Adler function, polarized Bjorken Sum Rule, the Crewther-Broadhurst-Kataev relation and the $\{\beta\}$-expansion with different fermion representations of the gauge group of order $O(\alpha_s^4)$

P. A. Baikov[1], S. V. Mikhailov[3]

based on

SVM JHEP 04(2017)169, K.Chetyrkin [arXiv:2206.12948],

PAB&SVM [arXiv:2206.14063]

[1]Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow 119991, Russia

[3]Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Russia

BLTP “Quantum Field Theory, High-Energy Physics, and Cosmology”

What are Adler function, Bjorken Sum Rule, the Crewther relation

There are renorm-group invariant single scale Q^2 quantities D, $C^{\rm Bjp}$:

Adler function

$$d_R D(a_s) = D_A = -12\pi^2 Q^2 \frac{d}{dQ^2}\Pi(Q^2); \quad Q^2 = -q^2$$

Bjorken polarized Sum Rule

$$\frac{1}{6}\left|\frac{g_A}{g_V}\right| C^{\rm Bjp}(a_s) + \text{high twist} = S_{\rm NS}^{\rm Bjp}(Q^2) = \int_0^1 \left[g_1^{lp}(x,Q^2) - g_1^{ln}(x,Q^2)\right] dx$$

Crewther relation

–a plausible **conjecture** **[Crewther 1972,1997]** inspired by conformal symmetry

$$D_{ns}(a_s)\cdot C^{\rm Bjp}(a_s) = 1 + \beta(a_s)K(a_s), \text{ where } K(a_s) - \text{polynom in } a_s = \frac{\alpha_s}{4\pi}$$

[D.Broadhurst,A.Kataev,PLB1993]-**Crucial** 3-loop analysis in $\overline{\textbf{MS}}$-scheme

---

[P.Baikov,K.Chetyrkin,J.Kühn, PRL2010]- confirmation in $O(a_s^4)$.

OUTLINE

1. **Intro**: What is the $\{\beta\}$**-expansion** for RG-invariants
   and what does it express?

2. How to apply the $\{\beta\}$**-expansion**?
   To understand and to control the corresponding PT series in each expansion order, etc.

3. How to obtain the $\{\beta\}$**-expansion** from **QCDe**
   (extended QCD with different fermion representations of gauge group),
   results for **Adler** $D_{ns}(a_s)$ and **Bjorken** $C^{\mathrm{Bjp}}(a_s)$.

4. **Crewther-Broadhurst-Kataev relation** and its corollaries
   from $\{\beta\}$**-expansion** point of view.

5. **Conclusions**

STORE

Motivation for the revision of series representation

we consider 1-scale Q^2 RG-INVARIANT quantities at $Q^2 = \mu_R^2$, e.g., D_{ns}

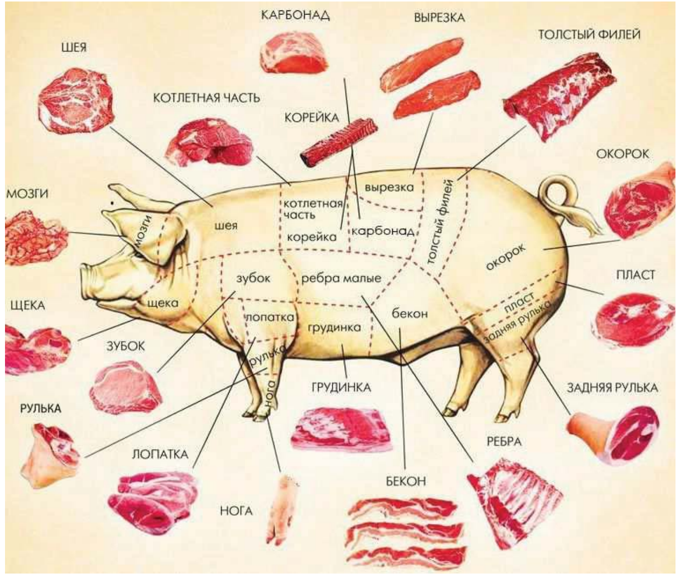
"Wild" approach: $\forall d_n$ - numbers, taken **wholly**

$$D(a_s) \sim 1 + a_s d_1 + a_s^2 d_2 + a_s^3 d_3 + \ldots$$

$$d_2 = 31.77 - 1.84 n_f;$$
$$d_3 = 1164.8 - 270.1 n_f - 5.5 n_f^2;$$
$$d_4 = 34765 - 8806.4 n_f + 481.3 n_f^2 - 2.56 n_f^3.$$

