# Anti-de Sitter neutron stars in the theory of gravity with nonminimal derivative coupling

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International Conference on Quantum Field Theory, High-Energy Physics, and Cosmology

*Dubna, July 18, 2022*

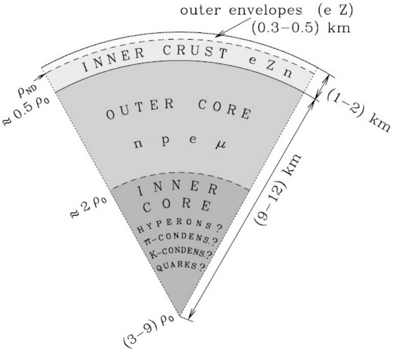
Based on

- P.E. Kashargin, S.V. Sushkov, Anti-de Sitter neutron stars in the theory of gravity with nonminimal derivative coupling, arxiv:2205.08949 (2022)

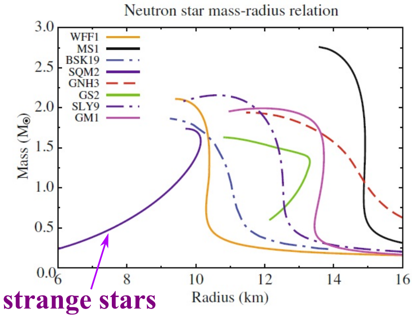
**Plan of the talk**

- Introduction
- Theory of gravity with nonminimal derivative coupling
- Spherically symmetric configuration
- Basic equation
- Results of numerical integration
- Summary

**Schematic structure of a neutron star**[1]




**Neutron star mass-radius relation**[2]




The available data suggests that the most neutron stars have masses close to 1.3 – 1.4 $M_{sun}$ and characteristic star radii is 10 – 13 $km$, but lower and higher masses exist and this ranges can be extended[3].

[1] Neutron Stars: Equation of State and Structure, P. Haensel, A. Y. Potekhin, and D. G. Yakovlev, Astrophysics and Space Science Library, (Springer, New York, 2007).

[2] The Physics and Astrophysics of Neutron Stars, ed L. Rezzolla et al. Springer, Cham 457 (2018).

[3] J. M. Lattimer, The Nuclear Equation of State and Neutron Star Masses, Annu. Rev. Nucl. Part. Sci. 62, pp.485-515 (2012).

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- Black hole solutions in the theory of gravity with a nonminimal derivative coupling of a scalar field with the Einstein tensor.
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**Action**[1,2]**:**

$$S = \int \left[L_2 + L_3 + L_4 + L_5\right] \sqrt{-g}\, d^4x + S^{(m)}, \tag{1}$$

where

$$\begin{aligned}
L_2 &= G_2(\phi, X), \\
L_3 &= -G_3(\phi, X)\Box\phi, \\
L_4 &= G_4(\phi, X)R + G_{4,X}(\phi, X)\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right], \\
L_5 &= G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5,X}(\phi, X)\times \\
&\times\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)\left(\nabla^\beta\nabla_\mu\phi\right)\right],
\end{aligned}$$

$R$ is the Ricci scalar, $G_{\mu\nu}$ is the Einstein tensor, $S^{(m)}$ is the action for matter, $G_j$ depend on the scalar field $\phi$ and its kinetic term $X = -\nabla_\mu\phi\nabla^\mu\phi/2$, $G_{j,X} = \partial G_j/\partial X$, $\Box\phi = \nabla_\mu\nabla^\mu\phi$.

[1] G.W. Horndeski, Second-order scalar-tensor field equations in a four-dimensional space, Int. J. Theor. Phys. 10, 363-384 (1974).

[2] T. Kobayashi, M. Yamaguchi, J. Yokoyama, Generalized G-inflation: Inflation with the most general second-order field equations, Prog. Theor. Phys. 126, 511 (2011).

**Specific case**

$$G_2 = \alpha X - \Lambda_0/\kappa, \quad G_3 = 0, \quad G_4 = 1/(2\kappa), \quad G_5 = \beta\phi/2 \tag{2}$$

yields the following action:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}(R - 2\Lambda_0) - \frac{1}{2}\left(\alpha g_{\mu\nu} + \beta G_{\mu\nu}\right)\nabla^\mu\phi\nabla^\nu\phi\right] + S^{(m)}, \tag{3}$$

where $\Lambda_0$ is a «bare» cosmological constant, $\kappa = 8\pi G/c^4$, $\alpha$ and $\beta$ are real parameters.

