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Sunrise integrals in non-relativistic QCD with elliptics

OUTLINE

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Abstract

We consider a set of two-loop sunrise master integrals with two different internal masses at pseudo-threshold kinematics (i.e. $q^2 = m^2$ in Euclidean space) and we solve it including in terms of elliptic polylogarithms to all orders of the dimensional regulator.

0. Introduction

In the last decades much progress has been made in the understanding of the mathematical properties of Feynman integrals. Arguably many of the breakthroughs in this line of research originated from the identification of classes of special functions suited for the solution of Feynman integrals by means of various analytic methods. It is a well-known fact that while many Feynman integrals admit representations in terms of so-called multiple polylogarithms (MPLs) (Goncharov,1998); (Remiddi,Vermaseren,1999).

More recently, the scientific community has centered its attention to the study of Feynman integrals whose geometric properties are defined by elliptic curves. Following early investigations of (Sabry,1962), (Broadhurst,Fleischer,Tarasov,1993),
many integrals involving elliptic curves have been computed in the literature (see the review in (Weinzierl,2020)).

In a parallel line of research, a class of functions, the so-called Elliptic Multiple Polylogarithms (eMPLs), describing all iterated integrals on the torus has been identified in the mathematics literature

(Brown,Andrey,2011); (Beilinson,Levin,1994); (Levin,Racined,2007). While these functions formally solve the problem of generalising MPLs to more complicated geometries, their definition is not naturally suited for physical applications. Progress in this direction has been made in (Bloch,Vanhove,2013), (Weinzierl et al.,2013-2020); (Broedel,Duhr,Dulat,Tancredi,2017) [**below (Broedel,2017)**], where eMPLs are defined on the complex plane, and their structure naturally adapts to representations of Feynman integrals commonly used in the physics literature (e.g. Feynman parameters).

Special functions such as MPLs and eMPLs, frequently appear when computing Feynman integrals in dimensional regularisation. More specifically, Feynman integrals admit a Laurent expansion with respect to the dimensional regulator and the coefficients of this expansion can be often computed explicitly in terms of known special functions. In practice it is often possible to truncate the Laurent series, as the computation of physically relevant quantities requires only a few expansion orders. Nonetheless it is interesting to explore the analytic structure of these coefficients at higher orders or, more generally, to all orders of the dimensional regulator.

In this talk we consider a two-loop sunrise integral topology with two internal masses and pseudo-threshold kinematics (Kniehl,Kotikov,Onishchenko,Veretin,2005,2019), (Besuglov,Kotikov,Onishchenko,2022); (Kalmykov,Kniehl,2008) [**below (Kniehl,2005,2019); (Besuglov,2022); (Kalmykov,2008)**]. More precisely, we consider two different internal masses, denoted by m and M, and external kinematics $q^2 = m^2$ (in Euclidean space). This integral family appears when considering non-relativistic limits of Quantum Chromodynamics (NRQCD) and Quantum Electrodynamics (NRQED).

This integral family admits a closed-form solution in terms of ${}_4F_3$-hypergeometric functions, as shown in (Kalmykov,2008) (the corresponding off-shell diagrams with equal masses are considerably more complicated and their explicit solution requires Appell's F_2 hypergeometric functions (Tarasov,2006)). [But Oleg Tarasov comment of "Advances in Quantum Field Theory"].

Moreover, rather similar results (but with $O(\varepsilon)$ accuracy) exist for three-point and four-point two-loop Feynman diagrams in NRQCD kinematics (see (Kniehl,2019)). Some exact results are presented in (Besuglov,2022).

Here we consider the two-loop sunrise integral family discussed above and derive results in terms:

- of one- and two-fold integral representations.
- (in the first two ε-orders under consideration) multiple integrals containing the elliptic kernel and logarithms and dilogarithms in the integrands. In more general cases, such multiple integrals containing the elliptic kernel and Goncharov's MPLs in the integrand (see (Besuglov,Onishchenko,Veretin,2020);(Besuglov,2021); (Besuglov,Onishchenko,2021);(Besuglov,Kotikov,Onishchenko,2022)). [**below (Besuglov)**].
- of eMPLs (following (Broedel,2017)) valid to all orders of the dimensional regulator.

1. The sunrise integral

We study the sunrise integral topology defined as,

$$J_{i_1,i_2,i_3}(m^2, M^2) = \int \frac{d^D k_1 d^D k_2 \, (\mu^2)^{2\varepsilon}}{[k_2^2 + m^2]^{i_1}[k_1^2 + M^2]^{i_2}[(k_1 - k_2 - q)^2 + M^2]^{i_3}}\Bigg|_{q^2=m^2},$$

with $D = 4 - 2\epsilon$. This integral family has three master integrals, which can be chosen to be $J_{1,1,1}$, $J_{1,1,2}$, $J_{1,2,2}$ and which can be solved in closed form in terms of hypergeometric functions (Kalmykov,2008) as,

$$J_{1,2,2}(m^2,M^2) = -\frac{\hat{N}_1}{M^2}\frac{(1+\epsilon)}{\epsilon(1-\epsilon)} \times \left[\frac{1}{6}{}_4F_3\left(\begin{array}{c} 1+\frac{\varepsilon}{2}, \frac{3+\varepsilon}{2}, \frac{3}{2}, 1 \\ 2-\varepsilon, \frac{5}{4}, \frac{7}{4} \end{array} \middle| -\frac{m^4}{4M^4}\right)\right.$$

$$-\left(\frac{M^2}{m^2}\right)^{1-\epsilon} \frac{\epsilon}{(1+\epsilon)(1+2\epsilon)}{}_4F_3\left(\begin{array}{c} \varepsilon+\frac{1}{2}, 1+\varepsilon, 1+\frac{\varepsilon}{2}, 1 \\ \frac{3-\varepsilon}{2}, \frac{3+2\varepsilon}{4}, \frac{5+2\varepsilon}{4} \end{array} \middle| -\frac{m^4}{4M^4}\right)$$

$$\left.-\left(\frac{M^2}{m^2}\right)^{-\epsilon} \frac{1-\epsilon}{(2-\epsilon)(3+2\epsilon)}{}_4F_3\left(\begin{array}{c} \varepsilon+\frac{3}{2}, 1+\varepsilon, \frac{3+\varepsilon}{2}, 1 \\ \frac{4-\varepsilon}{2}, \frac{5+2\varepsilon}{4}, \frac{7+2\varepsilon}{4} \end{array} \middle| -\frac{m^4}{4M^4}\right)\right],$$

$$J_{1,1,2}(m^2,M^2) = \frac{N_1}{2\epsilon(1-\epsilon)} \times \left[{}_4F_3\left(\begin{array}{c} 1+\frac{\varepsilon}{2}, \frac{1+\varepsilon}{2}, \frac{1}{2}, 1 \\ 2-\varepsilon, \frac{3}{4}, \frac{5}{4} \end{array} \middle| -\frac{m^4}{4M^4}\right)\right.$$

$$-\left(\frac{M^2}{m^2}\right)^{1-\epsilon} \frac{1}{\epsilon}{}_4F_3\left(\begin{array}{c} \varepsilon+\frac{1}{2}, \varepsilon, \frac{\varepsilon}{2}, 1 \\ \frac{3-\varepsilon}{2}, \frac{1+2\varepsilon}{4}, \frac{3+2\varepsilon}{4} \end{array} \middle| -\frac{m^4}{4M^4}\right)$$

$$\left.-2\left(\frac{M^2}{m^2}\right)^{-\epsilon} \frac{1-\epsilon}{(2-\epsilon)(1+2\epsilon)}{}_4F_3\left(\begin{array}{c} \varepsilon+\frac{1}{2}, 1+\varepsilon, \frac{1+\varepsilon}{2}, 1 \\ \frac{4-\varepsilon}{2}, \frac{3+2\varepsilon}{4}, \frac{5+2\varepsilon}{4} \end{array} \middle| -\frac{m^4}{4M^4}\right)\right],$$

$$J_{1,1,1}(m^2,M^2) = \frac{-M^2\hat{N}_1}{\epsilon^2(1-\epsilon)} \times \left[{}_4F_3\left(\begin{array}{c} \frac{\varepsilon}{2}, \frac{1+\varepsilon}{2}, \frac{1}{2}, 1 \\ 2-\varepsilon, \frac{3}{4}, \frac{5}{4} \end{array} \middle| -\frac{m^4}{4M^4}\right)\right.$$

$$+\left(\frac{M^2}{m^2}\right)^{1-\epsilon} \frac{1}{(1-2\epsilon)}{}_4F_3\left(\begin{array}{c} \varepsilon-\frac{1}{2}, \varepsilon, \frac{\varepsilon}{2}, 1 \\ \frac{3-\varepsilon}{2}, \frac{1+2\varepsilon}{4}, \frac{3+2\varepsilon}{4} \end{array} \middle| -\frac{m^4}{4M^4}\right)$$

$$\left.-\left(\frac{M^2}{m^2}\right)^{-\epsilon} \frac{1-\epsilon}{(2-\epsilon)(1+2\epsilon)}{}_4F_3\left(\begin{array}{c} \varepsilon+\frac{1}{2}, \varepsilon, \frac{1+\varepsilon}{2}, 1 \\ \frac{4-\varepsilon}{2}, \frac{3+2\varepsilon}{4}, \frac{5+2\varepsilon}{4} \end{array} \middle| -\frac{m^4}{4M^4}\right)\right].$$