Delicate approach: $\forall d_n$ has an intrinsic structure due to a_s-renorm.

$$D(a_s) \sim 1 + \hat{M}(a_s, \{\beta_i\}) \leftarrow \text{2D matrix}$$



$$d_2 = \beta_0 d_2[1] + d_2[0];$$
$$d_3 = \beta_0^2 d_3[2] + \beta_1 d_3[0,1] + \beta_0 d_3[1] + d_3[0]$$
$$d_4 = \beta_0^3 d_4[3] + \beta_2 d_3[0,0,1] + \ldots$$
$$\rightarrow \text{ series becomes "thick"}$$

the decomposition is named **$\{\beta\}$-expansion[MS2005-7]**
it shows the **dynamic** knowledge of D exhibited how a_s-renorm.

How to apply the $\{\beta\}$-expansion 1.(If we already have it)

[MS2007, A.Kataev&MS PRD2015] $(a_s, \mu^2) \xrightarrow{RG} (a'_s, \mu'^2)$

$$\ln(\mu^2/\mu'^2) = t - t' \equiv \Delta(a') = \Delta_0 + a'\beta_0\Delta_1 + (a'\beta_0)^2\Delta_2 + \ldots, \quad \boxed{\Delta_0 = d_2[1]/d_1}$$

↑ **BLM**

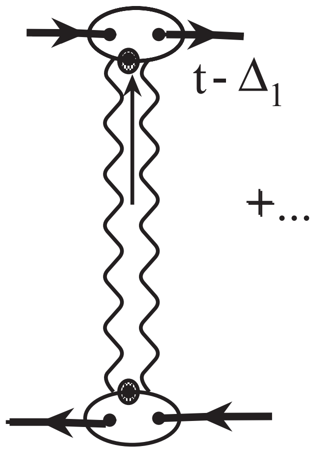
$$\begin{aligned}
\textbf{Each} \mid a^2 \cdot d_2 &\to a'^2 \cdot \left[d'_2 = \beta_0\,(d_2[1] - \Delta_0) + d_2[0]\right];\\
\textbf{order} \mid a^3 \cdot d_3 &\to a'^3 \cdot \Big[d'_3 = \beta_0^2\Big(d_3[2] - 2d_2[1]\Delta_0 + \Delta_0^2 - \Delta_1\Big) + \beta_1\,(d_3[0,1] - \Delta_0)\\
\textbf{can be} \mid &\qquad + \beta_0\,(d_3[1] - 2d_2[0]\Delta_0) + d_3[0]\Big];\\
\textbf{controlled} \mid a^4 \cdot d_4 &\to a'^4 \cdot \left[d'_4 = \beta_0^3(d_4[3] - 3d_3[2]\Delta_0 \ldots - \Delta_2) + \ldots + d_4[0]\right]
\end{aligned}$$

Fitting components $\Delta_0, \Delta_1, \Delta_2, \ldots$ of the normalization scale μ'^2 to adjust the elements $d'_2, d'_3, d'_4, \ldots$ following to any **optimization procedure**. **Higher-order calculations are laborious and require bloody efforts**, so they should be used **with maximum efficiency**, **i.e., optimized**.

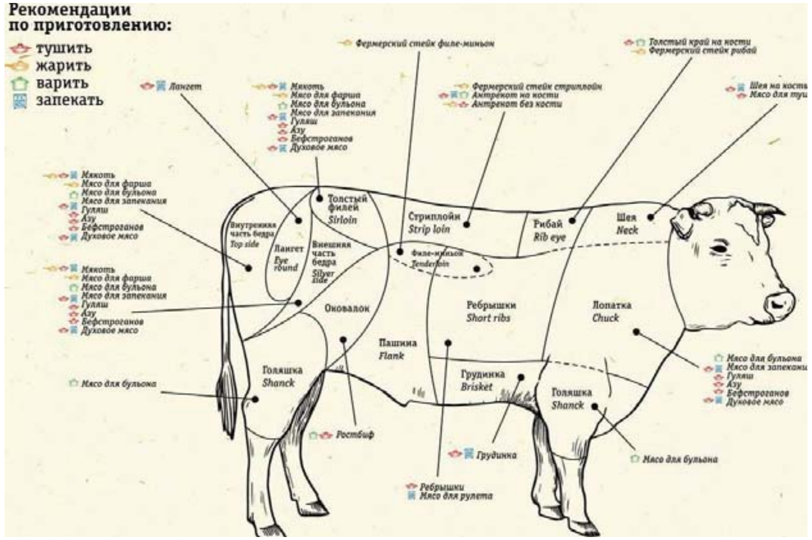
An optimization as **numerical minimum** of all QCD corr. to $C^{\text{Bjp}}(a_s) \approx c_0$, $+ \left[a_s c_1 + a_s^2 c_2 + a_s^3 c_3 + a_s^4 c_4\right]$, up to $O(a_s^4)$ was realized in **[D.Kotlorz&MS2019]**. Effect is about -20% at $\mu^2 = m_\tau^2 \approx 3\text{GeV}^2$.

