Chern-Simons action, ‘t Hooft-Polyakov monopole, and Point disclinations in the Geometric Theory of Defects

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Notation

$\mathbb{R}^3$ - continuous elastic media = Euclidean three-dimensional space

$x^i, y^i \quad i=1,2,3$ - Cartesian coordinates

δ_{ij} - Euclidean metric

$u^i(x)$ - displacement vector field

$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ - strain tensor

σ^{ij} - stress tensor

Elasticity theory of small deformations

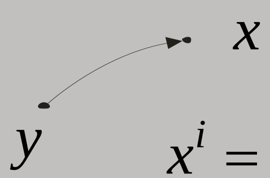
$\partial_i \sigma^{ij} + f^j = 0$ - Newton’s law

$\sigma^{ij} = \lambda \delta^{ij} \varepsilon_k{}^k + 2\mu \varepsilon^{ij}$ - Hooke’s law

$f^i(x)$ - density of nonelastic forces $(f^i = 0)$

λ, μ - Lame coefficients

Differential geometry of elastic deformations



$$x^i = y^i + u^i(x)$$

$y^i \to x^i(y)$ - diffeomorphism: $\mathbb{R}^3 \to \mathbb{R}^3$

$$y^i \mapsto x^i$$

$$\delta_{ij} \qquad g_{ij}$$

$$g_{ij}(x) = \frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j}\delta_{kl} \approx \delta_{ij} - \partial_i u_j - \partial_j u_i = \delta_{ij} - 2\varepsilon_{ij}$$ - induced metric $(*)$

$$\tilde{\Gamma}_{ijk} = \tfrac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \neq 0$$ - Christoffel's symbols

$$\tilde{R}_{ijk}{}^{l} = \partial_i \tilde{\Gamma}_{jk}{}^{l} - \tilde{\Gamma}_{ik}{}^{m}\tilde{\Gamma}_{jm}{}^{l} - (i \leftrightarrow j) = 0$$ - curvature tensor

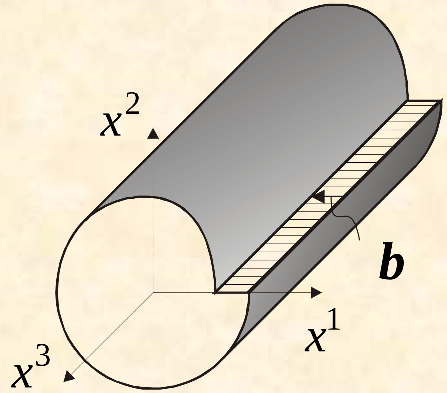
$$\ddot{x}^i = -\tilde{\Gamma}_{jk}{}^{i}\dot{x}^j\dot{x}^k$$ - extremals (geodesics)

$R_{ijk}{}^{l} = 0$ - Saint-Venant integrability conditions of $(*)$

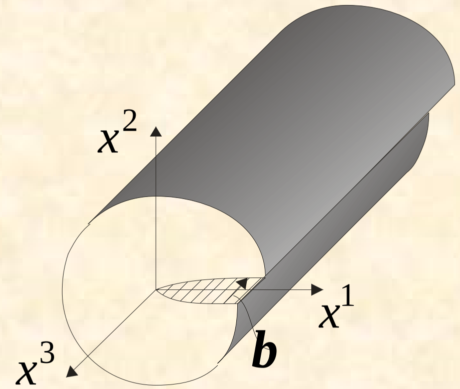
$$T_{ij}{}^{k} = \tilde{\Gamma}_{ij}{}^{k} - \tilde{\Gamma}_{ji}{}^{k} = 0$$ - torsion tensor

Dislocations

Linear defects:



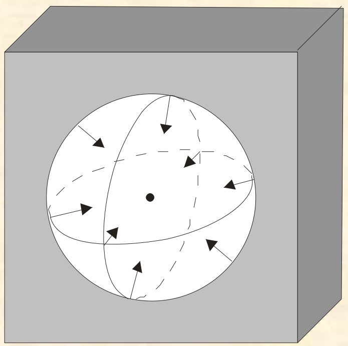
Edge dislocation



Screw dislocation

$\boldsymbol{b}$ - Burgers vector

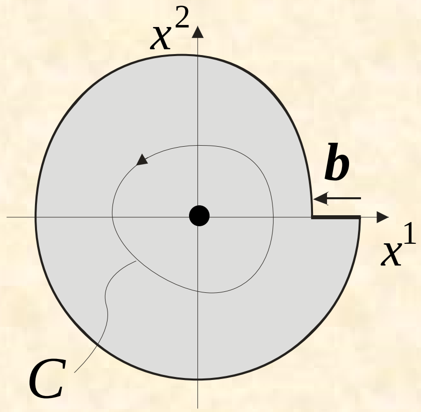
Point defects:



