Stability and bifurcations in holographic RG flows of 3d gauged supergravity

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Anastasia Golubtsova (BLTP JINR), Marina Usova (MI RAS)

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Outline

Introduction

The AdS/CFT conjecture

The strongest version of the conjecture $4d$ $\mathcal{N}=4$ SYM with $SU(N)$ is dynamically equivalent to type IIB superstring theory (contains strings and D-branes) on $AdS_5 \times S^5$ with a string length $\ell_s = \sqrt{\alpha'}$ and coupling constant g_s with the radius L and N units of $F_{(5)}$ flux on S^5. (Maldacena'97)

$$g_{YM}^2 = 2\pi g_s, \quad 2g_{YM}^2 N = \frac{L^4}{\alpha'^2}, \quad \lambda = g_{YM}^2 N.$$

	$\mathcal{N}=4$ SYM	IIB theory on $AdS_5 \times S^5$
Strongest form	any N and λ	Quantum string theory, $g_s \neq 0$, $\alpha'/L^2 \neq 0$
Strong form	$N \to \infty$,λ fixed but arbitrary	Classical string theory, $g_s \to 0$,$\alpha'/L^2 \neq 0$
Weak form	$N \to \infty$,λ large	Classical supergravity, $g_s \to 0$,$\alpha'/L^2 \to 0$

The holographical principle

The information of a gravity theory in AdS_{d+1} is mapped to a d theory which lives on the conformal boundary of the $(d+1)$-dimensional spacetime.

The AdS/CFT correspondence

- $d = 2$ CFT has a description in terms of $3d$-gravity in AdS_3:

$$S = \int dx^2 dw \sqrt{-g}(R - \Lambda)$$

.

- An operator $\mathcal{O}(w)$ corresponds to a dynamical bulk field $\phi(x, w)$
- $\phi(x, 0)$ – a source for the $\mathcal{O}$ in the CFT

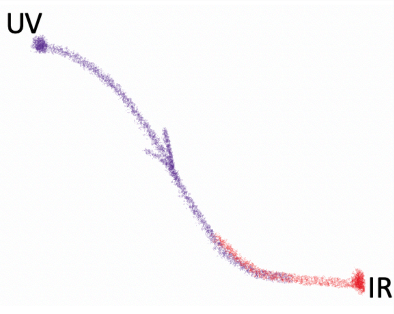
$$S = \int dx^2 dw \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right].$$

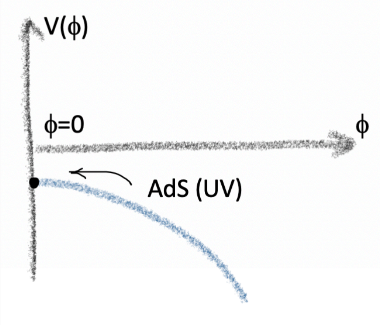
- $\phi(x, w) = \alpha w^{d-\Delta} + \ldots \Leftrightarrow$

$$S = S_{CFT} + \int d^2 x \alpha \mathcal{O}(x)$$

- $\alpha = 0$ – undeformed CFT, bulk scalar – const., spacetime is AdS
- $\alpha \neq 0$ corresponds to relevant coupling for the CFT; deform. AdS

Holographic picture for deviations from conformality





Holographic Renormalization Group

Akhmedov'98; de Boer et. al.'98, Boonstra et. al.'98;Skenderis'99

The domain wall solution

$$ds^2 = e^{2\mathcal{A}(w)}\eta_{ij}dx^i dx^j + dw^2, \quad \phi = \phi(w)$$

- AdS isometry group $\Leftrightarrow$ Poincaré isometry group of DW
- the conformal symmetry at UV and/or IR fixed points
- $e^{\mathcal{A}}$ – measures the field theory energy scale
- $\phi(w)$ identifies with the running coupling along the flow
- The β-function

$$\beta = \frac{d\lambda}{d\log E}|_{QFT} = \frac{d\phi}{dA}_{Holo}$$

The holographic model

The holographic model Sezgin & Deger'99, Deger'02

The action $3d\ \mathcal{N}=2$ supergravity is given by

$$S=\frac{1}{16\pi G_3}\int d^3x\sqrt{|g|}\left(R-\frac{1}{a^2}(\partial\phi)^2-V(\phi)\right)+G.H.Y.,$$

where $G.H.Y.$ – Gibbons-Hawking-York term.