**Action:**

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}(R - 2\Lambda_0) - \frac{1}{2}\left(\varepsilon_1 g_{\mu\nu} + \varepsilon_2\ell^2 G_{\mu\nu}\right)\nabla^\mu\phi\nabla^\nu\phi\right] + S^{(m)}, \tag{4}$$

where $\varepsilon_i = \pm 1$, $\ell$ is a characteristic length.

**In this work we study the neutron star configurations in the scalar-tensor theory of gravity (4) with the coupling between the kinetic term of a scalar field and the Einstein tensor.**

**Gravitational field equations:**

$$\frac{1}{\kappa}\left(G_{\mu\nu}+g_{\mu\nu}\Lambda_0\right)=\varepsilon_1 T^{(\phi)}_{\mu\nu}+\varepsilon_2\ell^2\Theta_{\mu\nu}+T^{(m)}_{\mu\nu}, \tag{5}$$

where

$$T^{(\phi)}_{\mu\nu}=\nabla_\mu\phi\nabla_\nu\phi-\frac{1}{2}g_{\mu\nu}\left(\nabla\phi\right)^2, \tag{6}$$

$$\Theta_{\mu\nu}=-\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi R+2\nabla_\alpha\phi\nabla_{(\mu}\phi R^\alpha_{\nu)}+\nabla^\alpha\phi\nabla^\beta\phi R_{\mu\alpha\nu\beta}+\nabla_\mu\nabla^\alpha\phi\nabla_\nu\nabla_\alpha\phi- \tag{7}$$

$$\nabla_\mu\nabla_\nu\phi\Box\phi-\frac{1}{2}\left(\nabla\phi\right)^2 G_{\mu\nu}+g_{\mu\nu}\left[-\frac{1}{2}\nabla^\alpha\nabla^\beta\phi\nabla_\alpha\nabla_\beta\phi+\frac{1}{2}\left(\Box\phi\right)^2-\nabla_\alpha\phi\nabla_\beta\phi R^{\alpha\beta}\right],$$

$$T^{(m)}_{\mu\nu}=(\epsilon+p)u_\mu u_\nu+pg_{\mu\nu}, \tag{8}$$

$u_\mu$ is a unit timelike 4-vector, $\epsilon$ is an energy density, $p$ is an isotropic pressure.

**Equation of motion of the perfect fluid:**

$$\nabla^\mu T^{(m)}_{\mu\nu}=0. \tag{9}$$

**Equation of the scalar field:**

$$\nabla_\mu J^\mu=0, \tag{10}$$

where

$$J^\mu=\left(\varepsilon_1 g^{\mu\nu}+\varepsilon_2\ell^2 G^{\mu\nu}\right)\nabla_\nu\phi. \tag{11}$$

**Spacetime metric:**

$$ds^2 = -A(r)c^2dt^2 + \frac{dr^2}{B(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{12}$$

**Gravitational field equations:**

$$\frac{1}{\kappa}\left(-\frac{B'}{r} + \frac{1-B}{r^2}\right) = \epsilon + \frac{1}{\kappa}\Lambda_0 + \frac{1}{2}\varepsilon_1 B\psi^2 - \varepsilon_2\ell^2\frac{B\psi^2}{2r^2}\left(1 + B + 3rB' + 4rB\frac{\psi'}{\psi}\right), \tag{13}$$

$$\frac{1}{\kappa}\left(\frac{BA'}{rA} - \frac{1-B}{r^2}\right) = p - \frac{1}{\kappa}\Lambda_0 + \frac{1}{2}\varepsilon_1 B\psi^2 - \varepsilon_2\ell^2\frac{B\psi^2}{2r^2}\left(1 - 3B - 3rB\frac{A'}{A}\right), \tag{14}$$

where a prime means a derivative with the respect of $r$, and $\psi = \phi'$.

**Equation of the scalar field:**

$$\left[\varepsilon_1 r^2 - \varepsilon_2\ell^2\left(1 - B - rB\frac{A'}{A}\right)\right]\psi\sqrt{AB} = C, \tag{15}$$

where $C$ is a constant of integration.

**Equation of motion of the perfect fluid:**

$$\frac{A'}{A} = -\frac{2p'}{\epsilon + p}. \tag{16}$$

**Equation of state**

$$p = K\rho_0^\Gamma, \quad \epsilon = \rho_0 c^2 + \frac{p}{\Gamma - 1}, \tag{17}$$

where $\rho_0$ is a baryonic mass density, $\Gamma = 2$ is the adiabatic index and $K = 1.79 \times 10^5$ (CGS) is the polytropic constant [1-3].