How to apply the $\{\beta\}$-**expansion**? 2.(If we already have it)

Different pieces are appropriate for miscellaneous and can be cooked differently



(a)



Evident usage of the $\{\beta\}$-**expansion** is the different kinds of **optimization**: one can change the contributions of different origins of a_s-renorm playing with the choice of μ_R^2. E.g., well-known **BLM approach [Brodsky et al 1983]**:

$$\boldsymbol{d_2} \;=\; \underline{\beta_0 d_2[1]} + d_2[0] \to \boldsymbol{d_2}[\mathbf{0}] \text{ at } \mu_R^2 \to \mu_{BLM}^2 = \exp(-d_2[1]/d_1)\mu_R^2$$

$$\beta_0 = 11/3C_A - 4/3T_R \boldsymbol{n_f} \Big| \text{ profit at } \underline{\beta_0 d_2[1]} \gg \boldsymbol{d_2}[\mathbf{0}]$$

$$\boldsymbol{d_3} \;=\; \underline{\beta_0^2 d_3[2]} + \beta_1 d_3[0,1] + \beta_0 d_3[1] + d_3[0] \to \boldsymbol{d_3}[\mathbf{0}]$$

$$\boldsymbol{d_n} \;=\; \underline{\beta_0^{n-1} d_n[n-1]} + \ldots + d_n[0] \to \boldsymbol{d_n}[\mathbf{0}]$$

generalization BLM: $D = 1 + \sum_1^N a_s^n(\mu_R^2)\boldsymbol{d_n} \to D_0 = 1 + \sum_1^N a_s^n(\mu_{opt}^2)\boldsymbol{d_n}[\mathbf{0}]$

[PMC, Brodsky et al, 2013 – 2021]

But this “conformal limit” may be not an **optimized series** in any sense, for $R(s)$ it makes the PT convergence **even worse** in $O(a_s^{3\div4})$ **[A. Kataev&MS PRD2015]**

How to obtain the elements of $\{\beta\}$-expansion? 1. QCDe

We need in extended QCD with additional degrees of freedom-**d.o.f.**,

e.g., fermion multiplets

These **d.o.f. $\{R\}$ interact following the universal gauge principle** entering only in intrinsic loops **[K.Chetyrkin PLB1997, D with MSSM gluinos $n_{\tilde{g}}$]**.

$$T_R n_f,\ \frac{C_A}{2} n_{\tilde{g}}, \ldots \xrightarrow{general} \{R\} \quad \text{[M.Zoller 2016] for } \beta(a_s, \{R\}),$$

$$D(a_s, \{R\}) \text{ or } C^{\text{Bjp}}(a_s, \{R\}) : \textbf{QCDe} \text{ [K.Chetyrkin 2206.12948]}$$

d.o.f.:

$\{R\}$- any numbers of different quark representations **[K.Chetyrkin 2206.12948]**

$$\mathcal{L}_{QCD} = \ldots + \sum_{r=1}^{N_{\text{rep}}} \sum_{q=1}^{n_{f,r}} \left\{ \frac{i}{2} \bar{\psi}_{q,r} \overleftrightarrow{\hat{\partial}} \psi_{q,r} - m_{q,r} \bar{\psi}_{q,r} \psi_{q,r} + g_s \bar{\psi}_{q,r} \hat{A}^a T^{a,r} \psi_{q,r} \right\},$$

$$\mathrm{R} = (q - \text{flavors}, r - \text{Representation})$$

Lie Algebra:

$$\left[T^{a,r}, T^{b,r}\right] = i f^{abc} T^{c,r}; \ T^{a,r}_{ik} T^{a,r}_{kj} = \delta_{ij} C_{F,r}; \ T_{F,r} \delta^{ab} = \mathbf{Tr}\left(T^{a,r} T^{b,r}\right);$$

$$d_R^{a_1 a_2 \ldots a_n} = \frac{1}{n!} \sum_{\text{perm } \pi} \text{Tr}\left\{ T^{a_{\pi(1)},R} T^{a_{\pi(2)},R} \ldots T^{a_{\pi(n)},R} \right\},$$

How to obtain the elements of $\{\beta\}$-expansion? 2.