The potential of the scalar field $V(\phi)$ is

$$V(\phi)=2\Lambda_{uv}\cosh^2\phi\left[(1-2a^2)\cosh^2\phi+2a^2\right],$$

where $\Lambda_{uv}<0$ is a cosmological constant, a is a constant (the curvature of the scalar manifold $\mathcal{M}$).
$n=1$ (one scalar):

$$\mathcal{M}=SU(1,1)/U(1).$$

The behaviour of the diltaton potential

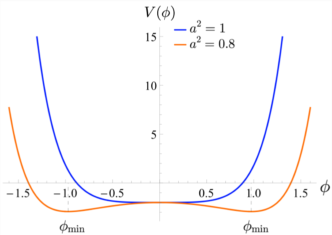


Figure 1: The dependence of the dilaton potential $V(\phi)$ for different a^2; blue curve - for a^2=1, orange curve – $a^2 = 0.8$;

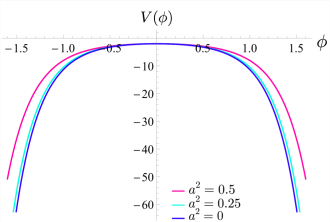


Figure 2: The dependence of the dilaton potential $V(\phi)$ for different a^2: rose curve - for $a^2 = 0.5$, light blue curve – for $a^2 = 0.25$, blue – for $a^2 = 0$

$$\phi_1 = 0, \quad \phi_{2,3} = \frac{1}{2} \ln \left(\frac{1 \pm |a|\sqrt{1-a^2}}{2a^2 - 1} \right).$$

The superpotential of the model

The superpotential reads

$$W = \sqrt{-\Lambda_{uv}}\cosh^2\phi, \quad V(\phi) = \frac{a^2}{4}\left(\frac{\partial W}{\partial \phi}\right)^2 - \frac{1}{2}W^2.$$

For the RG flows W always increases, thus its minimum corresponds to a UV fixed point, while the maximum - to an IR.

$$\mathcal{C}\textbf{-function} \quad \mathcal{C} \sim \frac{1}{W}$$

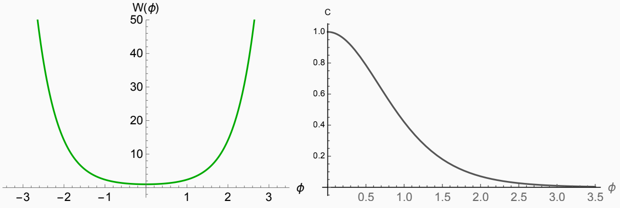


Figure 3: a) The behaviour of $W(\phi)$; b)The behaviour of $\mathcal{C}$-function.

EOM and exact solutions

The ansatz for the metric and for the scalar field is given by

$$ds^2 = e^{2A(w)}(-dt^2 + dx^2) + dw^2, \quad \phi = \phi(w).$$

The equations of motion are

$$\begin{aligned} 2\dot{A}^2 + V - \frac{\dot{\phi}^2}{a^2} &= 0, \\ \ddot{A} + \frac{\dot{\phi}^2}{a^2} &= 0, \\ \ddot{\phi} + 2\dot{A}\dot{\phi} - \frac{a^2}{2}V_\phi &= 0. \end{aligned}$$

The exact solution to the dilaton Deger'02

$$\phi = \frac{1}{2}\log\left(\frac{1 + e^{-4ma^2 w}}{1 - e^{-4ma^2 w}}\right), \quad 0 \le w < \infty, \quad m^2 = -\frac{\Lambda_{uv}}{4}.$$

The metric can be represented as follows:

$$ds^2 = (e^{8ma^2 w} - 1)^{\frac{1}{2a^2}}(-dt^2 + dx^2) + dw^2.$$

The conformal dimension of the operator

CFT side: The deformation of the fixed point $L_{CFT} + \int d^2x\phi_0\mathcal{O}$,

- $\Delta = 2$ marginal operator
- $\Delta < 2$ relevant operator
- $\Delta > 2$ irrelevant operator

Gravity dual: The scalar field in AdS_3

$$\boxed{S \sim \int d^3x\sqrt{-g}\left(g^{\mu\nu}(\partial\phi)^2 + m^2\phi^2\right),}$$

$$ds^2 = \frac{-dt^2 + dx^2 + dz^2}{z^2}, \quad z = e^{w-w_0}, \quad ds^2_{DW} = e^{w-w_0}(-dt^2+dx^2)+dw^2.$$

The Breitenlohner-Freedman bound:

The equation for the scalar field

$$\partial_w^2\phi - 2\partial_w\phi - m^2\phi = 0, \quad \phi \sim e^{\Delta(w-w_0)},$$