Vacancy

$$u^i(x) \begin{cases} \text{is continuous} & = \text{elastic deformations} \\ \text{is not continuous} & = \text{dislocations} \end{cases}$$

Edge dislocation



$$\oint_C dx^\mu \partial_\mu u^i = -\oint_C dx^\mu \partial_\mu y^i = -b^i \qquad (*)$$

$x^\mu,\ \mu = 1,2,3$ - arbitrary curvilinear coordinates

$y^i(x)$ - is not continuous !

$$e_\mu{}^i(x) = \begin{cases} \partial_\mu y^i & \text{- outside the cut} \\ \lim \partial_\mu y^i & \text{- on the cut} \end{cases}$$

- triad field (continuous on the cut)

$$(*) \Rightarrow b^i = \oint_C dx^\mu e_\mu{}^i = \iint_S dx^\mu \wedge dx^\nu (\partial_\mu e_\nu{}^i - \partial_\nu e_\mu{}^i)$$ - Burgers vector in elasticity

$$T_{\mu\nu}{}^i = \partial_\mu e_\nu{}^i - \omega_\mu{}^{ij} e_{\nu j} - (\mu \leftrightarrow \nu)$$ - torsion

$$R_{\mu\nu}{}^{ij} = \partial_\mu \omega_\nu{}^{ij} - \omega_\mu{}^{ik} \omega_{\nu k}{}^j - (\mu \leftrightarrow \nu)$$ - curvature

$$\omega_\mu{}^{ij} = -\omega_\mu{}^{ji}$$

SO(3)-connection

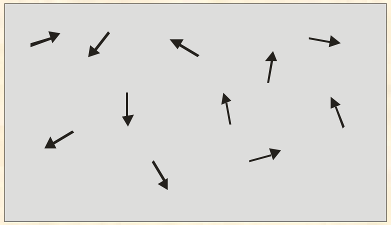
$$b^i = \iint dx^\mu \wedge dx^\nu T_{\mu\nu}{}^i$$

- definition of the Burgers vector in the geometric theory

Back to elasticity: if $R_{\mu\nu}{}^{ij} = 0$ then $\omega_\mu{}^{ij} \to 0$

Disclinations

Ferromagnets



$n^i(x)$ - unit vector field $\qquad n_0^i$ - fixed unit vector

$$n^i = n_0^j S_j{}^i(\omega)$$

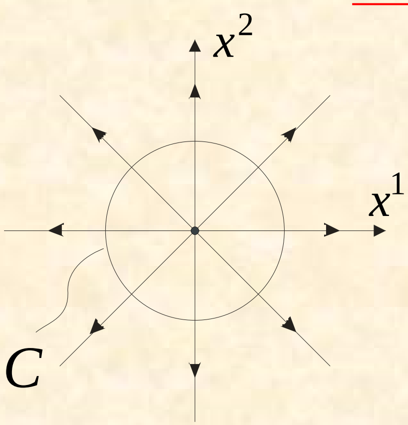
$S_i{}^j \in \mathbb{SO}(3)$ - orthogonal matrix

$\omega^{ij} = -\omega^{ji} \in \mathfrak{so}(3)$ - Lie algebra element (spin structure)

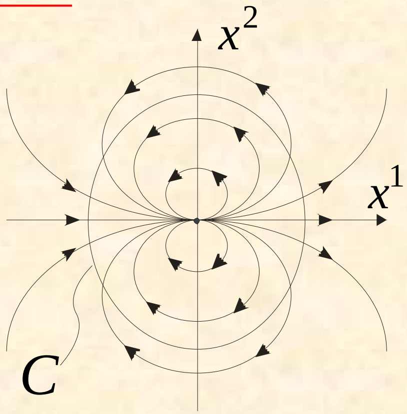
$\omega_i = \frac{1}{2}\varepsilon_{ijk}\omega^{jk}$ - rotational angle