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1. R.F. Tooper, Adiabatic Fluid Spheres in General Relativity. The Astrophysical Journal 142, 1541-1562 (1965).
2. J.M. Lattimer, The Nuclear Equation of State and Neutron Star Masses, Annu. Rev. Nucl. Part. Sci. 62, pp.485-515 (2012).
3. A. Cisterna, T. Delsate, M. Rinaldi, Neutron stars in general second order scalar-tensor theory: The case of nonminimal derivative coupling, Phys. Rev. D 92, 044050 (2015).

**Dimensionless values:** $\xi = \Lambda_0 \ell^2,\ x = \frac{r}{\ell},\ \mathcal{E} = \kappa\ell^2\epsilon,\ \mathcal{P} = \kappa\ell^2 p,\ \Psi^2 = \kappa\ell^2\psi^2.$ (18)

**Basic equations:**

$$\frac{dB}{dx} = -\frac{1}{\Delta}\Big[\Big((1+\varepsilon\xi)x^4 + (\varepsilon - 5\xi)x^2 + 2\Big)B + \Big((1-\varepsilon x^2)\mathcal{E} + 2(3-\varepsilon x^2)\mathcal{P}\Big)x^2 B - (1-\varepsilon x^2)^2\left(2 - (\varepsilon+\xi)x^2 - x^2\mathcal{E}\right)\Big], \quad (19)$$

$$\frac{d\mathcal{P}}{dx} = -\frac{(\mathcal{E}+\mathcal{P})(1-B-\varepsilon x^2)}{2xB}, \quad (20)$$

$$\frac{dA}{dx} = \frac{A(1-B-\varepsilon x^2)}{xB}, \quad (21)$$

$$\Psi^2 = -\frac{x^2(\varepsilon - \xi + \mathcal{P})}{\varepsilon_2 B\,(1-\varepsilon x^2)}, \quad (22)$$

$$\mathcal{E} = \kappa\ell^2 c^2 \left(\frac{\mathcal{P}}{\kappa\ell^2 K}\right)^{1/\Gamma} + \frac{\mathcal{P}}{\Gamma - 1} \quad \text{(equation of state)}, \quad (23)$$

where $\varepsilon = \varepsilon_1/\varepsilon_2$ and $\Delta = x(1-\varepsilon x^2)\big(2 - (\varepsilon+\xi)x^2 + x^2\mathcal{P}\big)$, $\varepsilon = \pm 1$, $\xi$ and $\ell$ – free parameters.

**Regularity conditions:** $\Psi^2 \neq \infty$ and $\Psi^2 \geqslant 0$.

- $\varepsilon = \varepsilon_1/\varepsilon_2 = -1$;
- in the case $\varepsilon_2 = +1$: $\max\{p\} \leqslant (1+\xi)/(\kappa\ell^2)$, $\xi > -1$;
- in the case $\varepsilon_2 = -1$: $\xi \leqslant -1$.

**Boundary conditions for $A(r)$, $B(r)$, $\Psi(r)$ and $\mathcal{P}(r)$:**

$$\begin{aligned} p(0) &= K\rho_{0c}^{\Gamma}, & p'(0) &= 0, \\ B(0) &= 1, & B'(0) &= 0, \\ A'(0) &= 0, & A(R) &= A_{out}(R), \\ \psi(0) &= 0, \end{aligned}$$

where $\rho_{0c}$ – central barionic mass density, $A_{out}(r)$ is the external vacuum solution, $R$ is the radius of the star.

**Radius $R$ of the star:**

$$\mathcal{P}(R) = 0.$$

**Consequence of the boundary conditions:**

$$C = 0 \quad \Rightarrow \quad J^{\mu} = 0 \quad \Rightarrow \quad \text{no hair solution.}$$

**Vacuum solution[1] in the case $\rho = 0$ and $p = 0$:**

$$B(x) = \frac{(x^2+1)^2}{\left((1-\xi)x^2+2\right)^2} F(x), \tag{24}$$

$$A(x) = 3C_2 F(x), \tag{25}$$

$$\Psi^2(x) = \frac{x^2(1+\xi)}{\varepsilon_2 B\,(1+x^2)}, \tag{26}$$

where $C_1$ and $C_2$ are constants of integration and

$$F(x) = (1-\xi)(3+\xi) + \frac{1}{x}\left((1+\xi)^2 \arctan x + C_1\right) + \frac{x^2}{3}(1-\xi)^2.$$

**The case $\xi = -1$ (anti-de Sitter-Schwarzschild black hole):**

$$A(r) = B(r) = 1 - \frac{r_g}{r} + \frac{|\Lambda_{AdS}|}{3} r^2, \quad \Psi^2(r) = 0. \tag{27}$$

[1] M. Rinaldi, Phys. Rev. D 86, 084048 (2012).