The solution:

$$\Delta(\Delta - 2) - m^2 = 0, \quad \Delta_\pm = 1 \pm \sqrt{1+m^2}.$$

At the same time the expansion of the dilaton potential of the quadratic order gives

$$m^2 = -4\Lambda_{uv} a^2(a^2 - 1).$$

The Breitenlohner-Freedman bound:

$$\boxed{\Delta = \Delta_+ = 1 + |1 - 2a^2|}.$$

The conformal dimensions using holography

Possible conformal dimensions

1. for $a^2 = 0$, $\Delta = 2$, the operator is marginal
2. for $0 < a^2 < 1/2$, $1 < \Delta < 2$, the operator is relevant,
3. for $a^2 = 1/2$, $\Delta = 1$,i,e. the operator is relevant,
4. for $1/2 < a^2 < 1$, $1 < \Delta < 2$,i.e. the operator is relevant,
5. for $a^2 = 1$, $\Delta = 2$, the operator is marginal.

The general solution to the scalar field ϕ we can represent using $\Delta_+ = \Delta$ and $\Delta_- = 2 - \Delta$:

$$\boxed{\phi = \phi_0^- e^{-(2-\Delta)w} + \phi_0^+ e^{-\Delta w}.}$$

Holographic RG flows and dynamical system

The autonomous dynamical system

We introduce new variables (Aref'eva, Policastro, AG'19):

$$X = \frac{\dot{\phi}}{\dot{A}}, \qquad Z = e^{-\phi},$$

$Z \in (0, +\infty)$ for $\phi \in (-\infty; \infty)$.

$$\lambda = e^{\phi} \to +\infty, \quad \phi \to +\infty$$

The dynamical system is represented by

$$\frac{dZ}{dA} = f(Z, X),$$
$$\frac{dX}{dA} = g(Z, X),$$

where the functions f and g are defined as:

$$f(Z, X) = -ZX,$$
$$g(Z, X) = \left(\frac{X^2}{a^2} - 2\right)\left(X + \frac{a^2}{2} \times \frac{4\left(2a^2(Z^8 - 1) - (Z^2 - 1)(Z^2 + 1)^3\right)}{(Z^2 + 1)^4 - 2a^2(Z^4 - 1)^2}\right).$$

The points of equilibrium

$$\begin{cases} f(Z,X)\Big|_{Z_c,X_c} = 0, \\ g(Z,X)\Big|_{Z_c,X_c} = 0. \end{cases}$$

The stationary points are

1. $Z_c = 0,\ X_c = a\sqrt{2}$,
2. $Z_c = 0,\ X_c = -a\sqrt{2}$,
3. $Z_c = 0,\ X_c = -2a^2$,
4. $Z_c = 1,\ X_c = 0$,
5. $Z_c = \sqrt{\frac{1-2|a|\sqrt{1-a^2}}{2a^2-1}},\ X_c = 0$,
6. $Z_c = \sqrt{\frac{1+2\sqrt{1-a^2}}{2a^2-1}},\ X_c = 0$.

Stability analysis of equilibrium points

We perturbe near Z_c, X_c: $Z = Z_c + \delta Z$, $X = X_c + \delta X$.

$$\frac{d}{dA}\begin{pmatrix}\delta Z\\ \delta X\end{pmatrix} = \mathcal{M}\begin{pmatrix}\delta Z\\ \delta X\end{pmatrix},$$

where $\mathcal{M}$ – the Jacobian matrix

$$\mathcal{M} = \begin{pmatrix}\frac{\partial f}{\partial Z} & \frac{\partial f}{\partial X}\\ \frac{\partial g}{\partial Z} & \frac{\partial g}{\partial X}\end{pmatrix}\Big|_{Z=Z_c, X=X_c}$$

$$\mathcal{M}_{11} = -X_c, \quad \mathcal{M}_{12} = -Z_c,$$

$$\mathcal{M}_{21} = -\frac{8Z_c(2a^2 - X_c^2)\left(8a^4 Z_c^2(Z_c^2-1)^2 + 2a^2(Z_c^2+1)^2(Z_c^4+1) - (Z_c^2+1)^4\right)}{(Z_c^2+1)^2\left((Z_c^2+1)^2 - 2a^2(Z_c^2-1)^2\right)^2},$$

$$\mathcal{M}_{22} = \frac{3X_c^2}{a^2} - 2 - \frac{4X_c\left((Z_c^2-1)(Z_c^2+1)^3 - 2a^2(Z_c^8-1)\right)}{(Z_c^2+1)^4 - 2a^2(Z_c^4-1)^2}.$$