The normalization constant is,

$$\hat{N}_1 = \frac{\Gamma^2(1+\epsilon)(\mu^2)^{2\epsilon}}{(m^2 M^2)^{\epsilon}} \frac{m^2}{M^2}\,.$$

2. Integral representations

Here we show that the above hypergeometric functions admit one-fold and two-fold integral representations:

$$J_{1,2,2} = \frac{\hat{N}_1}{M^2}[J^{(1)}_{1,2,2}(t) + (2t)^{\varepsilon-1}\, J^{(2)}_{1,2,2}(t) + (2t)^{\varepsilon}\, J^{(3)}_{1,2,2}(t)]\,,$$
$$J_{1,1,2} = \hat{N}_1[J^{(1)}_{1,1,2}(t) + (2t)^{\varepsilon-1}\, J^{(2)}_{1,1,2}(t) + (2t)^{\varepsilon}\, J^{(3)}_{1,1,2}(t)]\,,$$
$$J_{1,1,1} = M^2\hat{N}_1[J^{(1)}_{1,1,2}(t) + (2t)^{\varepsilon-1}\, J^{(2)}_{1,1,2}(t) + (2t)^{\varepsilon}\, J^{(3)}_{1,1,2}(t)]\,,$$

where $t = m^2/(2M^2)$ and

$$J^{(1)}_{1,2,2}(t) = -\frac{1+\varepsilon}{6\varepsilon(1-\varepsilon)}\, {}_4F_3\left(\begin{array}{c} 1+\frac{\varepsilon}{2}, \frac{3+\varepsilon}{2}, \frac{3}{2}, 1 \\ 2-\varepsilon, \frac{5}{4}, \frac{7}{4} \end{array}\middle| -t^2\right) = \frac{\hat{K}}{2^{2+2\epsilon}\varepsilon t^2}\, I^{(1)}_1(t)\,,$$

$$J^{(1)}_{1,1,2}(t) = \frac{1}{2\varepsilon(1-\varepsilon)}\, {}_4F_3\left(\begin{array}{c} 1+\frac{\varepsilon}{2}, \frac{1+\varepsilon}{2}, \frac{1}{2}, 1 \\ 2-\varepsilon, \frac{3}{4}, \frac{5}{4} \end{array}\middle| -t^2\right) = \frac{\hat{K}}{2^{1+2\epsilon}(1-2\varepsilon)\varepsilon t^2}\, I^{(1)}_2(t)\,,$$

$$J^{(1)}_{1,1,1}(t) = -\frac{1}{\varepsilon^2(1-\varepsilon)}\, {}_4F_3\left(\begin{array}{c} \frac{\varepsilon}{2}, \frac{1+\varepsilon}{2}, \frac{1}{2}, 1 \\ 2-\varepsilon, \frac{3}{4}, \frac{5}{4} \end{array}\middle| -t^2\right) = -\frac{1}{\varepsilon^2(1-\varepsilon)} - \frac{\hat{K}}{2^{2\epsilon-1}\varepsilon(1-2\varepsilon)^2 t^2}\, \tilde{I}^{(1)}_3(t)\,,$$

$$J^{(2)}_{1,2,2}(t) = \frac{1}{(1+2\varepsilon)(1-\varepsilon)}\, {}_4F_3\left(\begin{array}{c} \varepsilon+\frac{1}{2}, 1+\varepsilon, 1+\frac{\varepsilon}{2}, 1 \\ \frac{3-\varepsilon}{2}, \frac{3+2\varepsilon}{4}, \frac{5+2\varepsilon}{4} \end{array}\middle| -t^2\right) = \frac{\hat{K}}{2^{1+4\epsilon}t^{1-\varepsilon}}\, I^{(2)}_1(t)\,,$$

$$J^{(2)}_{1,1,2}(t) = -\frac{1}{2\varepsilon^2(1-\varepsilon)}\, {}_4F_3\left(\begin{array}{c} \varepsilon+\frac{1}{2}, \varepsilon, \frac{\varepsilon}{2}, 1 \\ \frac{3-\varepsilon}{2}, \frac{1+2\varepsilon}{4}, \frac{3+2\varepsilon}{4} \end{array}\middle| -t^2\right) = -\frac{1}{2\varepsilon^2(1-\varepsilon)} + \frac{1}{(1-2\varepsilon)}\, \frac{\hat{K}}{2^{4\epsilon}t^{1-\varepsilon}}\, \tilde{I}^{(2)}_2(t)\,,$$

$$J^{(2)}_{1,1,1}(t) = -\frac{1}{\varepsilon^2(1-\varepsilon)(1-2\varepsilon)}\,{}_4F_3\left(\begin{array}{c} \varepsilon-\frac{1}{2},\varepsilon,\frac{\varepsilon}{2},1 \\ \frac{3-\varepsilon}{2},\frac{1+2\varepsilon}{4},\frac{3+2\varepsilon}{4} \end{array}\middle| -t^2\right) = -\frac{1}{(1-\varepsilon)(1-2\varepsilon)\varepsilon^2} - \frac{2^{2-4\epsilon}\hat{K}}{(1-2\varepsilon)^2 t^{1-\varepsilon}}\,\tilde{I}^{(2)}_3(t)\,,$$

$$J^{(3)}_{1,2,2}(t) = \frac{1+\varepsilon}{\varepsilon(2-\varepsilon)(3+2\varepsilon)}\,{}_4F_3\left(\begin{array}{c} \varepsilon+\frac{3}{2},1+\varepsilon,\frac{3+\varepsilon}{2},1 \\ \frac{4-\varepsilon}{2},\frac{5+2\varepsilon}{4},\frac{7+2\varepsilon}{4} \end{array}\middle| -t^2\right) = -\frac{\hat{K}}{2^{2+4\epsilon}\varepsilon t^2}\,I^{(3)}_1(t)\,,$$

$$J^{(3)}_{1,1,2}(t) = -\frac{1}{\varepsilon(2-\varepsilon)(1+2\varepsilon)}\,{}_4F_3\left(\begin{array}{c} \varepsilon+\frac{1}{2},1+\varepsilon,\frac{1+\varepsilon}{2},1 \\ \frac{4-\varepsilon}{2},\frac{3+2\varepsilon}{4},\frac{5+2\varepsilon}{4} \end{array}\middle| -t^2\right) = -\frac{\hat{K}}{2^{1+4\epsilon}(1-2\varepsilon)\varepsilon t^2}\,I^{(3)}_2(t)\,,$$

$$J^{(3)}_{1,1,1}(t) = -\frac{1}{\varepsilon^2(2-\varepsilon)(1+2\varepsilon)}\,{}_4F_3\left(\begin{array}{c} \varepsilon+\frac{1}{2},\varepsilon,\frac{1+\varepsilon}{2},1 \\ \frac{4-\varepsilon}{2},\frac{3+2\varepsilon}{4},\frac{5+2\varepsilon}{4} \end{array}\middle| -t^2\right) = \frac{1}{\varepsilon^2(2-\varepsilon)(1+2\varepsilon)} + \frac{2^{1-4\epsilon}\hat{K}}{(1-2\varepsilon)^2\varepsilon t^2}\,\tilde{I}^{(3)}_3(t)\,,$$

where $\hat{K}$ is defined as,

$$\hat{K} = \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)\Gamma(1+\varepsilon)},$$

