Spherically symmetric black holes and physical vacuum

Ivanova I.D.

Institute for Nuclear Research of the Russian Academy of Sciences, Moscow, Russia

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Motivation

- The importance of investigation of particle production by strong fields and the back reaction of these processes on a space-time metric which includes not only the influence of the created particles but also the contribution due to the vacuum polarization.

- The main obstacle in accounting for the back reaction is that the rigorous solution of the quantum problem requires the knowledge of the boundary conditions, while the latter can be imposed only after solving the field equations. To avoid this difficulty an attempt has been made to describe the particle creation process phenomenologically on the classical level.

- Using the example of the action for an ideal fluid with a variable number of particles, it can be shown that spherically symmetric vacuum solutions of the black hole type in GR (and also for some cases of QG) cannot describe the physical vacuum, i.e. a state when particles can be produced, but they are not still present.

Action for the perfect fluid in Eulerian variables

$$S_m = \int \{ -E(n, X) + \mu_0 (u_a u^a - 1) + \mu_1 \nabla_a (n u^a) + \\ + \mu_2 \partial_a X u^a \} \sqrt{|g|}\, d^4x, \quad (1)$$

J.R.Ray J.Math.Phys. 13 (1972)

$\mu_0(x)$, $\mu_1(x)$ and $\mu_2(x)$ are the Lagrange multipliers.

The dynamical variables are: the number density $n(x)$, the four velocity vector of fluid's flow u^a and some auxiliary field $X(x)$ for enumeration of the world-lines.

The invariant energy density of the fluid: $E(n, X) = n\,(m(X) + \Pi(n))$,

where $\Pi(n)$ is the potential energy describing the (self)interaction between the constituent particles, and $m(X)$ is their mass distribution.

The hydrodynamic pressure is $p = n^2 \frac{d\Pi}{dn} = -E + n \frac{dE}{dn}$.

Motion equations and constraints

$$\delta n : -\frac{\partial E}{\partial n} - \partial_a \mu_1 \, u^a = 0 \,, \tag{2}$$

$$\delta u^a : \; \mu_2 \, \partial_a X + 2\mu_0 \, u_a - \partial_a \mu_1 \, n = 0 \,, \tag{3}$$

$$\delta X : \; -\frac{\partial E}{\partial X} - \nabla_a \left(\mu_2 \, u^a \right) = 0 \,, \tag{4}$$

$$\delta \mu_0 : \; u_a \, u^a = 1 \,, \tag{5}$$

$$\delta \mu_1 : \; \nabla_a \left(n \, u^a \right) = 0, \tag{6}$$

$$\delta \mu_2 : \; \partial_a X \, u^a = 0 \,, \tag{7}$$

$$\delta g_{ab} : \; T^{ab} = (p + E) \, u^a u^b - p \, g^{ab} \,. \tag{8}$$

The phenomenological description of particle creation

$$\mu_1 \nabla_a (n\, u^a) \to \mu_1 \, \{\nabla_a (n\, u^a) - \Phi)\}$$

$\Phi(inv)$ is some function of the invariants characterizing the field(s) that causes the particle creation

V,A.Berezin Int.J.Mod.Phys. A 2 (1987)

The continuity equation is replaced by the law of particle creation:

$$\nabla_a (n\, u^a) = 0 \to \nabla_a (n\, u^a) = \Phi \tag{9}$$

Energy-momentum tensor:

$$T^{ab} = (p + E)\, u^a u^b - p\, g^{ab} + \frac{2}{\sqrt{|g|}} \frac{\delta \left(\mu_1 \, \Phi \, \sqrt{|g|} \right)}{\delta g_{ab}}$$