The characteristic equation is:

$$\lambda^2 - \lambda\left(\mathcal{M}_{11} + \mathcal{M}_{22}\right) + \mathcal{M}_{11}\mathcal{M}_{22} - \mathcal{M}_{12}\mathcal{M}_{21} = 0.$$

Point	$a^2=0$	$0<a^2<\frac{1}{2}$	$a^2=\frac{1}{2}$	$\frac{1}{2}<a^2<1$	$a^2=1$
1	none	$a\in(-\frac{1}{\sqrt{2}};0)$ unst. node $a\in(0;\frac{1}{\sqrt{2}})$ saddle	$a=\frac{1}{\sqrt{2}}$ saddle $a=-\frac{1}{\sqrt{2}}$ none	saddle	saddle
2	none	$a\in(0;\frac{1}{\sqrt{2}})$ unst. node $a\in(-\frac{1}{\sqrt{2}};0)$ saddle	$a=\frac{1}{\sqrt{2}}$ none $a=-\frac{1}{\sqrt{2}}$ saddle	saddle	saddle
3	none	saddle	none	unst.node	unst. node
4	none	stable node	stable node	stable node	none
5,6	saddle	saddle	saddle	saddle	none

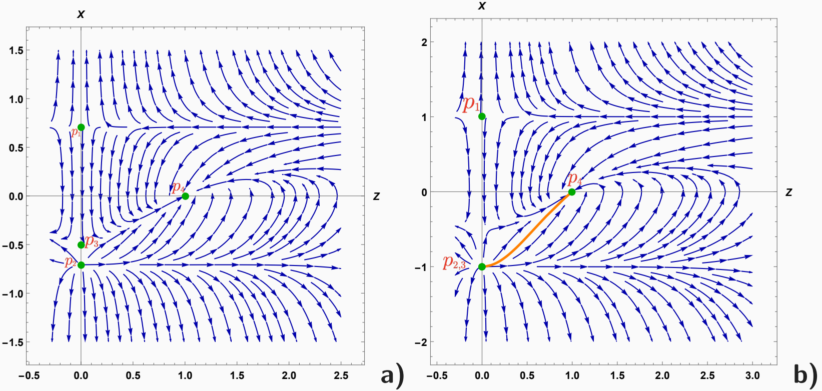


Figure 4: **a)** Phase portrait for $a^2 = 0.25$; **b)** Phase portrait for $a^2 = 0.5$.

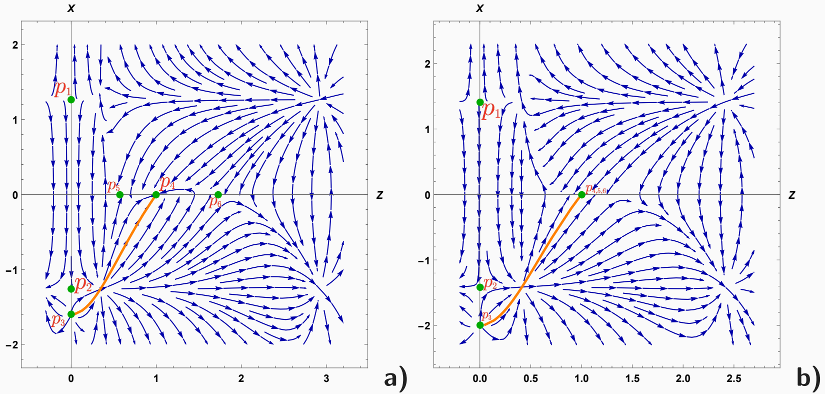


Figure 5: a) Phase portrait for $a^2 = 0.8$; **b)** Phase portrait for $a^2 = 1$.