Then one can decompose all β**-terms** explicitly following an **algebraic procedure** **[MS2017]**: all **7** elements of $\boldsymbol{d_4}$ and $\boldsymbol{c_4}$ are **explicitly** obtained here,

$$d_4 = \beta_0^3 d_4[3] + \beta_2 d_4[0,0,1] + \beta_1\beta_0 d_4[1,1] + \beta_0^2 d_4[2] + \beta_1 d_4[0,1] + \beta_0 d_4[1] + d_4[0]$$

$$d_n = \underbrace{\beta_0^{n-1} d_n[n-1] + \ldots + \beta_0 d_n[1] + d_n[0]}_{\boldsymbol{N}(n)},$$ **series** D becomes **matrix** $\hat{D}$

$$\hat{D}(a_s, \{\beta_i\}) = \begin{pmatrix} \ldots & a_s^{n-1} & & a_s^n & \ldots \\ & & & d_n[0] & \\ \beta_0 & \vdots & \boldsymbol{N}\left\{\begin{matrix} \\ \\ \\ \end{matrix}\right. & \beta_0 d_n[1] & \\ & & & \vdots & \\ & & & & \end{pmatrix}$$

$$\boldsymbol{N}(n) = \sum_{l=0}^{(n-1)} p(l) = \{\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{7}, \mathbf{12}, .., \mathbf{97}, ..\} \sim \frac{\sqrt{6n}}{\pi} \cdot (p(n) \leftarrow \text{ partition of numbers}) + \ldots$$

$$\uparrow \qquad \uparrow \qquad \uparrow$$

$$n = \{\mathbf{1}, .. \quad , \mathbf{4}, .. \quad , \mathbf{10}, ..\}$$

Hardy-Ramanujan asymptotic for partition of numbers $p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{2n/3}\right)$

Important! We need new d.o.f. Only to perform the decomposition, after that we return from QCDe to the standard QCD, $\{R\} \to T_{\mathrm{R}} n_f$.

The marked traces of **the general gauge principle** saved as $\{\beta\}$**-expansion**.

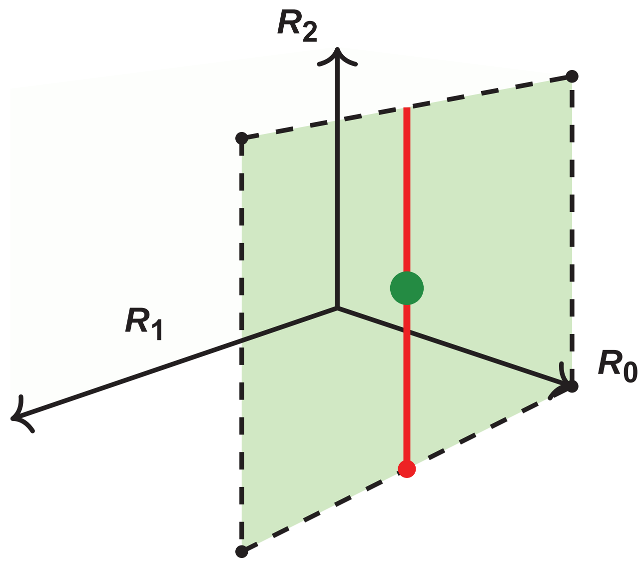
How to obtain the $\{\beta\}$-**expansion**? 3. Solving a set of equations

The key role plays the **set of zeros** of $\beta_0(\{R\}), \beta_1(\{R\}), \beta_2(\{R\}), \ldots$ and **zeros** of sets of these β_k.