**Asymptotics at $x \to \infty$:**

$$B(x) = \frac{x^2}{3} + \frac{7+\xi}{3(1-\xi)} + \frac{C_1 + \frac{1}{2}(1+\xi)^2\pi}{(1-\xi)^2}\,\frac{1}{x} + \mathcal{O}(x^{-2}), \tag{28}$$

$$A(x) = 3C_2(1-\xi)(3+\xi)\left[1 + \frac{1-\xi}{3(3+\xi)}x^2 + \frac{C_1 + \frac{\pi}{2}(1+\xi)^2}{(3+\xi)(1-\xi)}\,\frac{1}{x}\right] + \mathcal{O}(x^{-2}), \tag{29}$$

$$\Psi^2(x) = \frac{3(1+\xi)}{\varepsilon_2 x^2} + \mathcal{O}(x^{-4}). \tag{30}$$

**Conditions:**

- $t$ is the time of a distant observer $\Rightarrow$ $C_2 = [3(1-\xi)(3+\xi)]^{-1}$;
- Metric signature $\Rightarrow$ $C_2 > 0$ $\Rightarrow$ $-3 < \xi < 1$.

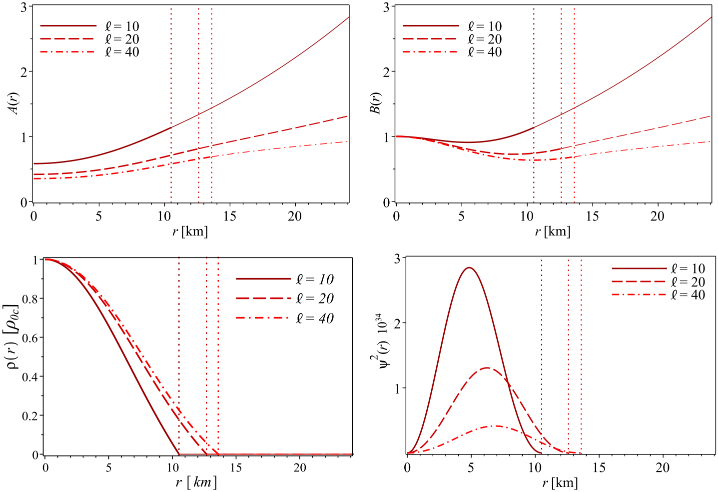
**Anti-de Sitter-Schwarzschild asymptotics:**

$$A(r) \approx 1 - \frac{r_g}{r} + \frac{|\Lambda_{AdS}|}{3}\,r^2, \tag{31}$$

$$r_g = \frac{2GM}{c^2} = -\ell\,\frac{C_1 + \frac{\pi}{2}(1+\xi)^2}{(3+\xi)(1-\xi)}, \quad \Lambda_{AdS} = -\frac{1-\xi}{3+\xi}\,\frac{1}{\ell^2}, \tag{32}$$

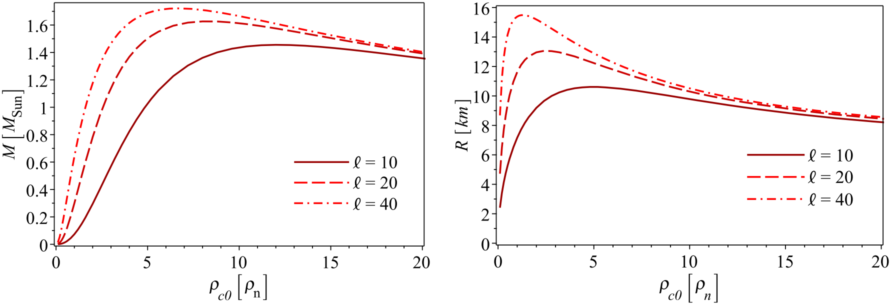
where $r_g$ is the Schwarzschild radius, $M$ is the Schwarzschild mass and $\Lambda_{AdS}$ is the effective negative cosmological constant, $\xi = \Lambda_0\ell^2$.