The factors $I_j^{(i)}(t)$ and $\tilde{I}_3^{(i)}(t)$ $(j=1,2)$, $(i=1,3)$ are

$$I_j^{(i)}(t) = I_{j,1}^{(i)}(t) - \frac{2\epsilon}{1+i} I_{j,2}^{(1)}(t),\;\; \tilde{I}_3^{(i)}(t) = I_{3,1}^{(i)}(t) - \frac{2\epsilon}{1+i} I_{3,2}^{(i)}(t)\,,$$

with,

$$
\begin{aligned}
I_{1,1}^{(1)}(t) &= \int_0^1 dp\, p^{\epsilon-1}(1-p)^{-\epsilon-\frac{1}{2}}\left((p^2t^2+1)^{-\frac{1}{2}}-1\right),\\
I_{1,2}^{(1)}(t) &= \int_0^1 dp\, p^{\epsilon-1}\,(1-p)^{-\epsilon-\frac{1}{2}}\,(p^2t^2+1)^{-\frac{1}{2}}\,J^{(1)}(q(p)),\\
I_{2,1}^{(1)}(t) &= \int_0^1 dp\, p^{\epsilon-2}(1-p)^{-\epsilon-\frac{1}{2}}\left((p^2t^2+1)^{\frac{1}{2}}-1\right),\\
I_{2,2}^{(1)}(t) &= \int_0^1 dp\, p^{\epsilon-2}\,(1-p)^{-\epsilon-\frac{1}{2}}\,(p^2t^2+1)^{\frac{1}{2}}\,J^{(1)}(q(p)),\\
\tilde{I}_{3,1}^{(1)}(t) &= \int_0^1 dp\, p^{\epsilon-3}(1-p)^{-\epsilon+\frac{1}{2}}\left((p^2t^2+1)^{\frac{1}{2}}-1-\frac{(pt)^2}{2}\right),\\
\tilde{I}_{3,2}^{(1)}(t) &= \int_0^1 dp\, p^{\epsilon-3}\,(1-p)^{-\epsilon+\frac{1}{2}}\,(p^2t^2+1)^{\frac{1}{2}}\,\tilde{J}^{(1)}(q(p)),\\
I_1^{(2)}(t) &= \int_0^1 dp\, p^{3\epsilon-1}(1-p)^{-\epsilon-\frac{1}{2}}\left(p^2t^2+1\right)^{-\epsilon-\frac{1}{2}}\,J^{(2)}(pt),\\
I_2^{(2)}(t) &= \int_0^1 dp\, p^{3\epsilon-2}(1-p)^{-\epsilon-\frac{1}{2}}\left(p^2t^2+1\right)^{-\epsilon+\frac{1}{2}}\,J^{(2)}(pt),\\
\tilde{I}_2^{(2)}(t) &= \int_0^1 dp\, p^{3\epsilon-2}(1-p)^{-\epsilon-\frac{1}{2}}\left(p^2t^2+1\right)^{-\epsilon+\frac{1}{2}}\,\tilde{J}^{(2)}(pt),\\
\tilde{I}_3^{(2)}(t) &= \int_0^1 dp\, p^{3\epsilon-3}(1-p)^{-\epsilon+\frac{1}{2}}\left(p^2t^2+1\right)^{-\epsilon+\frac{1}{2}}\,\tilde{J}^{(2)}(pt),\\
I_{1,1}^{(3)}(t) &= \int_0^1 dp\, p^{2\epsilon-1}(1-p)^{-\epsilon-\frac{1}{2}}\left(\left(p^2t^2+1\right)^{-\frac{\epsilon}{2}-\frac{1}{2}}-1\right),\\
I_{1,2}^{(3)}(t) &= \int_0^1 dp\, p^{2\epsilon-1}(1-p)^{-\epsilon-\frac{1}{2}}\left(p^2t^2+1\right)^{-\frac{\epsilon}{2}-\frac{1}{2}}\,J^{(3)}(q(p)),\\
I_{2,1}^{(3)}(t) &= \int_0^1 dp\, p^{2\epsilon-2}(1-p)^{-\epsilon-\frac{1}{2}}\left((p^2t^2+1)^{\frac{1}{2}-\frac{\epsilon}{2}}-1\right),
\end{aligned}
$$

$$I^{(3)}_{2,2}(t) = \int_0^1 dp\, p^{2\epsilon-2}\,(1-p)^{-\epsilon-\frac{1}{2}}\,(p^2t^2+1)^{\frac{1}{2}-\frac{\epsilon}{2}}\,J^{(3)}(q(p)),$$

$$\tilde{I}^{(3)}_{3,1}(t) = \int_0^1 dp\, p^{2\epsilon-3}(1-p)^{-\epsilon+\frac{1}{2}}\left((p^2t^2+1)^{\frac{1}{2}-\frac{\epsilon}{2}} - 1 - (1-\varepsilon)\,\frac{(pt)^2}{2}\right),$$

$$\tilde{I}^{(3)}_{3,2}(t) = \int_0^1 dp\, p^{2\epsilon-3}\,(1-p)^{-\epsilon+\frac{1}{2}}\,(p^2t^2+1)^{\frac{1}{2}-\frac{\epsilon}{2}}\,\tilde{J}^{(3)}(q(p)),$$

where

$$J^{(1)}(q) = q^{\epsilon}\int_0^q dz\left((1-z)^{-\frac{1}{2}} - 1\right) z^{-\epsilon-1},$$

$$J^{(2)}(pt) = \int_0^{pt} dz\, z^{-\epsilon}\left(z^2+1\right)^{\epsilon-\frac{1}{2}},$$

$$J^{(3)}(q) = q^{\frac{\epsilon}{2}}\int_0^q dz\left((1-z)^{-\frac{\epsilon}{2}-\frac{1}{2}} - 1\right) z^{-\frac{\epsilon}{2}-1},$$

$$\tilde{J}^{(1)}(q) = J^{(1)}(q) - \frac{(pt)^2}{2(1-\varepsilon)},$$

$$\tilde{J}^{(2)}(pt) = J^{(2)}(pt) - \frac{1}{(p^2t^2+1)^{\frac{1}{2}-\epsilon}}\,\frac{(pt)^{1-\varepsilon}}{1-\varepsilon},$$

$$\tilde{J}^{(3)}(q) = J^{(3)}(q) - \frac{(1+\varepsilon)}{(2-\varepsilon)}(pt)^2$$

and

$$q(p) = \frac{p^2t^2}{p^2t^2+1}.$$

Integrals with tildes are used when the corresponding integrals have singularities for small p values and for small z values. They are constructed from the corresponding integrals by extracting the leading asymptotics of subintegral expressions for small p and for small z, respectively, and, therefore, they are finite.

3. Integral representations (continuation)

Using above avaluations, we have more convenient integral representations.

3.1. $J_{1,2,2}$

$$M^2\, J_{1,2,2} = \hat{N}_1\, \frac{\hat{K}_1}{4t^2}\, \hat{J}_{1,2,2},$$

$$\hat{J}_{1,2,2} = [\frac{1}{\varepsilon} I_1^{(1)}(t) + \left(\frac{t^2}{2}\right)^{\varepsilon} I_1^{(2)}(t) - \frac{1}{\varepsilon}\left(\frac{t}{2}\right)^{\varepsilon} I_1^{(3)}(t)]$$

$$= \left[\frac{1}{\varepsilon} I_{1,1}^{(1)}(t) - I_{1,2}^{(1)}(t) + \left(\frac{t^2}{2}\right)^{\varepsilon} I_1^{(2)}(t) - \left(\frac{t}{2}\right)^{\varepsilon} \left(\frac{1}{\varepsilon} I_{1,1}^{(3)}(t) - \frac{1}{2} I_{1,2}^{(3)}(t)\right)\right],$$

where

$$\hat{K}_1 = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} - \epsilon\right)\Gamma(\epsilon + 1)} = \frac{\hat{K}}{2^{2\varepsilon}}$$

and $\hat{K}$ is defined above.