ε_{ijk} - totally antisymmetric tensor $(\varepsilon_{123} = 1)$

Examples



$\Theta = 2\pi$



$\Theta = 4\pi$

$$\Omega^{ij} = \oint_C dx^\mu \partial_\mu \omega^{ij}$$

$\Theta_i = \varepsilon_{ijk}\Omega^{jk}$ - Frank vector (total angle of rotation)

$$\Theta = \sqrt{\Theta^i \Theta_i}$$

Frank vector

$\omega^{ij}(x)$ - is not continuous !

$$\omega_\mu{}^{ij}(x) = \begin{cases} \partial_\mu \omega^{ij} & \text{- outside the cut} \\ \lim \partial_\mu \omega^{ij} & \text{- on the cut} \end{cases}$$

- SO(3)-connection (continuous on the cut)

$$\Omega^{ij} = \oint dx^\mu \omega_\mu{}^{ij} = \iint dx^\mu \wedge dx^\nu (\partial_\mu \omega_\nu{}^{ij} - \partial_\nu \omega_\mu{}^{ij})$$ - the Frank vector

$$R_{\mu\nu}{}^{ij} = \partial_\mu \omega_\nu{}^{ij} - \omega_\mu{}^{ik} \omega_{\nu k}{}^{j} - (\mu \leftrightarrow \nu)$$ - curvature

$$\boxed{\Omega^{ij} = \iint dx^\mu \wedge dx^\nu R_{\mu\nu}{}^{ij}}$$ - definition of the Frank vector in the geometric theory

Back to the spin structure: if $n \in \mathbb{R}^2$ then $\mathbb{SO}(3) \to \mathbb{SO}(2)$

Summary of the geometric approach
(physical interpretation)

Media with dislocations and disclinations $=$

$= \mathbb{R}^3$ with a given Riemann-Cartan geometry

Independent variables $\begin{cases} e_\mu{}^i & \text{- triad field} \\ \omega_\mu{}^{ij} & \text{- SO(3)-connection} \end{cases}$

$$T_{\mu\nu}{}^i = \partial_\mu e_\nu{}^i - \omega_\mu{}^{ij} e_{\nu j} - (\mu \leftrightarrow \nu)$$ - torsion (surface density of the Burgers vector)

$$R_{\mu\nu}{}^{ij} = \partial_\mu \omega_\nu{}^{ij} - \omega_\mu{}^{ik} \omega_{\nu k}{}^j - (\mu \leftrightarrow \nu)$$ - curvature (surface density of the Frank vector)

Elastic deformations: $R_{\mu\nu}{}^{ij} = 0, \quad T_{\mu\nu}{}^i = 0$

Dislocations: $R_{\mu\nu}{}^{ij} = 0, \quad T_{\mu\nu}{}^i \neq 0$

Disclinations: $R_{\mu\nu}{}^{ij} \neq 0, \quad T_{\mu\nu}{}^i = 0$

Dislocations and disclinations: $R_{\mu\nu}{}^{ij} \neq 0, \quad T_{\mu\nu}{}^i \neq 0$

The free energy

$$S = S_{\mathrm{HE}}[e] + S_{\mathrm{CS}}[\omega]$$ - the total free energy

No elastic stresses: $g_{\mu\nu} = \delta_{\mu\nu} \quad \left(e_\mu{}^i = \delta_\mu^i\right)$ - Euclidean metric $\quad S_{\mathrm{HE}} \to 0$

Three dimensions:

$$\omega_\mu{}^{ij} = \omega_{\mu k}\varepsilon^{kij}, \qquad \omega_{\mu k} := \tfrac{1}{2}\omega_\mu{}^{ij}\varepsilon_{ijk},$$ - SO(3) connection (1-form, the only variable)

$$\mu, \nu, \ldots = 1,2,3; \qquad i, j, \ldots = 1,2,3$$ - indices

$$R_{\mu\nu k}(\omega) := R_{\mu\nu}{}^{ij}\varepsilon_{ijk} = 2\left(\partial_\mu \omega_{\nu k} - \partial_\nu \omega_{\mu k} + \omega_\mu{}^i \omega_\nu{}^j \varepsilon_{ijk}\right)$$ - curvature

$$S_{\mathrm{CS}} = \int_{\mathbb{R}^3} \left(\tfrac{1}{2}\omega^i \wedge d\omega_i + \tfrac{1}{3}\varepsilon_{ijk}\omega^i \wedge \omega^j \wedge \omega^k - \omega_i \wedge J^i\right)$$ - free energy for disclinations