Bifurcations

- The bifurcation occurs when a control parameter change causes a change of stability properties of critical points of the dynamical system.
- A local bifurcation is a bifurcation of a dynamic system that can be identified by analyzing the stability of fixed points.
- Global bifurcations cannot be detected only by a stability analysis of the fixed points and often occur when an invariant sets of the system 'collide' with each other, or with fixed points of the system.
- Typically the bifurcations are characterized by a vanishing eigenvalue of Jacobian matrix.

$$Z_c = 0: \ \dot{X} = (\frac{X^2}{a^2} - 2)(X - 2a^2), \quad X_c = -\sqrt{2}a, \quad X_c = \sqrt{2}a, \quad X_c = 2a^2.$$

1) $a = \frac{1}{\sqrt{2}}$ $X_c = \sqrt{2}a$ none $\lambda_1 = \sqrt{2}a, \quad \lambda_2 = 4(1 - a\sqrt{2})$
$\det \mathcal{M} = 4\sqrt{2}a(1 - \sqrt{2}a)$; $X_c = -\sqrt{2}a$ saddle(unstable), $X_c = 2a^2$ (none), **2)** while for $a = -\frac{1}{\sqrt{2}}$ $X_c = \sqrt{2}a$ unstable, for $X_c = -\sqrt{2}a$ and $X_c = 2a^2$ none.

Bifurcations

Gukov'17

$$\Delta = |1 - 2a^2| + 1; \quad \Delta - d = |1 - 2a^2| - 1.$$

1. for $a^2 = 0$ $\Delta - d = 0$,
2. for $0 < a < \frac{1}{\sqrt{2}}$, $\Delta - d = -2a^2$,
3. for $-\frac{1}{\sqrt{2}} < a < 0$, $\Delta - d = -2a^2$,
4. for $a^2 = \frac{1}{2}$, $\Delta - d = -1$,
5. for $a^2 = 1$, $\Delta - d = 0$,
6. $\frac{1}{\sqrt{2}} < a < 1$, $\Delta - d = 2(a^2 - 1)$,
7. $-1 < a < -\frac{1}{\sqrt{2}}$, $\Delta - d = 2(a^2 - 1)$.

Asymptotic solutions near the fixed points

Asymptotic gravitational solutions near the fixed points

Recall:

$$\frac{\ddot{A}}{\dot{A}^2} = -\frac{X_c^2}{a^2}, \quad X_c = \frac{\dot{\phi}}{\dot{A}}.$$

The generic form of the solution for the metric and the dilaton:

$$\boxed{A = \frac{a^2}{X_c^2} \ln\Big[\frac{X_c^2 \dot{A}_0 (w - w_0) + a^2}{X_c^2 \dot{A}_0 (w_1 - w_0) + a^2}\Big] + A_0,}$$

and

$$\boxed{\phi = \frac{a^2}{X_c} \ln\Big[\frac{\dot{A}_0 X_c^2 (w - w_0) + a^2}{\dot{A}_0 X_c^2 (w_1 - w_0) + a^2}\Big] + \phi_0,}$$

where $A_0 = A(w_0), \phi_0 = \phi(w_0)$, $\dot{A}_0 = \dot{A}(w_0)$, w_0, w_1 are constants of integration.

- $Z_c = 0$, $X_c = \sqrt{2}a$. The metric and the dilaton are given by

$$ds^2 \cong \left|\frac{2\dot{A}_0(w-w_0)+1}{2\dot{A}_0(w_1-w_0)+1}\right|(-dt^2+dx^2)+dw^2, \phi = \frac{a}{\sqrt{2}}\ln\left|\frac{2\dot{A}_0(w-w_0)+1}{2\dot{A}_0(w_1-w_0)+1}\right|+\phi_0.$$

Since $Z_c = 0$, then $\phi \to +\infty$, so $a > 0$ and $w \to w_0 - \frac{1}{2\dot{A}_0}$, or $w \to +\infty$ and $a < 0$. The potential $\phi \to +\infty$: $V \to \pm\infty$, however from the EOM $V = 0, \quad \frac{dV}{d\phi} = 0$.
NOT A SOLUTION TO EOM.

- $Z_c = 0, \quad X_c = -a\sqrt{2}$. The metric and the dilaton are given by

$$ds^2 \cong \left|\frac{2\dot{A}_0(w-w_0)+1}{2\dot{A}_0(w_1-w_0)+1}\right|(-dt^2+dx^2)+dw^2, \phi = -\frac{a}{\sqrt{2}}\ln\left|\frac{2\dot{A}_0(w-w_0)+1}{2\dot{A}_0(w_1-w_0)+1}\right|+\phi_0.$$

NOT A SOLUTION TO EOM.