3.2. $J_{1,1,1}$ and $J_{1,1,2}$

We split the expressions for $J_{1,1,2}$ and $J_{1,1,1}$ to singular and regular parts:

$$J_{1,1,j} = J^{\text{sing}}_{1,1,j} + J^{reg}_{1,1,j} \quad (j=1,2),$$

$$J^{\text{sing}}_{1,1,2} = -\frac{\hat{N}_2}{2(1-\varepsilon)\varepsilon^2}, \quad J^{reg}_{1,1,2} = \frac{\hat{K}_1\,\hat{N}_1}{2(1-2\varepsilon)t^2}\,\hat{J}_{1,1,2},$$

$$\hat{J}^{\text{reg}}_{1,1,2} = [\frac{1}{\varepsilon} I_2^{(1)}(t) + \left(\frac{t^2}{2}\right)^{\varepsilon} \tilde{I}_2^{(2)}(t) - \frac{1}{\varepsilon}\left(\frac{t}{2}\right)^{\varepsilon} I_2^{(3)}(t)]\,,$$

$$J^{\text{sing}}_{1,1,1} = \left\{\frac{m^2}{\varepsilon^2}\left[\frac{1}{(2-\varepsilon)(1+2\varepsilon)} - \frac{(2t)^{-\varepsilon}}{(1-\varepsilon)}\right] - \frac{M^2}{(1-\varepsilon)(1-2\varepsilon)\varepsilon^2}\right\}\hat{N}_2,$$

$$J^{\text{reg}}_{1,1,1} = -M^2\,\frac{2\hat{K}_1\,\hat{N}_1}{(1-2\varepsilon)^2 t^2}\,\hat{J}_{1,1,1},$$

$$\hat{J}_{1,1,1} = [\frac{1}{\varepsilon}\tilde{I}_3^{(1)}(t) + \left(\frac{t^2}{2}\right)^{\varepsilon} \tilde{I}_3^{(2)}(t) - \frac{1}{\varepsilon}\left(\frac{t}{2}\right)^{\varepsilon} \tilde{I}_3^{(3)}(t)]\,,$$

where the new normalization constant is

$$\hat{N}_2 = (2t)^{\varepsilon-1}\,\hat{N}_1 = \frac{\Gamma^2(1+\varepsilon)(\mu^2)^{2\varepsilon}}{(M^2)^{2\varepsilon}}\,.$$

3.3. Example: the first hypergeometric function in $J_{1,2,2}$.

The first ${}_4F_3$-hypergeometric function of $J_{1,2,2}$, i.e.

$$J^{(1)}_{1,2,2}(t) = -\frac{1+\varepsilon}{6\varepsilon(1-\varepsilon)}\,{}_4F_3\left(\left.\begin{matrix} 1+\frac{\varepsilon}{2}, \frac{3+\varepsilon}{2}, \frac{3}{2}, 1 \\ 2-\varepsilon, \frac{5}{4}, \frac{7}{4} \end{matrix}\right| -t^2\right) \equiv -\frac{1+\varepsilon}{6\varepsilon(1-\varepsilon)}\,F^{(1)}_1(t)\,,$$

admits the following series representation,

$$F^{(1)}_1(t) \equiv {}_4F_3\left(\left.\begin{matrix} 1, \frac{3}{2}, 1+\frac{\varepsilon}{2}, \frac{3}{2}+\frac{\varepsilon}{2} \\ 2-\varepsilon, \frac{5}{4}, \frac{7}{4} \end{matrix}\right| -t^2\right),$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{3}{2})\Gamma(m+1+\frac{\varepsilon}{2})\Gamma(m+\frac{3}{2}+\frac{\varepsilon}{2})}{\Gamma(m+2-\varepsilon)\Gamma(m+\frac{5}{4})\Gamma(m+\frac{7}{4})}\,\frac{\Gamma(2-\varepsilon)\Gamma(\frac{5}{4})\Gamma(\frac{7}{4})}{\Gamma(\frac{3}{2})\Gamma(1+\frac{\varepsilon}{2})\Gamma(\frac{3}{2}+\frac{\varepsilon}{2})}\,(-t^2)^m\,,$$

where $t = m^2/(2M^2)$. The product $\Gamma(\alpha)\Gamma(1/2+\alpha)$ can be written as,

$$\Gamma(\alpha)\Gamma(1/2+\alpha) \;=\; 2^{1-2\alpha}\,\sqrt{\pi}\,\Gamma(2\alpha)\,,$$

which results in the following simplified expression for $F_1^{(1)}(t)$,

$$\sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{3}{2})\Gamma(2m+2+\varepsilon)}{\Gamma(m+2-\varepsilon)\Gamma(2m+\frac{5}{2})}\,\frac{\Gamma(2-\varepsilon)\Gamma(\frac{5}{2})}{\Gamma(\frac{3}{2})\Gamma(2+\varepsilon)}\,(-t^2)^m\,.$$

It is convenient to use the following integral representations for the ratio of gamma functions,

$$\frac{\Gamma(2m+2+\varepsilon)}{\Gamma(2m+\frac{5}{2})} = \int_0^1 dp\,\frac{p^{2m+1+\varepsilon}(1-p)^{-1/2-\varepsilon}}{\Gamma(\frac{1}{2}-\varepsilon)}\,.$$

We find,

$$F_1^{(1)}(t) = \int_0^1 dp\,\frac{p^{1+\varepsilon}(1-p)^{-1/2-\varepsilon}}{\Gamma(\frac{1}{2}-\varepsilon)}\sum_{m=0}^{\infty}\frac{\Gamma(m+\frac{3}{2})}{\Gamma(m+2-\varepsilon)}\,\frac{\Gamma(2-\varepsilon)\Gamma(\frac{5}{2})}{\Gamma(\frac{3}{2})\Gamma(2+\varepsilon)}\,(-(tp)^2)^m\,.$$

In order to proceed with our analysis it is convenient to consider first the series on the right hand side in the limit $\varepsilon = 0$,

$$\sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{3}{2})}{(m+1)!} \left(-(tp)^2\right)^m = \sum_{m=1}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{m!} \left(-(tp)^2\right)^{m-1}$$

$$= -\frac{\Gamma(\frac{1}{2})}{(tp)^2} \left[\frac{1}{\left(1+t^2p^2\right)^{1/2}} - 1\right]$$

In the general case we have,

$$\sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{3}{2})}{\Gamma(m+2-\varepsilon)} \left(-(tp)^2\right)^m = \sum_{m=1}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1-\varepsilon)} \left(-(tp)^2\right)^{m-1}$$

$$= -\frac{\Gamma(\frac{1}{2})}{\Gamma(1-\varepsilon)(tp)^2} \left[{}_2F_1\left(1, \frac{1}{2}; 1-\varepsilon; -p^2t^2\right) - 1\right]$$

Using standard properties of the ${}_2F_1$-function,

$$ {}_2F_1(a,b;c;z) = (1-z)^b\, {}_2F_1\left(c-a,b;c;\frac{z-1}{z}\right) $$

we obtain,

$$ {}_2F_1\left(1,\frac{1}{2};1-\varepsilon;-p^2t^2\right) = \frac{1}{\left(1+t^2p^2\right)^{1/2}}\, {}_2F_1\left(\frac{1}{2},-\varepsilon;1-\varepsilon;q(p)\right)\ , $$

where $q = p^2t^2/(1+p^2t^2)$ and

$$ {}_2F_1\left(\frac{1}{2},-\varepsilon;1-\varepsilon;q\right) = \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{m!\Gamma(\frac{1}{2})}\frac{-\varepsilon}{m-\varepsilon}q^m = 1-\varepsilon\sum_{m=1}^{\infty}\frac{\Gamma(m+\frac{1}{2})}{m!\Gamma(\frac{1}{2})}\frac{q^m}{m-\varepsilon}\,. $$

Using the integral representation for the factor $1/(m-\varepsilon) = \int_0^1 dz\, z^{m-1-\varepsilon}$, we have,

$$ {}_2F_1\left(\frac{1}{2},-\varepsilon;1-\varepsilon;q\right) = 1-\varepsilon\int_0^1\frac{dz}{z^{1+\varepsilon}}\left[\frac{1}{\sqrt{1-zq}}-1\right] $$

$$ = 1-\varepsilon\int_0^q\frac{dz_1 q^{\varepsilon}}{z_1^{1+\varepsilon}}\left[\frac{1}{\sqrt{1-z_1}}-1\right] \equiv 1-\varepsilon J^{(1)}(q)\,. $$

Combining these results, we have

$$\sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{3}{2})}{\Gamma(m+2-\varepsilon)} \left(-(tp)^2\right)^m = -\frac{\Gamma(\frac{1}{2})}{\Gamma(1-\varepsilon)(tp)^2} f_1^{(1)}(pt) \,,$$

where

$$f_1^{(1)}(pt) = \frac{1}{\left(1+t^2p^2\right)^{1/2}} \left\{1 - \varepsilon J^{(1)}(q)\right\} - 1 \,.$$

The final result for $F_1^{(1)}(t)$ reads,

$$F_1^{(1)}(t) = -\frac{3(1-\varepsilon)}{2(1+\varepsilon)t^2} \hat{K}_1 \, I_1^{(1)}(t) \,, \quad I_1^{(1)}(t) = \int_0^1 dp \, p^{\varepsilon-1} (1-p)^{-1/2-\varepsilon} \, f_1^{(1)}(pt) \,,$$

where the normalization $\hat{K}_1$ was determined above. The elliptic structure is carried by the product $(1-p)^{-1/2-\varepsilon}(1+t^2p^2)^{-1/2}$.