E.g., in $O(a_s^4)$ $\beta_0, \beta_1, \beta_2$ are defined on the axes of variables R_0, R_1, R_2:

1) $\exists$ 3D point $\bar{R}_{0,1,2} : \beta_0(\{\bar{R}_{0,1,2}\}) = \beta_1(\{\bar{R}_{0,1,2}\}) = \beta_2(\{\bar{R}_{0,1,2}\}) = 0$,
Then $D(\bar{R}_{0,1,2}) = (a_s^2\, d_2[0], a_s^3\, d_3[0], a_s^4\, d_4[0], \ldots)$, step by step

2) $\exists$ line in 3D $\beta_0(\{\bar{R}_{0,1}\}) = \beta_1(\{\bar{R}_{0,1}\}) = 0$,
Then $d_4(\bar{R}_{0,1}) = \beta_2(\{\bar{R}_{0,1}\})\underline{d_4[0,0,1]} + d_4[0]$



3) $\exists$ 2D surface in 3D $\beta_0(\{\bar{R}_0\}) = 0$,
Then $d_4(\bar{R}_0) = \beta_2(\{\bar{R}_0\})\sharp + \beta_1(\{\bar{R}_0\})\sharp + \sharp$

4) $\exists$ curve in 3D $\beta_0(\{\bar{R}_{0,2}\}) = \beta_2(\{\bar{R}_{0,2}\}) = 0$,
Then $d_4(\bar{R}_{0,2}) = \beta_1(\{\bar{R}_{0,2}\})\underline{d_4[0,1]} + d_4[0]$

. . .

Finally this reduces to the solvable set of equations.
In $O(a_s^5)$ at 6-loop we also have single-valued solution of the similar set of equation [MS2017].

Crewther-Broadhurst-Kataev relation and its corollaries

The elements of $\{\beta\}$**-expansion** for $\boldsymbol{D_{ns}}$ and $\boldsymbol{C}^{\mathrm{Bjp}}$ were independently obtained. They provide the *appropriate "bricks"* to analyse **C-B-K relation**, which is **our main subject** here.

First time it was applied to **C-B-K** **[A.Kataev&MS TMP2012]**

$$\boldsymbol{D_{ns}}(\boldsymbol{a_s}) \cdot \boldsymbol{C}^{\mathrm{Bjp}}(\boldsymbol{a_s}) = \mathbf{1} + \beta(\boldsymbol{a_s}) \times \sum_{n=1} a_s^{n-1} K_n \quad (\clubsuit)$$

Structure of K_n

$$K_1 = K_1[1],\ K_2 = K_2[1] + \beta_0 K_2[2], \quad K_3 = K_3[1] + \beta_0 K_3[2] + \beta_0^2 K_3[3] + \beta_1 K_3[1,1],$$
$$K_4 = K_4[1] + \ldots$$

- ►Products of $\boldsymbol{d_k}[.]$, $\boldsymbol{c_j}[.]$ elements are already presented in the LHS of ($\clubsuit$)
- ►While the structure of the RHS ($\clubsuit$) orders combinations of the elements that leads to the equations for them.
- ►So, we can **Confirm** the values of obtained $\boldsymbol{d_k}[.]$, $\boldsymbol{c_j}[.]$ elements satisfied these equations (**C-B-K relation**) and **Predict** the elements in next orders.

Another way to obtain $\boldsymbol{d_k}[.]$, $\boldsymbol{c_j}[.]$ just based on (**C-B-K relation**)($\clubsuit$) is presented in **[Cvetic&Kataev PRD2016], [Kataev&Molokoedov JHEP2022]**, the results do not agree with ours.

Crewther-Broadhurst-Kataev relation, its corollaries for 1 term.

1. The $\boldsymbol{D_0} \cdot C_0^{\rm Bjp} = \mathbf{1}$ –"conformal" part of the relation, here $d_n \to d_n[0] \in \boldsymbol{D_0}$,

$$\boldsymbol{c_k}[\mathbf{0}] + \boldsymbol{d_k}[\mathbf{0}] = (-)^k \det[\boldsymbol{D_0^{(k)}}] \equiv (-)^k \begin{vmatrix} d_1 & 1 & 0 & & \dots & 0 \\ d_2 & d_1 & 1 & & \dots & 0 \\ d_3 & d_2 & d_1 & & \dots & 0 \\ \dots & & & & \dots & 0 \\ d_{k-1} & \dots & \dots & & d_1 & 1 \\ \mathbf{0} & d_{k-1} & d_{k-2} & \dots & d_2 & d_1 \end{vmatrix},$$

$\boldsymbol{c_k}[\mathbf{0}] + \boldsymbol{d_k}[\mathbf{0}] = \text{Polynom}(\dots d_{k-1})$, $k = 2, 3, 4$ – **Confirmations!**

$$\begin{aligned} d_4[0] + c_4[0] &= 2d_1 d_3[0] - 3d_1^2 d_2[0] + d_2[0]^2 + d_1^4 \\ &= 3C_F^2 \left[132 C_F C_A - \frac{111}{4} C_F^2 + \left(\frac{175}{2} - 432\zeta_3\right) C_A^2\right], \end{aligned}$$