J^i - the source term (2-form)

$$R_{\mu\nu}{}^k = J_{\mu\nu}{}^k$$ - equations of equilibrium

One linear disclination

$q^\mu(t) \in \mathbb{R}^3, \qquad t \in \mathbb{R}$ - the core of disclination

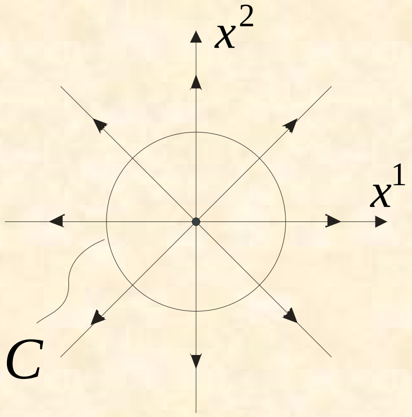
$S_{\text{int}} = \int dq^\mu \omega_{\mu i} J^i = \int dt \dot{q}^\mu \omega_{\mu i} J^i =$ - the interaction term

$$= \int dt d^3x \, \dot{q}^\mu \omega_{\mu i} J^i \delta^3(x-q) = \int d^3x \frac{\dot{q}^\mu}{\dot{q}^3} \omega_{\mu i} J^i \delta^2(\boldsymbol{x}-\boldsymbol{q})$$

$$\frac{\delta S_{\text{int}}}{\delta \omega_{\mu i}} = \frac{\dot{q}^\mu}{\dot{q}^3} J^i \delta^2(\boldsymbol{x}-\boldsymbol{q})$$

- the source term $\boldsymbol{x} = (x^1, x^2)$

One disclination along x^3 axis



Notation: $x^1 = x, \quad x^2 = y, \qquad z := x + iy$

$\omega_x{}^3, \omega_y{}^3$ - the only nontrivial components

$\omega_z{}^3 = \frac{1}{2}\omega_x{}^3 - \frac{i}{2}\omega_y{}^3$ - one complex component

$R_{z\bar{z}}{}^3 = 2\left(\partial_z \omega_{\bar{z}}{}^3 - \partial_{\bar{z}} \omega_z{}^3\right)$ - the curvature tensor

One straight linear disclination

Fixing the source term: $R_{z\bar{z}}{}^{3} = 4\pi i A\delta(z),$ $A \in \mathbb{R}$ - new kind of defect

The solution: $\omega_z{}^{3} = -\frac{iA}{z}$ $\partial_z \frac{1}{\bar{z}} = \pi\delta(z)$ - important formula

$$\omega_x{}^{3} = -\frac{2Ay}{x^2+y^2}, \qquad \omega_y{}^{3} = \frac{2Ax}{x^2+y^2}$$ - real components

Rotational angle field $\omega(\mathbf{x})$

$$\partial_x\omega = -\frac{2Ay}{x^2+y^2}, \qquad \partial_y\omega = \frac{2Ax}{x^2+y^2}$$

The integrability conditions $\partial_{xy}\omega = \partial_{yx}\omega$ are fulfilled

$$\omega = -2A\arctan\frac{x}{y} + C$$ - a general solution

$$C = \pi A \quad \Rightarrow \quad \tan\varphi = \frac{x}{y}, \quad \varphi := \frac{\omega}{2A}$$ - polar angle

One straight linear disclination

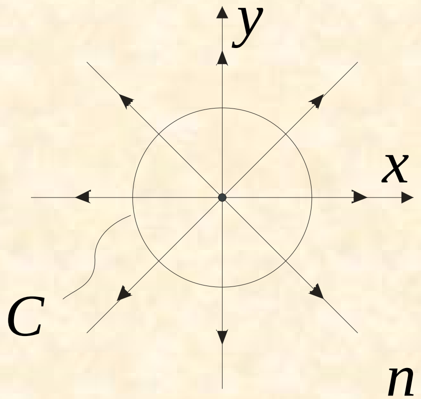
To make the rotational angle field $\omega(x, y)$ well defined, we must impose the quantization condition:

$$A = \frac{n}{2}, \quad n \in \mathbb{Z} \qquad \Rightarrow \qquad \omega = n\varphi$$

