from χ_0^-), $\mathcal{D}_{\alpha_1}^{m_1}$ (acting from χ_1^+), $\mathcal{D}_{\alpha_2}^{m_2}$ (acting from χ_2^+).

8.2 Induction from maximal parabolics

When inducing from the maximal parabolic $\mathcal{P}_1 = \mathcal{M}_1 \oplus \mathcal{A}_1 \oplus \mathcal{N}_1$ there is one $\mathcal{M}_1$-compact root, namely, α_1. We take again the Verma module with $\Lambda_{HC} = \Lambda_0^{1-}$. We take $\chi_0^{1-} = \chi_{HC}$. Altogether, the main multiplet in this case includes the same number of ERs/GVMs as in (50), so we may use the same notation only adding super index 1, but in order to avoid coincidence with (50) we must impose the conditions: $m_1 \notin \mathbb{N}$, $m_1 \notin \mathbb{N}/2$.

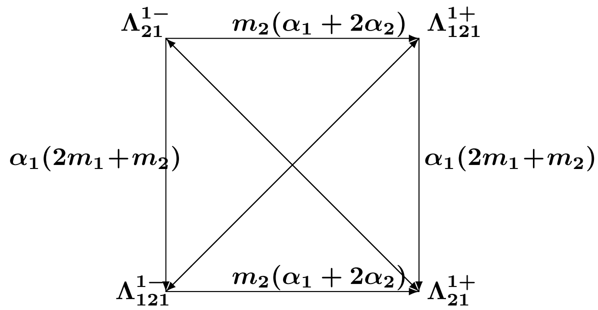
What is peculiar is that the ERs/GVMs of the main multiplet here actually consists of three submultiplets with intertwining diagrams as follows:

$$\begin{array}{ccc} \Lambda_0^{1-} & \stackrel{\mathcal{D}^{m_2}_{\alpha_2}}{\longrightarrow} & \Lambda_2^{1-} \\ \updownarrow & & \updownarrow \\ \Lambda_0^{1+} & \stackrel{\mathcal{D}^{m_2}_{\alpha_2}}{\longleftarrow} & \Lambda_2^{1+} \end{array} \quad \text{subtype } (\mathrm{A}_1) \tag{52a}$$

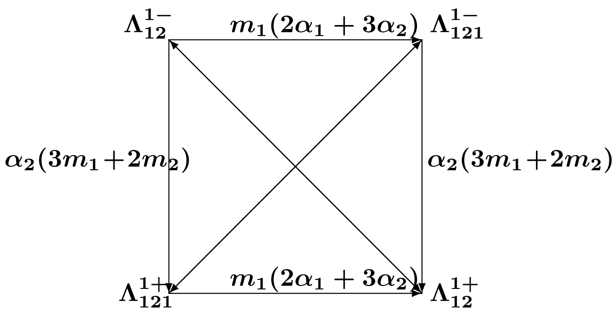
$$\begin{array}{ccc} \Lambda_1^{1-} & \stackrel{\mathcal{D}^{m_2}_{\alpha_5}}{\longrightarrow} & \Lambda_{21}^{1-} \\ \updownarrow & & \updownarrow \\ \Lambda_1^{1+} & \stackrel{\mathcal{D}^{m_2}_{\alpha_5}}{\longleftarrow} & \Lambda_{21}^{1+} \end{array} \quad \text{subtype } (\mathrm{B}_1) \tag{52b}$$

$$\begin{array}{ccc} \Lambda_{12}^{1-} & \stackrel{\mathcal{D}^{m_2}_{\alpha_6}}{\longrightarrow} & \Lambda_{121}^{1+} \\ \updownarrow & & \updownarrow \\ \Lambda_{12}^{1+} & \stackrel{\mathcal{D}^{m_2}_{\alpha_6}}{\longleftarrow} & \Lambda_{121}^{1-} \end{array} \quad \text{subtype } (\mathrm{C}_1) \tag{52c}$$

Next we relax one of the conditions, namely, we allow $m_1 \in \mathbb{N}/2$, still keeping $m_2 \in \mathbb{N}$, $m_1 \notin \mathbb{N}$. This changes the diagram of subtype (C_1), (52c), as given in the next figure:


Λ_{21}^{1-}
$m_2(\alpha_1 + 2\alpha_2)$
Λ_{121}^{1+}
$\alpha_1(2m_1+m_2)$
$\alpha_1(2m_1+m_2)$
Λ_{121}^{1-}
$m_2(\alpha_1 + 2\alpha_2)$
Λ_{21}^{1+}