4. Leading terms of the ε-expansion and one-fold integrals

Here we derive a one-fold integral representation for the first two orders of the ϵ-expansion of $J_{1,2,2}$, $J_{1,1,2}$ and $J_{1,1,1}$. The first two ϵ orders considered here can be expressed as one-fold integrals over logarithms and dilogarithms (in General Goncharov MPLs as it was shown in (Besuglov)) with algebraic prefactors. A similar analysis shows that to arbitrary order of the dimensional regulator the result is in terms of one-fold integrals over higher weight MPLs.

4.1. Inner integrals

We start by considering integral $J^{(2)}(p)$, at order ε^0,

$$J^{(2)}(pt, \varepsilon = 0) = \int_0^{tp} \frac{ds}{\sqrt{1+s^2}}\,,$$

which can be evaluated directly by means of the variable change,

$$s_2 = \frac{\sqrt{1+s^2} - s}{\sqrt{1+s^2} + s}\,,$$

leading to,

$$J^{(2)}(pt, \varepsilon = 0) = \frac{1}{2} \int_{R_2}^{1} \frac{ds_2}{s_2} = -\frac{1}{2} \log R_2 \equiv J_0^{(2)}(p)\,,$$

with

$$R_2 = \frac{\sqrt{1+t^2p^2} - tp}{\sqrt{1+t^2p^2} + tp} = \frac{1-\sqrt{q}}{1+\sqrt{q}}\,.$$

By means of the same variable change, we evaluate the next ε order,

$$J^{(2)}(pt) = \frac{1}{2^{1+\varepsilon}} \int_{R_2}^{1} \frac{ds_2}{s_2^{1+\varepsilon/2}} \frac{(1+s_2)^{2\varepsilon}}{(1-s_2)^{2\varepsilon}} = J_0^{(2)}(pt) + \varepsilon J_1^{(2)}(pt) + O(\varepsilon^2)\,,$$

where $J_0^{(2)}(pt)$ is given above and,

$$J_1^{(2)}(p) = \frac{1}{8}\log^2 R_2 + \zeta_2 + \mathrm{Li}_2(-R_2) - \frac{1}{2}\mathrm{Li}_2(R_2)\,.$$

We now consider integrals $J^{(1)}(p)$ and $J^{(3)}(p)$ at order ε^0,

$$J^{(1)}(p, \varepsilon = 0) = J^{(3)}(p, \varepsilon = 0) = \int_0^q \frac{dz}{z}\left(\frac{1}{\sqrt{1-z}} - 1\right)\,.$$

By introducing a regulator δ we have,

$$\int_\delta^y \frac{dz}{z} = \log q - \log\delta\,,$$

while the remaining term can be evaluated by the variable change,

$$z = 1 - s^2, \quad s = \frac{(1-s_1)}{(1+s_1)}\,,$$

and,

$$\int_{\delta}^{q} \frac{dz}{z\sqrt{1-z}} = \int_{\delta/4}^{R_1} \frac{ds_1}{s_1} = \log R_1 - \log \frac{\delta}{4}\,, \qquad R_1 = \frac{1-\sqrt{1-q}}{1+\sqrt{1-q}}\,.$$

The full result can be written as,

$$J^{(i)}(p, \varepsilon = 0) = \log(4R_1) - \log q = \log \frac{4R_1}{q} \equiv J_0^{(i)}(p), \qquad (i = 1, 3)\,.$$

At the next order we have,

$$J^{(i)}(p) = J_0^{(i)}(p) + \varepsilon J_1^{(i)}(p) + O(\varepsilon^2), \quad (i = 1, 3)\,,$$

where $J_0^{(1)}(p) = J_0^{(3)}(p)$ are given

$$\begin{aligned} J_1^{(1)}(p) &= \overline{J}_1^{(1)}(p) - 2\mathrm{Li}_2(-R_1), \\ J_1^{(3)}(p) &= \frac{1}{2}\overline{J}_1^{(1)}(p) + 2\mathrm{Li}_2(R_1) - 4\mathrm{Li}_2(-R_1)\,, \end{aligned}$$

with,

$$\overline{J}_1^{(1)}(p) = \log q \log(4R_1) - \frac{1}{2}\log^2 q - \log 4 \log R_1\,.$$

4.2. Results for the sunrise integrals

In order to obtain the ε-expansions of $J_{1,2,2}$, $J_{1,1,2}$ and $J_{1,1,1}$ up to and including $O(\varepsilon)$, we use the expressions for integrals $I^{(i)}(t)$ $(i=1,3)$ and their integral representations. We have, for example, for $J_{1,2,2}$

$$
\begin{aligned}
I^{(1)}_{1,1}(t) &= \int_0^1 \frac{dp}{p\sqrt{1-p}} \left(\frac{1}{\sqrt{1+p^2t^2}} - 1\right) \left[1+\varepsilon\, l_1 + \frac{\varepsilon^2}{2}\, l_1^2\right] + O(\varepsilon^2)\,, \\
I^{(1)}_{1,2}(t) &= \int_0^1 \frac{dp}{p\sqrt{1-p}} \frac{1}{\sqrt{1+p^2t^2}} \left[J_0^{(1)} + \varepsilon\,(l_1 J_0^{(1)} + J_1^{(1)})\right] + O(\varepsilon^2), \\
\left(\frac{t^2}{2}\right)^{\varepsilon} I^{(2)}_{1}(t) &= \int_0^1 \frac{dp}{p\sqrt{1-p}} \frac{1}{\sqrt{1+p^2t^2}} \left[J_0^{(2)} + \varepsilon\,(l_2 J_0^{(2)} + J_1^{(2)})\right] + O(\varepsilon^2)\,, \\
\left(\frac{t}{2}\right)^{\varepsilon} I^{(3)}_{1,1}(t) &= \int_0^1 \frac{dp}{p\sqrt{1-p}} \left(\frac{1}{\sqrt{1+p^2t^2}} \left[1+\varepsilon\, l_{32} + \frac{\varepsilon^2}{2}\, l_{32}^2\right] - \left[1+\varepsilon\, l_{31} + \frac{\varepsilon^2}{2}\, l_{31}^2\right]\right) + O(\varepsilon^2)\,, \\
\left(\frac{t}{2}\right)^{\varepsilon} I^{(3)}_{1,2}(t) &= \int_0^1 \frac{dp}{p\sqrt{1-p}} \frac{1}{\sqrt{1+p^2t^2}} \left[J_0^{(3)} + \varepsilon\,(l_{32} J_0^{(3)} + J_1^{(3)})\right] + O(\varepsilon^2)\,,
\end{aligned}
$$

where,

$$
\begin{aligned}
l_1 &= \log\left(\frac{p}{1-p}\right), \quad l_2 = \log\left(\frac{p^3t^2}{2(1-p)(1+p^2t^2)}\right) = \log\left(\frac{pq}{2(1-p)}\right), \\
l_{31} &= \log\left(\frac{p^2t}{2(1-p)}\right), \quad l_{32} = \log\left(\frac{p^2t}{2(1-p)\sqrt{1+p^2t^2}}\right) = \log\left(\frac{p\sqrt{q}}{2(1-p)}\right),
\end{aligned}
$$

with $J_0^{(i)}$ and $J_1^{(i)}$ $(i=1,2,3)$ given above.