Conformal invariance of particle production rate

Under a local conformal transformation of the metric: $g_{ab} = e^{2\omega}\, \widetilde{g}_{ab}$, the number density, the four-velocity, and the determinant of the metric change accordingly:

$$n = \frac{\widetilde{n}}{e^{3\omega}}, \quad u^a = \frac{\widetilde{u}^a}{e^{\omega}}, \quad \sqrt{-g} = \sqrt{-\widetilde{g}}\, e^{4\omega}\,,$$

which implies that:

$$\nabla_a \left(n\, u^a\right)\, \sqrt{-g} = \partial_a \left(n\, u^a\, \sqrt{-g}\right) = \partial_a \left(\widetilde{n}\, \widetilde{u}^a\, \sqrt{-\widetilde{g}}\right) =$$

$$= \widetilde{\nabla}_a \left(\widetilde{n}\, \widetilde{u}^a\right)\, \sqrt{-\widetilde{g}}$$

Then the relation (9) also implies the conformal invariance of $\Phi\, \sqrt{-g}$.

The particle production law in the absence of external fields

In the Riemannian geometry the combination $C^2\sqrt{-g} = C_{abcd}C^{abcd}\sqrt{-g}$ is conformally invariant, where C_{abcd} is the Weyl tensor defined as:

$$C_{abcd} = R_{abcd} + \frac{1}{2}\left(R_{ad}\, g_{bc} + R_{bc}\, g_{ad} - R_{ac}\, g_{bd} - R_{bd}\, g_{ac}\right) + \\ + \frac{1}{6} R\left(g_{ac}\, g_{bd} - g_{ad}\, g_{bc}\right) . \quad (10)$$

Without external classical fields the particle are created solely by the vacuum fluctuations due to the gravitational field thus Φ should depend on the gravitational invariants. Restricting ourselves to combinations of geometric invariants that are at most quadratic in curvature, we obtain: $\Phi = \beta C^2$.

The same result was obtained in: *Ya. B. Zel'dovich et al Sov. Phys. JETP 34 (1972)* for the particle creation by the vacuum fluctuations of the massless scalar field on the background metric of the homogeneous and slightly anisotropic cosmological space-time. Now it becomes fundamental for any Riemannian geometry, irrespective of the form of the gravitational Lagrangian and also the back reaction is taken into account.

Physical vacuum

Let's consider the vacuum solutions, when particles can be, in principle, produced, but they are not still present. It is called "the pregnant vacuum" (*arXiv:2203.04257 [gr-qc]*) and is an example of physical vacuum.

This case corresponds to the solution: $n = \Phi = 0$ of the equation (9). In addition, u^a is an arbitrary non-zero vector normalized by the condition (5) (the exception is cosmology), so for the equation (2):

$$\partial_a \mu_1 \, u^a = -\frac{p+E}{n}, \tag{11}$$

two options are possible:

1. matter that can potentially be born has non-zero pressure $p > 0$ and $\lim\limits_{n\to 0} \dfrac{E+p}{n} = 0$, therefore $\partial_a \mu_1 \, u^a = 0$. The arbitrariness of u^a implies $\mu_1 = const$.
2. dust is born, i.e. $p = 0$, $E = m_0 \, n$, then $\partial_a \mu_1 \, u^a = -m_0$. Similarly we obtain $m_0 = 0$ and $\mu_1 = const$.

It means that "the pregnant vacuum" cannot produce dust.

Gravitational action

Action of quadratic gravity:

$$S_q = -\frac{1}{16\pi}\int \sqrt{-g}\left(\alpha_1 R_{abcd}R^{abcd} + \alpha_2 R_{ab}R^{ab} + \alpha_3 R^2 + \alpha_4 R + \alpha_5 \Lambda\right)\, d^4x$$

Field equations:

$$2\left(4\alpha_1 + \alpha_2\right) B_{ab} + \frac{1}{3}\left(3\alpha_3 + \alpha_1 + \alpha_2\right) D_{ab} + \\ + \alpha_4 G_{ab} - \frac{\alpha_5}{2} g_{ab}\Lambda = 8\pi T_{ab}, \quad (12)$$

where
$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R, \quad D_{ab} = \left(2R_{ab} - \frac{1}{2}g_{ab}R + 2\,g_{ab}\Box - 2\,\nabla_b\nabla_a\right)R,$
$B_{ab} = \left(\nabla^c\,\nabla^d + \frac{1}{2}R^{cd}\right)C_{acbd}$ - Bach tensor.