$\boldsymbol{c_5}[\mathbf{0}] + \boldsymbol{d_5}[\mathbf{0}] = \text{Polynom}(\dots d_4)$, $\quad k = 5$ – **Prediction**

$$\begin{aligned} d_5[0] + c_5[0] &= 2d_1 d_4[0] + 2d_3[0] d_2[0] - 3d_3[0] d_1^2 + 4d_2[0] d_1^3 - 3d_2^2[0] d_1 - d_1^5 \\ &= d_1 \Big[C_A^2 C_F^2\, 27\,(43 + 128\zeta_3) + C_F^4 \left(\frac{2485}{2} + 192\zeta_3\right) - C_A C_F^3 (3097 + 864\zeta_3) \\ &\quad + C_A^3 C_F \left(\frac{206233}{72} + 7969\zeta_3 - 14220\zeta_5\right) + 2\delta d_4 \Big]. \end{aligned}$$

The sums $\boldsymbol{c_k}[\mathbf{0}] + \boldsymbol{d_k}[\mathbf{0}]$ agree with the results in **[Kataev&Molokoedov JHEP2022]**

Crewther-Broadhurst-Kataev relation, its corollaries for $\beta(a_s)$ term

2. The factorization of the $\beta(a_s)$, taken wholly, sets the chain of conditions

$$K_1[1] = \underset{\beta_0}{d_2[1] + c_2[1]} = \underset{\beta_1}{d_3[0,1] + c_3[0,1]} = \underset{\beta_2}{\underline{c_4[0,0,1] + d_4[0,0,1]}} = 3C_F\left(\frac{7}{2} - 4\zeta_3\right)$$

$$= d_n[\underbrace{0,0,\ldots,1}_{n-1}] + c_n[\underbrace{0,0,\ldots,1}_{n-1}] \quad \textbf{Confirmations/Prediction}$$

$$\begin{aligned}
K_2[1] &= c_3[1] + d_3[1] + d_1(c_2[1] - d_2[1]) = \\
&= \underline{c_4[0,1] + d_4[0,1]} + d_1(c_3[0,1] - d_3[0,1]) \quad \textbf{Confirmation} \\
&= C_F^2\left(-\frac{397}{6} - 136\zeta_3 + 240\zeta_5\right) + C_F C_A\left(\frac{47}{3} - 16\zeta_3\right) \\
&= \underline{c_5[0,0,1] + d_5[0,0,1]} + d_1(c_4[0,0,1] - d_4[0,0,1]) \quad \textbf{Prediction} \\
&= c_n[\underbrace{0,\ldots,1}_{n-2}] + d_n[\underbrace{0,\ldots,1}_{n-2}] + d_1(c_{n-1}[\underbrace{0,\ldots,1}_{n-2}] - d_{n-1}[\underbrace{0,\ldots,1}_{n-2}]).
\end{aligned}$$

$$\begin{aligned}
K_3[1] &= c_4[1] + d_4[1] + d_1\left(c_3[1] - d_3[1]\right) + d_2[0]c_2[1] + d_2[1]c_2[0] \quad \textbf{Prediction} \\
&= \underline{c_5[0,1] + d_5[0,1]} + d_1\left(c_4[0,1] - d_4[0,1]\right) + d_2[0]c_3[0,1] + c_2[0]d_3[0,1] = \ldots \\
&= c_{n+1}[\underbrace{0,\ldots,1}_{n-2}] + d_{n+1}[\underbrace{0,\ldots,1}_{n-2}] + d_1\left(c_n[\underbrace{0,\ldots,1}_{n-2}] - d_n[\underbrace{0,\ldots,1}_{n-2}]\right) + \\
&\quad d_2[0]\, c_{n-1}[\underbrace{0,\ldots,1}_{n-2}] + c_2[0]\, d_{n-1}[\underbrace{0,\ldots,1}_{n-2}]
\end{aligned}$$

Crewther-Broadhurst-Kataev relation, the structure of K – term

The universal form of the second term appears due to the cancellation of a_s^1- terms

$$K_{n\geqslant 3}[1] = c_{n+1}[1]+d_{n+1}[1]+d_1\left(c_n[1]-d_n[1]\right)+\sum_{k=2}^{n-2}\left(d_k[0]c_{n+1-k}[1]+c_k[0]d_{n+1-k}[1]\right).$$