**Graphs of the functions $A(r)$, $B(r)$, $\rho(r)$ and $\Psi^2(r)$**



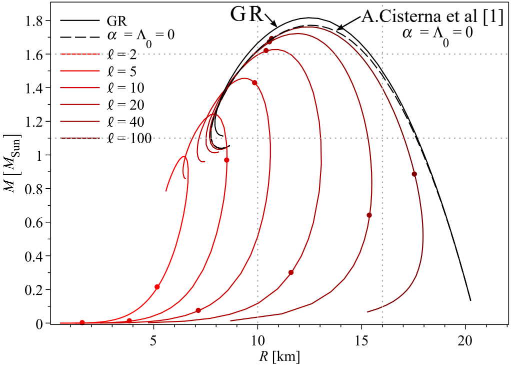

Graphs of the functions $A(r)$, $B(r)$, $\rho(r)$ and $\Psi^2(r)$ in the case $\xi = -1$ are shown for three values of the nonminimal derivative coupling parameter $\ell = 10, 20, 40$ $km$ and the central baryonic mass density $\rho_{0c} = 10^{15}$ $g/cm^3$. Vertical dotted lines mark the boundary of star ($\ell = 10, 20, 40$ from left to right).

**The dependence of the mass $M$ (left panel) and the radius $R$ (right panel) on the central baryonic mass density $\rho_{0c}$**




The dependence of the mass $M$ (left panel) and the radius $R$ (right panel) on the central baryonic mass density $\rho_{0c}$ in the case $\xi = -1$ is shown for three values of the nonminimal derivative coupling parameter $\ell = 10, 20, 40$ km. Values of $\rho_{0c}$ are given in term of the nuclear density $\rho_n = 2.5 \times 10^{14}\ \mathrm{g/cm^3}$.

**The mass-radius diagram in the case $\xi = -1$ ($\Lambda_{Ads} = -\ell^{-2}$, $\varepsilon_2 = -1$, $\varepsilon_1 = +1$)**




Solid red curves correspond to different values of the nonminimal derivative coupling parameter $\ell = 2, 5, 10, 20, 40, 100$ km (from left to right). All graphs are built in the range of values of the central baryonic mass density $\rho_{0c}$ from $0.1\rho_n$ to $100\rho_n$ (from bottom to up), where $\rho_n = 2.5 \times 10^{14}\ g/cm^3$ is the nuclear density. Small solid circles on the curves mark the values $\rho_{0c} = \rho_n$ and $\rho_{0c} = 10\rho_n$.

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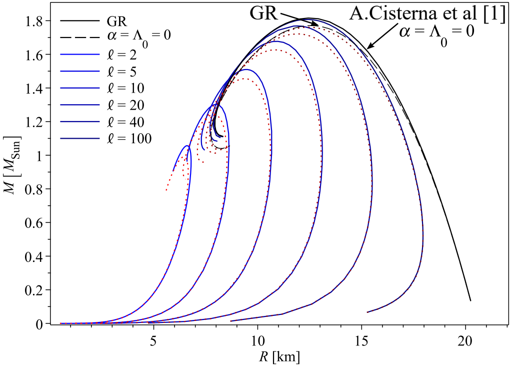
1. A. Cisterna, T. Delsate, M. Rinaldi, Neutron stars in general second order scalar-tensor theory: The case of nonminimal derivative coupling, Phys. Rev. D 92, 044050 (2015).

**The mass-radius diagram in the case $\varepsilon_1 = \varepsilon_2 = 0$, $\Lambda_0 = \Lambda_{Ads}$**




The mass-radius diagram in the theory

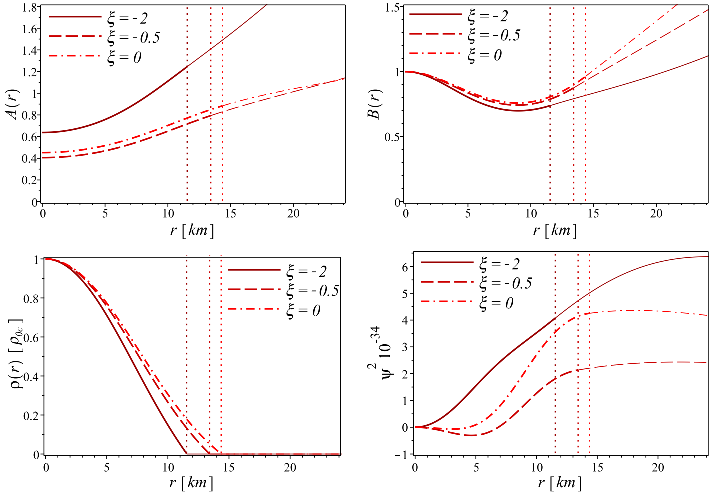
$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa} (R + 2|\Lambda_{AdS}|) \right] + S^{(m)}, \qquad (33)$$