C_1


Λ_{12}^{1-}
$m_1(2\alpha_1 + 3\alpha_2)$
Λ_{121}^{1-}
$\alpha_2(3m_1+2m_2)$
$\alpha_2(3m_1+2m_2)$
Λ_{121}^{1+}
$m_1(2\alpha_1 + 3\alpha_2)$
Λ_{12}^{1+}

C_2

Inducing from the other maximal parabolic $\mathcal{P}_2$ is partly dual to the previous one. The main multiplet is given as (50) only adding superscript 2 but in order to avoid coincidence with (50) we must impose the conditions: $m_2 \notin \mathbb{N}$, $m_2 \notin \mathbb{N}/2$, $m_2 \notin \mathbb{N}/3$.

Similarly to the $\mathcal{P}_1$ case the ERs/GVMs of the main miltiplet here actually consists of three submultiplets with intertwining diagrams as follows:

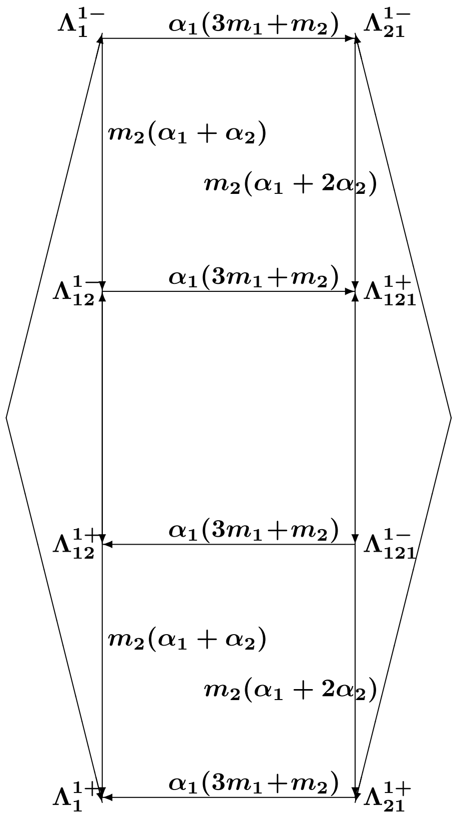
$$\begin{array}{ccc} \Lambda_0^{2-} & \stackrel{\mathcal{D}_{\alpha_1}^{m_1}}{\longrightarrow} & \Lambda_1^{2-} \\ & & \\ \updownarrow & & \updownarrow \qquad \text{subtype } (\mathrm{A}_2) \\ & & \\ \Lambda_0^{2+} & \stackrel{\mathcal{D}_{\alpha_1}^{m_1}}{\longleftarrow} & \Lambda_1^{2+} \end{array} \tag{53a}$$

$$\begin{array}{ccc} \Lambda_2^{2-} & \stackrel{\mathcal{D}_{\alpha_3}^{m_1}}{\longrightarrow} & \Lambda_{12}^{2-} \\ & & \\ \updownarrow & & \updownarrow \qquad \text{subtype } (\mathrm{B}_2) \\ & & \\ \Lambda_2^{2+} & \stackrel{\mathcal{D}_{\alpha_3}^{m_1}}{\longleftarrow} & \Lambda_{12}^{2+} \end{array} \tag{53b}$$

$$\begin{array}{ccc} \Lambda_{21}^{2-} & \stackrel{\mathcal{D}_{\alpha_4}^{m_1}}{\longrightarrow} & \Lambda_{121}^{2-} \\ & & \\ \updownarrow & & \updownarrow \qquad \text{subtype } (\mathrm{C}_2) \\ & & \\ \Lambda_{21}^{2+} & \stackrel{\mathcal{D}_{\alpha_4}^{m_1}}{\longleftarrow} & \Lambda_{121}^{2+} \end{array} \tag{53c}$$

Next we relax one of the conditions, namely, we allow $m_2 \in \mathbb{N}/2$, still keeping $m_2 \notin \mathbb{N}$, $m_2 \notin \mathbb{N}/3$. This changes the diagram of subtype (C_2), (53c), as given in the figure above.

Next we relax another condition, namely, we allow $m_2 \in \mathbb{N}/3$, still keeping $m_2 \notin \mathbb{N}$, $m_2 \notin \mathbb{N}/2$. This changes the diagrams of subtypes (B_1) and (C_1) combining them as given in the next figure:



Submultiplets type (B_1)+(C_1)

Thanks for the attention!