Partial results for K-term in order $O(a_s^4)$

$$K_1[1] = d_2[1]+c_2[1] = 3C_F\left(\frac{7}{2}-4\zeta_3\right)$$

$$\begin{aligned} K_2[1] &= c_3[1]+d_3[1]+d_1\left(c_2[1]-d_2[1]\right) \\ &= C_F^2\left(-\frac{397}{6}-136\zeta_3+240\zeta_5\right)+C_FC_A\left(\frac{47}{3}-16\zeta_3\right) \\ K_2[2] &= c_3[2]+d_3[2] = 3C_F\left(\frac{163}{6}-\frac{76}{3}\zeta_3\right) \end{aligned}$$

$$\begin{aligned} K_3[1] &= c_4[1]+d_4[1]+d_1\left(c_3[1]-d_3[1]\right)+d_2[0]c_2[1]+d_2[1]c_2[0] \\ K_3[2] &= c_4[2]+d_4[2]+d_1\left(c_3[2]-d_3[2]\right)+d_2[1]c_2[1], \\ K_3[3] &= c_4[3]+d_4[3], \\ K_3[1,1] &= c_4[1,1]+d_4[1,1] \end{aligned}$$

CONCLUSIONS

1. The **$\{\beta\}$-expansion** for PT series is invented and analyzed for the **Renormalization Group invariant** quantities.
   This allows to perform different optimizations of the PT series.

2. The elements of **$\{\beta\}$-expansion** can be determined within **QCDe** following to an algebraic procedure.

3. The **Crewther-Broadhurst-Kataev relation** is reproduced in order $O(a_s^4)$. The interesting relations between the elements of Adler $\boldsymbol{D_{ns}}$, and Bjorken polarized SR $\boldsymbol{C}^{\mathrm{Bjp}}$ are established in any orders of a_s.

STORE, explicit form of D-elements. 1

$$d_1 = 3C_F;\ d_2[1] = d_1\left(\frac{11}{2} - 4\zeta_3\right);\ d_2[0] = d_1\left(\frac{C_A}{3} - \frac{C_F}{2}\right);$$

$$d_3[2] = d_1\left(\frac{302}{9} - \frac{76}{3}\zeta_3\right);\ d_3[0,1] = d_1\left(\frac{101}{12} - 8\zeta_3\right);$$

$$d_3[1] = d_1\left(C_A\left(-\frac{3}{4} + \frac{80}{3}\zeta_3 - \frac{40}{3}\zeta_5\right) - C_F\left(18 + 52\zeta_3 - 80\zeta_5\right)\right);$$

$$d_4[3] = C_F\left(\frac{6131}{9} - 406\zeta_3 - 180\zeta_5\right);$$

$$d_4[1,1] = C_F\left(385 - \frac{1940}{3}\zeta_3 + 144\zeta_3^2 + 220\zeta_5\right);$$

$$d_4[2] = -C_F\Big[C_F\left(\frac{6733}{8} + 1920\zeta_3 - 3000\zeta_5\right) + C_A\left(\frac{20929}{144} - \frac{12151}{6}\zeta_3 + 792\zeta_3^2 + 1050\zeta_5\right)\Big];$$

$$d_4[0,0,1] = C_F\left(\frac{355}{6} + 136\zeta_3 - 240\zeta_5\right);$$

$$d_4[1] = C_F\Big[-C_F^2\left(\frac{447}{2} - 42\zeta_3 - 4920\zeta_5 + 5040\zeta_7\right) + C_A C_F\left(\frac{3301}{4} - 678\zeta_3 - 2280\zeta_5 + 2520\zeta_7\right) + C_A^2\left(\frac{16373}{36} - \frac{17513}{3}\zeta_3 + 2592\zeta_3^2 + 3030\zeta_5 - 420\zeta_7\right)\Big],$$

$$d_4[0,1] = -C_F\Big[C_A\left(\frac{139}{12} + \frac{1054}{3}\zeta_3 - 460\zeta_5\right) + C_F\left(\frac{251}{4} + 144\zeta_3 - 240\zeta_5\right)\Big],$$