Energy-momentum tensor for $\Phi = \beta\, C^2$:

$$T^{ab} = \left(p + E\right) u^a u^b - p\, g^{ab} - 8\beta\left(\nabla_c\,\nabla_d + \frac{1}{2}R_{cd}\right)\left(\mu_1\, C^{acbd}\right),$$

For physical vacuum $\mu_1 = const$, $n = 0$ therefore: $T^{ab} = -8\beta\,\mu_1\, B^{ab}$.

Spherical symmetry

General case of spherically symmetric metric:

$$ds^2 = r^2(x)\left(\widetilde{\gamma}_{\alpha\beta}\, dx^\alpha dx^\beta - d\Omega^2\right),\ \alpha, \beta = 0, 1. \quad (13)$$

Invariants:
$\Delta = g^{ab}\partial_a r \partial_b r$, $\widetilde{R}$ - scalar curvature of $\widetilde{\gamma}_{\alpha\beta}$, $\sigma = \Box\, r - \frac{2}{r}\,\Delta$,
connection with four-dimensional scalar curvature:
$R = \frac{1}{r^2}\left(\widetilde{R} - 2 - 6\, r\, \sigma\right)$

For the spherically symmetric geometry $C^2 = \frac{(\widetilde{R}-2)^2}{3r^4}$, therefore "the pregnant vacuum" corresponds to $\widetilde{R} = 2$.

The Bach tensor of the spherically symmetric metric with $\widetilde{R} = 2$ is zero (*V. A. Berezin et al J. Cosmol. Astropart. Phys. JCAP01(2016)019*), so the condition $T^{ab} = 0$ is automatically satisfied.

Physical vacuum in quadratic gravity

In GR($\alpha_1 = \alpha_2 = \alpha_3 = 0$) the general case of spherically symmetric vacuum is the Schwarzschild-de Sitter metric:

$$ds^2 = f(r)dt^2 - f^{-1}dr^2 - r^2 d\Omega^2, \quad f(r) = 1 - \frac{2M}{r} - \frac{r^2}{6}\frac{\alpha_5}{\alpha_4}\Lambda,$$

for which $\widetilde{R} = 2 - \frac{12M}{r}$.

Therefore for GR with the considered matter action only the de Sitter metric corresponds to the physical vacuum in case of spherical symmetry.

The same is true for the GR+C^2($\alpha_2 = -2\alpha_1$, $\alpha_3 = \frac{1}{3}\alpha_1$) since the Bach tensor is zero.

As for the general case of quadratic gravity the QG version of trace no-hair theorem implies that for static spacetimes with R sufficiently quickly approaching a constant, $R = -2\frac{\alpha_5}{\alpha_4}\Lambda$ throughout the spacetime (*Vojtech P. et al Phys. Rev. D 103 (2021); Lu H. et al Phys. Rev. D 92(2015); Nelson W. Phys. Rev. D 82 (2010)*). For a spherically symmetric geometry with $\widetilde{R} = 2$ this condition again leads to the fact that the de Sitter metric is the only possible solution of (12), taking into account all the restrictions.

Let's introduce an external scalar field into the function Φ describing the law of particle production. One of the simplest combinations that gives a non-trivial equation of motion and conformally invariant when multiplied by $\sqrt{-g}$ is:
$\varphi\,\Box\varphi - \frac{1}{6}\,\varphi^2\,R + \Lambda_0\,\varphi^4$.

Let's consider $\Phi = \beta\,C^2 + \alpha\left(\varphi\,\Box\varphi - \frac{1}{6}\,\varphi^2\,R + \Lambda_0\,\varphi^4\right)$.