SO(3)-connection:

$$\omega_x^{12} = -\frac{ny}{x^2 + y^2} = -n \sin\varphi,$$

$$\omega_y^{12} = \frac{nx}{x^2 + y^2} = n \cos\varphi.$$



$n = 1$

‘t Hooft-Polyakov monopole

$(x^\alpha) \in \mathbb{R}^{1,3}, \ \alpha = 0,1,2,3$ $\quad \eta_{\alpha\beta} := \mathrm{diag}(+---)$ - Lorentz metric

$i, j = 1,2,3$ $\quad \delta_{ij} := \mathrm{diag}(+++)$ - metric in target space

$$L = -\frac{1}{4} F^{\alpha\beta i} F_{\alpha\beta i} + \frac{1}{2} \nabla^\alpha \varphi^i \nabla_\alpha \varphi_i - \frac{1}{4} \lambda (\varphi^2 - a^2)^2$$

- the geometric model

$$F_{\alpha\beta}{}^i := \partial_\alpha A_\beta{}^i - \partial_\beta A_\alpha{}^i + A_\alpha{}^j A_\beta{}^k \varepsilon_{jk}{}^i$$

$A_\alpha{}^i$ - $\mathbb{SU}(2)$ gauge field

$\varphi = (\varphi^i) \in \mathbb{R}^3$ - triplet of scalar fields in adjoint representation of $\mathbb{SU}(2)$ group

$\nabla_\alpha \varphi^i := \partial_\alpha \varphi^i + A_\alpha{}^j \varphi^k \varepsilon_{jk}{}^i$ - covariant derivative

$\lambda, a > 0$ - coupling constants

$$\mathbb{SO}(3) = \frac{\mathbb{SU}(2)}{\mathbb{Z}_2} \quad \Rightarrow \quad \omega_{\alpha i}{}^j = A_{\alpha i}{}^j := A_\alpha{}^k \varepsilon_{ki}{}^j$$

Static solutions

$$A_\alpha{}^i = 0, \qquad \varphi^i = \text{const}, \qquad \varphi^2 = a^2$$ - vacuum solution

$$E = \frac{1}{4} F^{\mu\nu i} F_{\mu\nu i} + \frac{1}{2} \nabla^\mu \varphi^i \nabla_\mu \varphi_i + \frac{1}{4} \lambda (\varphi^2 - a^2)^2$$ - the energy

$(1+3)$ decomposition:

$$(x^\alpha) = (x^0, x^\mu) = (x^0, \boldsymbol{x}), \qquad (A_\alpha{}^i) := (A_0{}^i, A_\mu{}^i), \quad \mu := 1, 2, 3$$

$$A_\alpha{}^i = A_\alpha{}^i(\boldsymbol{x}), \quad \varphi^i(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^3$$ - static solutions

$$A_0{}^i = 0$$ - additional requirement

$$\nabla_\nu F^{\nu\mu}{}_i + (\nabla^\mu \varphi^j) \varphi^k \varepsilon_{ikj} = 0$$

$$-\nabla^\mu \nabla_\mu \varphi_i - \lambda \left(\varphi^2 - a^2\right) \varphi_i = 0$$

- equilibrium equations

$$A_0{}^i \Big|_{r=\infty} = 0, \qquad \varphi^2 \Big|_{r=\infty} = a^2$$ - boundary conditions

Spherically symmetric solutions

Assumption: $\mathbb{SO}(3)$ acts simultaneously in coordinate and target spaces

$$A_\mu{}^i = \frac{\varepsilon_\mu{}^{ij} x_j}{r^2}(K-1), \qquad \varphi^i = \frac{x^i}{r^2} H$$ - spherically symmetric ansatz

$K(r),\ H(r)$ - unknown functions of radius r

$$r^2 K'' = K(K^2 + H^2 - 1)$$
$$r^2 H'' = 2HK^2 + \lambda\left(H^2 - a^2 r^2\right)$$
- nonlinear system of equations