- $Z_c = 0$, $X_c = -2a^2$. The metric and the dilaton are

$$ds^2 \cong \left| \frac{4a^2 \dot{A}_0 (w - w_0) + 1}{4a^2 \dot{A}_0 (w_1 - w_0) + 1} \right|^{\frac{1}{2a^2}} (-dt^2 + dx^2) + dw^2,$$

$$\phi = -\frac{1}{2} \ln \left| \frac{4a^2 \dot{A}_0 (w - w_0) + 1}{4a^2 \dot{A}_0 (w_1 - w_0) + 1} \right| + \phi_0.$$

Since $Z_c = 0$, $\phi \to +\infty$, for $\phi \to +\infty$ the potential behaves as

$$V \sim \begin{cases} -\infty, & \text{for} \quad 0 \leq a^2 \leq \frac{1}{2}, \\ +\infty, & \text{for} \quad a^2 > \frac{1}{2}. \end{cases}$$

SOLVES EOM for any a and $w \to w_0 - \frac{1}{4a^2 \dot{A}_0}$.

- $Z_c = 1$, $X_c = 0$. The metric and the dilaton are given by

$$ds^2 \approx e^{2\sqrt{-\Lambda_{uv}}(w-w_0)} \left(-dt^2 + dx^2\right) + dw^2, \quad \phi = 0, \quad V = 2\Lambda_{uv}.$$

where w_0 –a constant of integration.**SOLVES EOM for any** a.

- $Z_c = \sqrt{\frac{1-2|a|\sqrt{1-a^2}}{2a^2-1}}, \quad X_c = 0.$
  The metric and the scalar field are

$$s^2 \approx e^{2a^2\sqrt{-\frac{\Lambda_{uv}}{2a^2-1}}(w-w_0)}(-dt^2+dx^2)+dw^2, \quad \phi = \ln\sqrt{\frac{1-2|a|\sqrt{1-a^2}}{2a^2-1}}$$

$$V = \frac{2a^4\Lambda_{uv}}{2a^2-1},$$

where w_0 – the constant of integration.**SOLVES EOM for $a^2 > \frac{1}{2}$**.

- $Z_c = \sqrt{\frac{1+2|a|\sqrt{1-a^2}}{2a^2-1}}, \quad X_c = 0.$
  The metric and the scalar field are

$$ds^2 \approx e^{2a^2\sqrt{-\frac{\Lambda_{uv}}{2a^2-1}}(w-w_0)}(-dt^2+dx^2)+dw^2, \quad \phi = \ln(\sqrt{\frac{1+2|a|\sqrt{1-a^2}}{2a^2-1}})$$

$$V = \frac{2a^4\Lambda_{uv}}{2a^2-1},$$

where w_0 – the constant of integration.**SOLVES EOM for $a^2 > \frac{1}{2}$**.

Holographic RG flows

Point	$V(\phi)$	**Type with energy scale**	UV/IR
$p_3,\ Z_c = 0,\ X_c = -2a^2$	$V \to \pm\infty$	stable for all a	IR
$p_4,\ Z_c = 1,\ X_c = 0$	const	Unstable for all a	UV
$p_5,\ Z_c = \sqrt{\frac{1-2\|a\|\sqrt{1-a^2}}{2a^2-1}},\ X_c = 0$	const	Unstable for all a	UV
$p_6,\ Z_c = \sqrt{\frac{1+2\|a\|\sqrt{1-a^2}}{2a^2-1}},\ X_c = 0$	const	Unstable for all a	UV

Holographic RG flows

Possible RG flows:

- $a^2 < \frac{1}{2}$:
  - p_4 (UV, AdS_3, $\phi = 0$) to p_3 (IR, $\phi \to +\infty$)

- $a^2 > \frac{1}{2}$:
  - p_4 (UV, AdS_3, $\phi = 0$) to p_3 (IR, $\phi \to +\infty$);
  - p_5 (UV, AdS_3, $\phi = \ln(\sqrt{\frac{1-2|a|\sqrt{1-a^2}}{2a^2-1}})$) to p_3 (IR, $\phi \to +\infty$);
  - p_6 (UV, AdS_3, $\phi = \ln(\sqrt{\frac{1+2|a|\sqrt{1-a^2}}{2a^2-1}})$) to p_3 (IR, $\phi \to +\infty$).

Outlook

Outlook

Summary

- Relevant deformations related to holographic RG flows were studied
- Stability analysis of equilibrium points was done
- Classification of fixed points according to stability was done
- Holographic RG flows from UV fixed points with $\phi = const$ to IR fixed points with $\phi \to +\infty$ were found (AdS-hypescaling violating geometry flows, no AdS-AdS flows).

Questions

- Finite-temperature generalizations; Gubser's bound $(V \leq 0)$?
- Phase transitions as bifurcations? (Gukov'17)
- Are there irrelevant deformations? (the so-called Zamolodchikov $T\bar{T}$-deformations)

Thank you for attention!