Additional motion equation:

$$\delta\varphi: \quad \mu_1\Box\varphi + \Box\left(\mu_1\varphi\right) + 4\mu_1\,\Lambda_0\varphi^3 - \frac{1}{3}\mu_1\,\varphi\,R = 0\,. \tag{14}$$

Energy-momentum tensor:

$$T^{ab} = (p+E)\,u^a u^b - p\,g^{ab} - 8\beta\left(\nabla_c\,\nabla_d + \frac{1}{2}\,R_{cd}\right)\left(\mu_1\,C^{acbd}\right) + \\ + \alpha\,\mu_1 g^{ab}\Lambda_0\,\varphi^4 - \alpha\,g^{ab}\,\partial_c\left(\mu_1\varphi\right)\partial^c\varphi + \alpha\,\partial^a\left(\mu_1\varphi\right)\partial^b\varphi + \\ + \alpha\,\partial^b\left(\mu_1\varphi\right)\partial^a\varphi + \frac{\alpha}{3}\left\{\mu_1\varphi^2\,G^{ab} - \nabla^a\nabla^b\left(\mu_1\varphi^2\right) + g^{ab}\Box\left(\mu_1\varphi^2\right)\right\} \tag{15}$$

Let's consider also the situation of physical vacuum for this case:

$$\Phi = \beta\, C^2 + \alpha \left(\varphi\, \Box \varphi - \frac{1}{6}\, \varphi^2\, R + \Lambda_0\, \varphi^4 \right) = 0\,. \qquad (16)$$

Since $n = 0$ and $\mu_1 = const$ the energy-momentum tensor and the equation of motion for φ reduce to:

$$\frac{1}{\mu_1 \alpha}\, T^{ab} = -\frac{8\beta}{\alpha}\, B^{ab} + g^{ab} \left(\Lambda_0\, \varphi^4 - \partial_c \varphi\, \partial^c \varphi \right) +$$

$$+ 2\partial^a \varphi\, \partial^b \varphi + \frac{1}{3} \left\{ \varphi^2\, G^{ab} - \nabla^a \nabla^b \left(\varphi^2 \right) + g^{ab}\, \Box \left(\varphi^2 \right) \right\} = 0\,, \qquad (17)$$

$$2\Box \varphi + 4\Lambda_0\, \varphi^3 - \frac{1}{3}\, \varphi\, R = 0\,, \qquad (18)$$

If we express $\Box\varphi$ from the equation of motion (18) and substitute it into the condition (16), we get:

$$\varphi^4 = \frac{\beta}{\alpha \Lambda_0}\, C^2\,. \qquad (19)$$

For spherically symmetric geometry it follows from the equation (19) that:

$$\varphi = \pm \left(\frac{\beta}{3\alpha\Lambda_0} \right)^{\frac{1}{4}} \frac{\sqrt{|\widetilde{R} - 2|}}{r} \tag{20}$$

If we consider the Schwarzschild-de Sitter space-time then $\varphi \propto r^{-\frac{3}{2}}$ in this case:

$$\Box\varphi = -\frac{1}{r^2}\,\partial_r\left(r^2 f(r)\,\partial_r\varphi\right) \propto \left(-\frac{1}{2} r^{-\frac{7}{2}} + \frac{3M}{r} - \frac{\Lambda\,\alpha_5}{4\,\alpha_4}\, r^{-\frac{3}{2}} \right),$$

but this result contradicts the equation of motion (18).

Thus, in GR the physical vacuum for the considered model with a scalar field in the spherically symmetric case cannot be the Schwarzschild de Sitter space-time. An exception is pure de Sitter, but in this case $\varphi = 0$.

Conclusions

Let's summarize the results obtained:

- The conformal invariance of the particle production law is shown.

- It is demonstrated that "the pregnant vacuum" cannot produce dust but it can give birth to matter with non-zero pressure, for example, thermal radiation.

- In the absence of external fields in GR there are no spherically symmetric vacuum solutions of the black hole type corresponding to the physical vacuum. The same is true for quadratic gravity if we restrict ourselves to static spacetimes with R sufficiently quickly approaching a constant.

- In GR the physical vacuum for the model with the external scalar field in the spherically symmetric case cannot be described by vacuum solutions of the black hole type.

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