STORE, explicit form of $D_0,\ C_0$ elements. 2

$$
\begin{aligned}
d_3[0] &= d_1\left(\left(\frac{523}{36} - 72\zeta_3\right) C_A^2 + \frac{71}{3} C_A C_F - \frac{23}{2} C_F^2\right) . \\
d_4[0] &= \tilde{d}_4[0] + \delta d_4 \\
&= C_F^4\left(\frac{4157}{8} + 96\zeta_3\right) - C_A C_F^3\left(\frac{2409}{2} + 432\zeta_3\right) + C_A^2 C_F^2\left(\frac{3105}{4} + 648\zeta_3\right) + \\
&\quad C_A^3 C_F\left(\frac{68047}{48} + \frac{8113}{2}\zeta_3 - 7110\zeta_5\right) + \delta d_4 , \\
\delta d_4 &= -\frac{16}{dR}\left(d_F^{abcd} n_f d_F^{abcd}(13 + 16\zeta_3 - 40\zeta_5) + d_F^{abcd} d_A^{abcd}(-3 + 4\zeta_3 + 20\zeta_5)\right)
\end{aligned}
$$

$$
\begin{aligned}
c_3[0] &= c_1\left(\left(\frac{523}{36} - 72\zeta_3\right) C_A^2 + \frac{65}{3} C_F C_A + \frac{C_F^2}{2}\right) \\
c_4[0] &= \tilde{c}_4[0] - \delta d_4 \\
&= -C_F^4\left(\frac{4823}{8} + 96\zeta_3\right) + C_A C_F^3\left(\frac{3201}{2} + 432\zeta_3\right) - C_A^2 C_F^2\left(\frac{2055}{4} + 1944\zeta_3\right) - \\
&\quad C_A^3 C_F\left(\frac{68047}{48} + \frac{8113}{2}\zeta_3 - 7110\zeta_5\right) - \delta d_4 ;
\end{aligned}
$$

STORE, explicit form of β-function elements. 3

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}\mathbf{nT}; \tag{1}$$

$$\beta_1 = \frac{34}{3}C_A^2 - 4\left[\mathbf{nTC1} + \frac{5}{3}C_A(\mathbf{nT})\right]; \tag{2}$$

$$\begin{aligned}\beta_2 = {} & \frac{2857}{54}C_A^3 + 2(\mathbf{nTC2}) - \frac{205}{9}C_A(\mathbf{nT})(\mathbf{nTC1}) - \frac{1415}{27}C_A^2(\mathbf{nT}) + \\ & \mathbf{nT}\left[\frac{44}{9}(\mathbf{nTC1}) + \frac{158}{27}C_A(\mathbf{nT})\right];\end{aligned} \tag{3}$$

$$\begin{aligned}\beta_3 = {} & \left(\frac{150653}{486} - \frac{44}{9}\zeta_3\right)C_A^4 - \left(\frac{80}{9} - \frac{704}{3}\zeta_3\right) + d_{AA} \\ & \left[46(\mathbf{nTC3}) - \left(\frac{4204}{27} - \frac{352}{9}\zeta_3\right)C_A(\mathbf{nTC2}) + \left(\frac{7073}{243} - \frac{656}{9}\zeta_3\right)C_A^2(\mathbf{nTC1})\right. \\ & \left.- \left(\frac{39143}{81} - \frac{136}{3}\zeta_3\right)C_A^3(\mathbf{nT})\right] + \left(\frac{512}{9} - \frac{1664}{3}\zeta_3\right)\sum_i n_{f,i}\, d_{FA,i} + \\ & \left[\left(\frac{184}{3} - 64\zeta_3\right)(\mathbf{nTC1})^2 - \left(\frac{304}{27} + \frac{128}{9}\zeta_3\right)(\mathbf{nT})(\mathbf{nTC2})\right. \\ & \left.+ \left(\frac{17152}{243} + \frac{448}{9}\zeta_3\right)C_A(\mathbf{nT})(\mathbf{nTC1}) + \left(\frac{7930}{81} + \frac{224}{9}\zeta_3\right)C_A^2(\mathbf{nT})^2\right] - \\ & \left(\frac{704}{9} - \frac{512}{3}\zeta_3\right)\sum_{i,j} n_{f,i}n_{f,j}\, d_{FF,ij} + (\mathbf{nT})^2\left[\frac{1232}{243}(\mathbf{nTC1}) + \frac{424}{243}C_A(\mathbf{nT})\right].\end{aligned} \tag{4}$$

$$\mathbf{nT} = \sum_i n_{f,i}T_{F,i}, \quad \mathbf{nTC}k = \sum_i n_{f,i}T_{F,i}C_{F,i}^k, \quad \mathbf{nd}^{abcd} = \sum_i n_{f,i}d_{F,i}^{abcd},$$