The Bogomol'nyi-Prasad-Sommerfield (1975) solution (for $\lambda = 0$):

$$K = \frac{ar}{\mathrm{sh}(ar)}, \quad H = \frac{ar}{\mathrm{th}(ar)} - 1$$

Disclinations and dislocations

$\mathbb{SU}(2) \to \mathbb{SO}(3)$ acts simultaneously in coordinate and target spaces

Media without elastic stresses: $\mathbb{R}^3, \quad g_{\mu\nu} = \delta_{\mu\nu}, \quad e_\mu{}^i = \delta_\mu^i$

$$E = \frac{1}{4} F^{\mu\nu i} F_{\mu\nu i} + \frac{1}{2} \nabla^\mu \varphi^i \nabla_\mu \varphi_i + \frac{1}{4} \lambda (\varphi^2 - a^2)^2$$

- the free energy

φ^i - sources for defects

$$\omega_\mu{}^{ij} = A_\mu{}^k \varepsilon_k{}^{ij} = (\delta_\mu^i x^j - \delta_\mu^j x^i) \frac{K-1}{r^2}$$

- spherically symmetric solution

$$R_{\mu\nu}{}^k = \varepsilon_{\mu\nu}{}^k \frac{K'}{r} - \frac{\varepsilon_{\mu\nu}{}^j x_j x^k}{r^3} \left(K' - \frac{K^2 - 1}{r} \right)$$

$$T_{\mu\nu}{}^k = (\delta_\mu^k x_\nu - \delta_\nu^k x_\mu) \frac{K-1}{r^2}$$

- continuous distribution of disclinations and dislocations

Point disclinations

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$$A_\mu{}^i = \frac{\varepsilon_\mu{}^{ij} x_j \left(K(r)-1\right)}{r^2} + \delta^i_\mu V(r) + \frac{x_\mu x^i U(r)}{r^2}$$

-the most general spherically symmetric SO(3) connection

$K(r),\ V(r),\ U(r)$ - arbitrary functions

(*) $F_{\mu\nu}{}^i = 0$ ⇒

$$K' + rV(V+U) = 0,$$

$$-K' + \frac{K^2-1}{r} - rVU = 0,$$

$$rV' - U - (K-1)(V+U) = 0.$$

- equilibrium equations outside the origin

Theorem. The most general solution for (*) is

$$K = \cos f,$$

$$V = \frac{\sin f}{r},$$

$$U = \frac{rf' - \sin f}{r},$$

where $f(r)$ is an arbitrary function.

Point disclinations

$$F_{\mu\nu}{}^{i} = 0 \quad \Longrightarrow \quad A_{\mu i}{}^{j} := A_{\mu}{}^{k}\varepsilon_{ki}{}^{j} = \partial_{\mu} S^{-1k}{}_{i} S_{k}{}^{j}, \qquad S = (S_{i}{}^{j}) \in \mathbb{SO}(3)$$

- pure gauge

$\gamma = \left(x^{\mu}(t)\right)$ - a curve $\qquad \dot{S} = \dot{x}^{\mu} A_{\mu} S$

$$S\left(x(t)\right) = \mathrm{P}\exp\left(\int_{0}^{t} ds\, \dot{x}^{\mu}(s) A_{\mu}(s)\right) S_{0}$$ - ordered exponent $\qquad S_{0} := S(0)$

General form of rotational matrix

$$S_{i}{}^{j} = \delta_{i}^{j}\cos f + \frac{f^{k}\varepsilon_{ki}{}^{j}}{f}\sin f + \frac{f_{i} f^{j}}{f^{2}}(1-\cos f), \qquad f^{k} := \frac{x^{k}}{r} f(r)$$