Combining all terms, we obtain the following finite expression,

$$\hat{J}_{1,2,2} = \int_0^1 \frac{dp}{p\sqrt{1-p}} \left[b_0 + \frac{1}{\sqrt{1+p^2t^2}} B_0 + \varepsilon \left(b_1 + \frac{1}{\sqrt{1+p^2t^2}} B_1 \right) \right] + O(\varepsilon^2) \,,$$

$$\hat{J}^{\text{reg}}_{1,1,2} = \int_0^1 \frac{dp}{p^2\sqrt{1-p}} \left[b_0 - (pt) + \frac{1}{\sqrt{1+p^2t^2}} B_0 + \varepsilon \Big(b_1 - (pt)(1+l_{31}) + \frac{1}{\sqrt{1+p^2t^2}} B_1 \Big) \right] + O(\varepsilon^2) \,,$$

$$\hat{J}^{\text{reg}}_{1,1,1} = \int_0^1 \frac{dp\sqrt{1-p}}{p^3} \Bigg[b_0 - (pt) + \frac{(pt)^2}{4} (2b_0 - 1) + \frac{1}{\sqrt{1+p^2t^2}} B_0$$
$$+ \varepsilon \Big(b_1 - (pt)(1+l_{31}) + \frac{(pt)^2}{4} \left[b_0^2 - 3b_0 + \frac{1}{2} + l_1(2b_0 - 1) \right] + \frac{1}{\sqrt{1+p^2t^2}} B_1 \Big) \Bigg] + O(\varepsilon^2) \,,$$

where,

$$b_0 = \log\left(\frac{pt}{2}\right), \quad B_0 = \log(R_1 R_2), \quad b_1 = \log\left(\frac{pt}{2}\right) \log\left(\frac{p^3 t}{2(1-p)^2}\right),$$

$$B_1 = J_1^{(2)} + \text{Li}_2(R_1) + \frac{1}{4} \log^2(R_1) - \frac{1}{2} \log(R_1 R_2) \log\left(\frac{pq}{4(1-p)}\right)$$
$$+ \frac{1}{4} \log\left(\frac{q}{4}\right) \log(R_1) - \frac{1}{8} \log^2\left(\frac{q}{4}\right) - \log^2 2 \,,$$

where $J_1^{(2)}$ is given above as

$$J_1^{(2)}(p) = \frac{1}{8} \log^2 R_2 + \zeta_2 + \text{Li}_2(-R_2) - \frac{1}{2} \text{Li}_2(R_2) \,.$$

Up to $O(\varepsilon^0)$, the results are in full agreement with (Kniehl,2019). Here, however, the results are given up to $O(\varepsilon^1)$.

5. All orders result in terms of elliptic polylogarithms

In this section we derive eMPL representations for the sunrise integrals $J_{1,2,2}$, $J_{1,1,2}$ and $J_{1,1,1}$ valid to all orders of the dimensional regulator. Specifically, we start with a short review of eMPLs (following to (Brodel,2017)) , discussing their definition and the basic analytic properties.

We do not discuss the general structure of the integral representations (it was presented in (CMK,2020)) but we show a simple example by calculation of the integral $I_2^{(1)}$.

We express all considered integrals in terms of eMPLs to all orders of the dimensional regulator, and present our final results.

5.1. Elliptic polylogarithms

We are interested in the computation of iterated integrals of the form,

$$\int_0^x dx_1 R_1(x_1, y(x_1)) \int_0^{x_1} dx_2 R_2(x_2, y(x_2)) \ldots \int_0^{x_{n-1}} dx_n R_n(x_n, y(x_n)) \,,$$

where R_i are rational functions of their arguments and $y(x)$ is an elliptic curve,

$$y(x) = \sqrt{(x-a_1)(x-a_2)(x-a_3)(x-a_4)} \equiv y_4(x) \,,$$
$$y(x) = \sqrt{(x-a_1)(x-a_2)(x-a_3)} \equiv y_3(x) \,,$$

All iterated integrals can be expressed in terms of eMPLs. In the complex plane, eMPLs are defined as

$$E_l\left(\begin{matrix} n_1, \ldots, n_k \\ c_1, \ldots, c_k \end{matrix} ; x\right) = \int_0^x dt\, \varphi_{n_1}(c_1, t)\, E_l\left(\begin{matrix} n_2, \ldots, n_k \\ c_2, \ldots, c_k \end{matrix} ; t\right), \;\; E_l(\;; x) = 1, \;\; (l = 3, 4)$$

with $n_i \in Z$ and $c_i \in C$.

Elliptic polylogarithms are a generalisation of ordinary multiple polylogarithms (MPLs), defined recursively as,

$$G(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t),$$

with $G(;x) \equiv 1$ and,

$$G(\vec{0}, x) \equiv \frac{\log(x)^n}{n!} \,.$$

By definition we see that MPLs are a subset of eMPLs,

$$E_l \left(\begin{matrix} 1, ..., 1 \\ c_1, ..., c_n \end{matrix} ; x \right) = G(c_1, c_2, \ldots, c_n; x) \,,$$

where $c_i \neq \infty$.

The case $l = 4$.

The recursion starts at $E_4(\; ; x) = 1$. By construction, the kernels $\varphi_n(c, x)$ have at most simple poles, and they are (see (Broedel,2017) for a detailed discussion)

$$\varphi_0(0,x) = \frac{c_4}{y(x)}\,,$$

$$\varphi_1(c,x) = \frac{1}{x-c}\,, \qquad \varphi_{-1}(c,x) = \frac{y(c)}{(x-c)y(x)} - (\delta_{c0} + \delta_{c1})\frac{1}{x-c}\,,$$

$$\varphi_{-1}(\infty,x) = \frac{x}{y(x)}\,, \qquad \varphi_1(\infty,x) = \frac{c_4}{y(x)}\,Z_4(x)\,,$$

$$\varphi_n(\infty,x) = \frac{c_4}{y(x)}\,Z_4^{(n)}(x)\,, \qquad \varphi_{-n}(\infty,x) = \frac{x}{y(x)}\,Z_4^{(n-1)}(x) - \frac{\delta_{n2}}{c_4}\,,$$

$$\varphi_n(c,x) = \frac{1}{x-c}\,Z_4^{(n-1)}(x) - \delta_{n2}\,\Phi_4(x)\,,$$

$$\varphi_{-n}(c,x) = \frac{y(c)}{(x-c)y(x)}\,Z_4^{(n-1)}(x)\,, \quad (n>1),$$

where $y(c)$ and c_4 are independent of x with,

$$c_4 = \frac{1}{2}\sqrt{a_{13}a_{24}} \quad \text{with} \quad a_{ij} = a_i - a_j\,.$$

Some properties of ellitic integrals:

Periods:

$$\omega_1 = 2c_4 \int_{a_2}^{a_3} \frac{dx}{y} = 2K(\lambda), \quad \lambda = \frac{a_{14}a_{23}}{a_{13}a_{24}},$$
$$\omega_2 = 2c_4 \int_{a_1}^{a_2} \frac{dx}{y} = 2iK(1-\lambda),$$

where $K(\lambda)$ is complete integral of the first kind.

For other roots, the result is a linear combination of ω_1 and ω_1:

$$2c_4 \int_{a_i}^{a_j} \frac{dx}{y} = m_{ij}\,\omega_1 + n_{ij}\,\omega_2\,.$$

Quasi-periods:

$$\eta_1 = -\frac{1}{2}\int_{a_2}^{a_3} dx\,\widetilde{\Phi}_4(x) = E(\lambda) - \frac{2-\lambda}{3}K(\lambda),$$
$$\eta_2 = -\frac{1}{2}\int_{a_1}^{a_2} dx\,\widetilde{\Phi}_4(x) = -iE(1-\lambda) + i\frac{1+\lambda}{3}K(1-\lambda),$$

where $E(\lambda)$ is complete integral of the second kind and the function $\widetilde{\Phi}_4(x)$ has the following form:

$$\widetilde{\Phi}_4(x) = \frac{1}{c_4 y}\left(x^2 - \frac{s_1 x}{2} + \frac{s_2}{6}\right)$$

and $s_n(a_1, a_2, a_3, a_4)$ is elementary symmetric polynomial of defree n.

For other roots, the result is a linear combination of ω_1 and ω_1:

$$-\frac{1}{2}\int_{a_i}^{a_j} dx\,\widetilde{\Phi}_4(x) = m_{ij}\,\eta_1 + n_{ij}\,\eta_2\,.$$

Moreover we define,

$$E_l\left(\begin{matrix}\vec{1}\\ \vec{0}\end{matrix};x\right) = G(\vec{0};x) \equiv \frac{\log(x)^n}{n!}, \quad (l=3,4)$$

where $\vec{1}$ and $\vec{0}$ are vectors with entries equal to 1 and 0 respectively, and $n = \text{length}(\vec{1}) = \text{length}(\vec{0})$.

The function $Z_4(x)$ is defined by first introducing an auxiliary function $\Phi_4(x)$,

$$\Phi_4(x) \equiv \widetilde{\Phi}_4(x) + 4c_4\frac{\eta_1}{\omega_1}\frac{1}{y} = \frac{1}{c_4\,y}\left(x^2 - \frac{s_1}{2}x + \frac{s_2}{6}\right) + 4c_4\frac{\eta_1}{\omega_1}\frac{1}{y},$$

whose primitive is,

$$Z_4(x) = \int_{a_1}^{x} dt\,\Phi_4(t)\,.$$

In the next sections we will see that, in our integral representations, the function $\Phi_4(x)$ appears only in the last (outer) integration, and only the case $Z_4^{(1)}(x) = Z_4(x)$ need to be considered.

The case $l = 3$.