where $\Lambda_{AdS} = -1/\ell^2$ is a negative cosmological constant.

[1] A. Cisterna, T. Delsate, M. Rinaldi, Neutron stars in general second order scalar-tensor theory: The case of nonminimal derivative coupling, Phys. Rev. D 92, 044050 (2015).

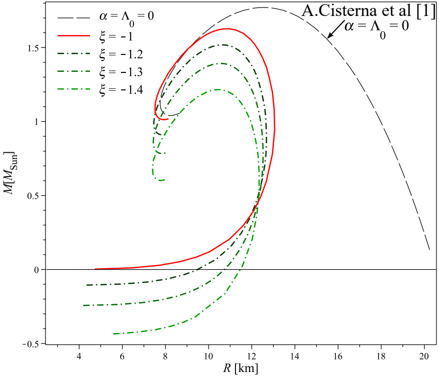
# Results of numerical integration: the case $\xi \neq -1$

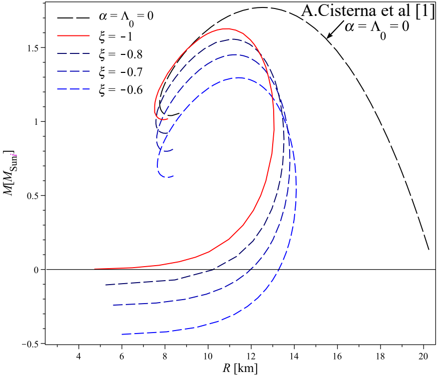
**Graphs of the functions $A(r)$, $B(r)$, $\rho(r)$ and $\Psi^2(r)$ in the case $\xi \neq -1$**




Graphs of the functions $A(r)$, $B(r)$, $\rho(r)$ and $\Psi^2(r)$ in the case $\xi \neq -1$ km are shown for three values of the parameter $\xi = -2, -0.5, 0$ and the central baryonic mass density $\rho_{0c} = 10^{15}\ g/cm^3$, $\ell = 20$ km. Vertical dotted lines mark the boundary of star ($\xi = -2, -0.5, 0$ from left to right).

**Mass-radius diagrams for various values of $\xi \neq -1$ ($\ell = 20$ and $-3 < \xi < 1$)**




[1] A. Cisterna, T. Delsate, M. Rinaldi, Neutron stars in general second order scalar-tensor theory: The case of nonminimal derivative coupling, Phys. Rev. D 92, 044050 (2015).

- We numerically constructed neutron star configurations in the theory of gravity with a nonminimal derivative coupling of a scalar field with the Einstein tensor and the cosmological constant $\Lambda_0$. The matter has a form of a perfect fluid and obeys the polytropic equation of state with the adiabatic index $\Gamma = 2$.
- We explore configurations of neutron stars for different sets of free model parameters $\ell$, $\varepsilon$ and $\xi$.
- In the case $\xi = -1$ the external vacuum solution has a form of the Schwarzschild-anti de Sitter black hole. We have the anti-de Sitter-Schwarzschild asymptotics outside the star in the case $\xi \in (-3, 1)$.
- In the case $\xi = -1$ the mass-radius diagram has an essential property, namely, radius decreases monotonically with decreasing mass. The specific «strange» relation between mass and radius corresponds to the so-called bare strange stars, and in our case it is forming due to the effective negative cosmological constant $\Lambda_{Ads}$.
- In the case $\ell \gg 1\,km$ the mass-radius diagram approaches the results without kinetic term of scalar field in the Lagrangian obtained earlier by A.Cisterna et al.
- In the general case $\xi \neq -1$ the mass-radius diagrams are shifted down and left in case $\xi < -1$, and down and right in case $\xi > -1$. The mass-radius diagrams are shifted in the region of negative asymptotic masses. Of course the baryonic mass of the star remains to be positive.

Thank you for your attention!