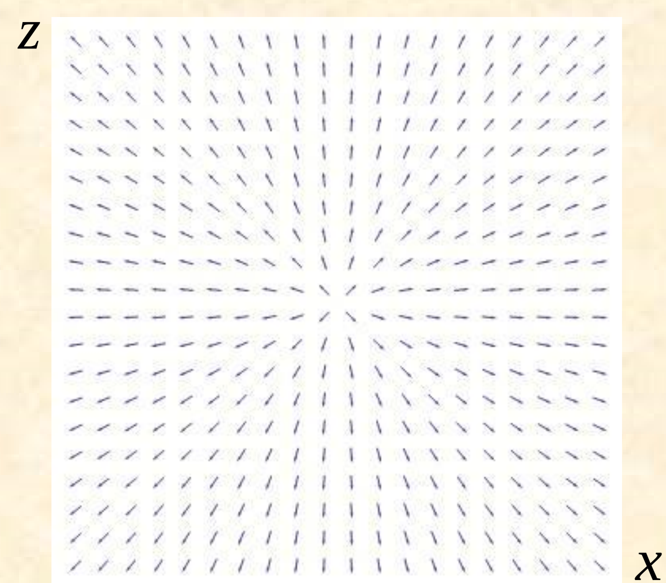
$f(0) = 0, \pi$ - no disclinations

Hedgehog disclination

$$n^{i}(r=\infty) = \frac{x^{i}}{r}$$

$$\Downarrow$$

$$n^{i}(x) := n^{j}(\infty) S_{j}{}^{i}(r) = \frac{x^{i}}{r}, \qquad \forall f$$



Essential singularity

$$n^i(\infty) = n_0 = (0,0,1) \quad \Rightarrow \quad n^i(x) = n_0{}^j S_j{}^i(r):$$

↳ breaks the spherical symmetry

$$n_x = -\frac{y}{r}\sin f + \frac{xz}{r^2}(1-\cos f),$$

$$n_y = \frac{x}{r}\sin f + \frac{yz}{r^2}(1-\cos f),$$

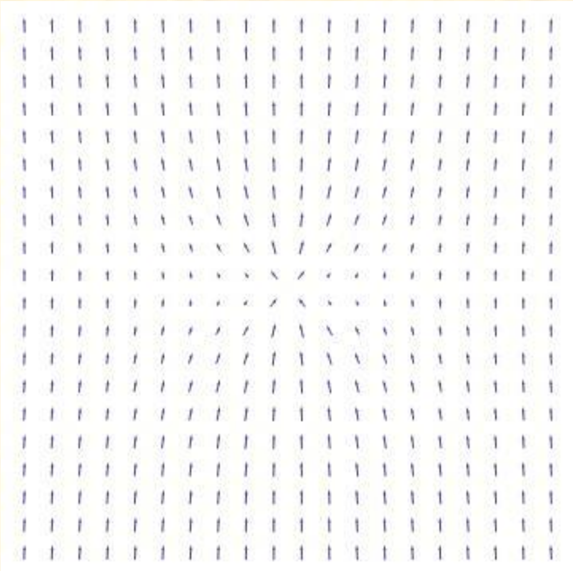
$$n_z = \cos f + \frac{z^2}{r^2}(1-\cos f).$$

Notation

$$(x^1, x^2, x^3) := (x, y, z)$$

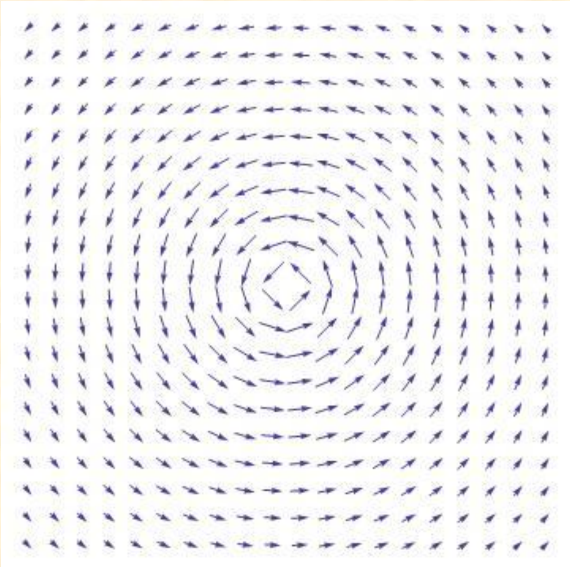
$$f(r) := \frac{\pi}{2}\mathrm{e}^{-r}, \quad \Rightarrow \quad f(0) = \frac{\pi}{2}, \quad f(\infty) = 0$$

z



x

y



x

Conclusion

1) The ‘t Hooft-Polyakov monopole has straightforward physical interpretation in the geometric theory of defects describing continuous distribution of dislocations and disclinations and, probably, can be observed in solids.

2) The Chern-Simons term is well suited for single disclinations in the geometric theory of defects.

3) We have described the straight linear disclination, hedgehog disclination, and point disclination with essential singularity at the origin. These are the first examples of disclinations described in the framework of geometric theory of defects.