The recursion starts at $E_3(\ ;x) = 1$. By construction, the kernels $\varphi_n(c,x)$ have at most simple poles, and they are (see (Broedel,2017) for a detailed discussion)

$$\begin{aligned}
\varphi_0(0,x) &= \frac{c_3}{y_3(x)}\,,\\
\varphi_1(c,x) &= \frac{1}{x-c}\,, \qquad \varphi_{-1}(c,x) = \frac{y_3(c)}{(x-c)y_3(x)}\,,\\
\varphi_1(\infty,x) &= \frac{c_3}{y_3(x)}\,Z_3(x)\,, \qquad \varphi_n(\infty,x) = \frac{c_3}{y_3(x)}\,Z_3^{(n)}(x)\,,\\
\varphi_n(c,x) &= \frac{1}{x-c}\,Z_3^{(n-1)}(x) - \delta_{n2}\,\Phi_3(x)\,,\\
\varphi_{-n}(c,x) &= \frac{y_3(c)}{(x-c)y_3(x)}\,Z_3^{(n-1)}(x)\,, \quad (n>1),
\end{aligned}$$

where $y_3(c)$ and c_3 are independent of x with,

$$c_3 = \frac{1}{2}\sqrt{a_{31}} \quad \text{with} \quad a_{ij} = a_i - a_j\,.$$

Some properties of ellitic integrals:

Periods:

$$\omega_1 = 2c_3 \int_{a_1}^{a_2} \frac{dx}{y_3} = 2K(\lambda), \quad \lambda = \frac{a_{21}}{a_{31}},$$
$$\omega_2 = 2c_3 \int_{a_3}^{a_2} \frac{dx}{y} = 2iK(1-\lambda),$$

where $K(\lambda)$ is complete integral of the first kind.

For other roots, the result is a linear combination of ω_1 and ω_1:

$$2c_3 \int_{a_i}^{a_j} \frac{dx}{y} = m_{ij}\,\omega_1 + n_{ij}\,\omega_2\,.$$

Quasi-periods:

$$\eta_1 = -\frac{1}{4} \int_{a_1}^{a_2} dx\, \widetilde{\Phi}_3(x) = E(\lambda) - \frac{2-\lambda}{3} K(\lambda),$$

$$\eta_2 = -\frac{1}{4} \int_{a_1}^{a_2} dx\, \widetilde{\Phi}_3(x) = -iE(1-\lambda) + i\frac{1+\lambda}{3} K(1-\lambda),$$

where $E(\lambda)$ is complete integral of the second kind and the function $\widetilde{\Phi}_4(x)$ has the following form:

$$\widetilde{\Phi}_3(x) = \frac{1}{c_3 y_3}\left(x - \frac{s_1}{3}\right)$$

and $s_n(a_1, a_2, a_3)$ is elementary symmetric polynomial of defree n.

For other roots, the result is a linear combination of ω_1 and ω_1:

$$-\frac{1}{2} \int_{a_i}^{a_j} dx\, \widetilde{\Phi}_4(x) = m_{ij}\, \eta_1 + n_{ij}\, \eta_2 \,.$$

The function $Z_3(x)$ is defined by first introducing an auxiliary function $\Phi_3(x)$,

$$\Phi_3(x) \equiv \widetilde{\Phi}_3(x) + 8c_3\frac{\eta_1}{\omega_1}\frac{1}{y} = \frac{1}{c_4\, y}\left(x - \frac{s_1}{3}\right) + 8c_3\frac{\eta_1}{\omega_1}\frac{1}{y}\,,$$

whose primitive is,

$$Z_3(x) = \int_{a_3}^{x} dt\, \Phi_3(t)\,.$$

As for all iterated integrals, eMPLs satisfy a shuffle algebra, with the shuffle product defined as, $(l = 3, 4)$

$$E_l\left(\begin{matrix} a_1, ..., a_n \\ a'_1, ..., a'_n \end{matrix}; x\right) E_l\left(\begin{matrix} b_1, ..., b_m \\ b'_1, ..., b'_m \end{matrix}; x\right) = \sum_{\vec{c}=\vec{a}\sqcup\vec{b}} E_l\left(\begin{matrix} c_1, ..., c_{n+m} \\ c'_1, ..., c'_{n+m} \end{matrix}; x\right).$$

The vector $\vec{c}$ is the vector obtained by performing all the shuffles of $\vec{a}$ and $\vec{b}$, preserving the ordering of the elements of $\vec{a}$ and $\vec{b}$.

Examples:

$$E_l\left(\begin{matrix} a_1, \\ a'_1 \end{matrix}; x\right) E_l\left(\begin{matrix} b_1, ..., b_m \\ b'_1, ..., b'_m \end{matrix}; x\right) = E_l\left(\begin{matrix} a_1, b_1, b_2, ..., b_m \\ a'_1, b'_1, b'_2, ..., b'_m \end{matrix}; x\right) + E_l\left(\begin{matrix} b_1, a_1, b_2, ..., b_m \\ b'_1, a'_1, b'_2, ..., b'_m \end{matrix}; x\right)$$
$$+E_l\left(\begin{matrix} b_1, b_2, a_1, ..., b_m \\ b'_1, b'_2, a'_1, ..., b'_m \end{matrix}; x\right) + ... + E_l\left(\begin{matrix} b_1, b_2, ..., a_1, b_m \\ b'_1, b'_2, ..., a'_1, b'_m \end{matrix}; x\right) + E_l\left(\begin{matrix} b_1, b_2, ..., b_m, a_1 \\ b'_1, b'_2, ..., b'_m, a'_1 \end{matrix}; x\right),$$

and

$$E_l\left(\begin{matrix} a_1, a_2 \\ a'_1, a'_2 \end{matrix}; x\right) E_l\left(\begin{matrix} b_1, b_2 \\ b'_1, b'_2 \end{matrix}; x\right) = E_l\left(\begin{matrix} a_1, a_2, b_1, b_2 \\ a'_1, a'_2, b'_1, b'_2 \end{matrix}; x\right) + E_l\left(\begin{matrix} a_1, b_1, a_2, b_2 \\ a'_1, b'_1, a'_2, , b'_m \end{matrix}; x\right)$$
$$+E_l\left(\begin{matrix} a_1, b_1, b_2, a_2 \\ a'_1, b'_1, b'_2, a'_2 \end{matrix}; x\right) + E_l\left(\begin{matrix} b_1, a_1, a_2, b_2 \\ b'_1, a'_1, a'_2, b'_2 \end{matrix}; x\right) + E_l\left(\begin{matrix} b_1, a_1, b_2, a_2 \\ b'_1, a'_1, b'_2, a'_2 \end{matrix}; x\right) + E_l\left(\begin{matrix} b_1, b_2, a_1, a_2 \\ b'_1, b'_2, a'_1, a'_2 \end{matrix}; x\right).$$

5.2. Regularisation

As we will see in the next sections we are interested in computing definite integrals of the form,

$$\int_0^1 f(x)dx = F(1) - F(0), \quad \frac{\partial F(x)}{\partial x} = f(x)\,.$$

In some cases individual functions inside the primitive develop divergences when evaluated at the integration bounds, and in order to compute the definite integral one needs to perform two limits,

$$\int_0^1 f(x)dx = \lim_{x\to 1} F(x) - \lim_{x\to 0} F(x) \equiv \mathrm{Reg}_{0,1} F(x).$$

5.3. Elliptic polylogarithms and all orders result

Contrary to general cases considered in [CMK] we consider here an example which clear up the obtained results.

5.3.1. Example

We show how the solution strategy of the previous section works in practice by considering integral $I_1^{(2)}(t)$. The dependence on the elliptic curve is made explicit by applying the variable change,

$$I_1^{(2)}(t) = \int_0^1 dx \, \frac{2}{t\left(1-x^2\right) y(x)} \left(\frac{\left(1-x^2\right)^3}{t^2 x^2 y(x)^2}\right)^{\varepsilon} J^{(2)}(x)$$

where

$$p(x) = 1 - x^2, \quad 0 < x < 1; \quad y(x) = \sqrt{\frac{1}{t^2} + (1-x^2)^2} \equiv \frac{1}{t}\sqrt{1 + t^2 p^2} \, .$$

The inner integral can be expressed as

$$J^{(2)}(x) = - \int_1^x dz_1 \frac{2z_1}{y(z_1)} \left(\frac{ty^2(z_1)}{1-z_1^2} \right)^{\varepsilon} .$$

All the ε-powers can be expanded in ε

$$\left(\frac{ty^2(x)}{1-x^2} \right)^{\varepsilon} = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \log^i \left(\frac{ty^2(x)}{1-x^2} \right) ,$$

The resulting logarithm can be expressed in terms of eMPLs,

$$\log \left(\frac{ty^2(x)}{1-x^2} \right) = \log(ty^2(0))) + \int_0^x dz \, \frac{d}{dz} \, \log \left(\frac{ty^2(z)}{1-z^2} \right)$$

$$= \log\left(t^2+1\right) - \log(t) + \int_0^x dz \frac{2z\left(t^2\left(z^2-1\right)^2 - 1\right)}{t^2\left(z^2-1\right)y(z)^2} .$$

The integrand above can be written in terms of the integration kernels as,

$$\frac{2z\left(t^2\left(z^2-1\right)^2-1\right)}{t^2\left(z^2-1\right)y(z)^2}=\sum_{i=1}^{4}\varphi_1\left(a_i,z\right)-\varphi_1(-1,z)-\varphi_1(1,z)\,,$$

where we denoted with a_i the four roots of the elliptic curve,

$$a_1=-\frac{\sqrt{t-i}}{\sqrt{t}}\,,\ a_2=\frac{\sqrt{t-i}}{\sqrt{t}}\,,\ a_3=-\frac{\sqrt{t+i}}{\sqrt{t}}\,,\ a_4=\frac{\sqrt{t+i}}{\sqrt{t}}\,.$$

Upon integration we find,

$$L_4\equiv\log\left(\frac{ty^2(x)}{1-x^2}\right)=\sum_{i=1}^{4}E_4\left(\begin{matrix}1\\ a_i\end{matrix};x\right)-E_4\left(\begin{matrix}1\\ -1\end{matrix};x\right)-E_4\left(\begin{matrix}1\\ 1\end{matrix};x\right)$$
$$+\log\left(t^2+1\right)-\log(t)\,.$$

The prefactor is

$$-\frac{2z_1}{y(z_1)} = -2\varphi_{-1}(\infty, z_1) \equiv k_5(z_1)$$

and, thus, the inner integral has the following form

$$-\int_1^x dz_1 \frac{2z_1}{y(z_1)} \left(\frac{ty^2(z_1)}{1-z_1^2}\right)^{\varepsilon}$$
$$= \int_1^x dz_1\, k_5(z_1) \sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!} L_4^j(z_1) \equiv \sum_{j=0}^{\infty} \frac{\varepsilon^j}{j!} \left[K_5 * L_4^j\right]_1^x ,$$

where the primitive K_5 is

$$K_5 = -2E_4\left(\begin{matrix}-1\\ \infty\end{matrix} ; x\right) .$$

The evalution of the rest is very siimilar and we have

$$I_1^{(2)}(t) = \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_4 * L_5^i \left[K_5 * L_4^j\right]_1^x ,$$

where,

$$\begin{aligned} L_5 = & -\sum_{i=1}^{4} E_4\begin{pmatrix} 1 \\ a_i \end{pmatrix}; x\Bigg) + 3E_4\begin{pmatrix} 1 \\ -1 \end{pmatrix}; x\Bigg) - 2E_4\begin{pmatrix} 1 \\ 0 \end{pmatrix}; x\Bigg) + E_4\begin{pmatrix} 1 \\ 1 \end{pmatrix}; x\Bigg) \\ & -\log\left(t^2+1\right), \\ K_4 = & \sum_{i=1}^{4} E_4\begin{pmatrix} -1 \\ -1 \end{pmatrix}; x\Bigg) - E_4\begin{pmatrix} -1 \\ 1 \end{pmatrix}; x\Bigg) - E_4\begin{pmatrix} 1 \\ 1 \end{pmatrix}; x\Bigg) . \end{aligned}$$

5.3.1. Same example

We consider the integral $I_1^{(2)}(t)$ when $y = y_3$. The dependence on the elliptic curve is made explicit by applying the variable change,

$$I_1^{(2)}(t) = \int_0^1 dx_3 \, \frac{2}{t\,(1-x_3)\,y(x_3)} \left(\frac{(1-x_3)^3}{t^2 y(x_3)^2} \right)^{\varepsilon} J^{(2)}(x_3)$$

where

$$p = 1 - x_3, \quad 0 < x_3 < 1; \quad y(x_3) = x_3 \sqrt{\frac{1}{t^2} + (1-x_3)^2} \equiv \frac{1-p}{t} \sqrt{1 + t^2 p^2}\,.$$

The inner integral can be expressed as

$$J^{(2)}(x_3) = -\int_1^{x_3} dz_3 \frac{t^{\varepsilon} (1-z_3)^{-\varepsilon} z_3^{1/2-\varepsilon}}{y^{1-3\varepsilon}(z_3)} = -\int_1^{x_3} dz_3 \frac{\sqrt{z_3}}{y(z_3)} + O(\varepsilon).$$

!!! The problem: the function

$$\frac{\sqrt{x}}{y_3}$$

is not an element of the Broedel et al basis!!!

We would like to remember that the Broedel et al basis has both parts: a nonelliptic one and an elliptic one

$$\varphi_0(0,x) = \frac{c_3}{y_3(x)}\,,$$

$$\varphi_1(c,x) = \frac{1}{x-c}\,, \qquad \varphi_{-1}(c,x) = \frac{y_3(c)}{(x-c)y_3(x)}\,, \ldots$$

where the nonelliptic one leads to

$$E_3\left(\begin{matrix}1\\ c\end{matrix};x\right) \equiv G(c;x)$$

But the term

$$\frac{\sqrt{x}}{y_3}$$

leads to $G(c; f(x))$ (with some known $f(x)$), which cannot be expressed as combination of

$$E_3\left(\begin{matrix} 1 \\ c_i \end{matrix}; x\right) \equiv G(c_i; x) \quad (c_i \text{ are some numbers})$$

and, thus, cannot be used as a term in shuffle product.

Indeed, in ε-expansion we had

$$J^{(2)}(pt) = \frac{1}{2^{1+\varepsilon}} \int_{R_2}^{1} \frac{ds_2}{s_2^{1+\varepsilon/2}} \frac{(1+s_2)^{2\varepsilon}}{(1-s_2)^{2\varepsilon}} = J_0^{(2)}(pt) + \varepsilon J_1^{(2)}(pt) + O(\varepsilon^2)\,,$$

where

$$J_0^{(2)}(p) = -\frac{1}{2}\log R_2\,, \quad R_2 = \frac{\sqrt{1+t^2p^2} - tp}{\sqrt{1+t^2p^2} + tp}$$

$$J_1^{(2)}(p) = \frac{1}{8}\log^2 R_2 + \zeta_2 + \mathrm{Li}_2(-R_2) - \frac{1}{2}\mathrm{Li}_2(R_2)\,.$$

!!! So, it is possible that in the case of Elliptic Polylogarithms these two representations (based on y_4 and y_3), are not completely interchangeable, as they were in the case of ellitic integrals. !!!

!!! But it needs additional investigations !!!

5.4. Sunsets

By applying the procedure described above we obtain one of the main results of this paper, i.e. an explicit expression for the above integrals in terms of eMPLs valid to all orders of the dimensional regulator.

5.4.1. $J_{1,2,2}$

We obtain the following expression for $J_{1,2,2}$,

$$M^2\, J_{1,2,2} = \hat{N}_1\, \frac{\hat{K}_1}{4t^2}\, \hat{J}_{1,2,2},$$

$$\hat{J}_{1,2,2} = [\frac{1}{\varepsilon} I_1^{(1)}(t) + \left(\frac{t^2}{2}\right)^{\varepsilon} I_1^{(2)}(t) - \frac{1}{\varepsilon}\left(\frac{t}{2}\right)^{\varepsilon} I_1^{(3)}(t)]$$

$$= [\frac{1}{\varepsilon} I_{1,1}^{(1)}(t) - I_{1,2}^{(1)}(t) + \left(\frac{t^2}{2}\right)^{\varepsilon} I_1^{(2)}(t) - \left(\frac{t}{2}\right)^{\varepsilon}\left(\frac{1}{\varepsilon}\, I_{1,1}^{(3)}(t) - \frac{1}{2}\, I_{1,2}^{(3)}(t)\right)]\,.$$

where,

$$
\begin{aligned}
I^{(1)}_{1,1}(t) &= \mathrm{Reg}_{0,1} \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} K_1 * L_1^i, \quad , \\
I^{(1)}_{1,2}(t) &= \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_2 * L_3^i \left[K_3 * L_2^j\right]_1^x , \\
I^{(2)}_{1}(t) &= \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_4 * L_5^i \left[K_5 * L_4^j\right]_1^x , \\
I^{(3)}_{1,1}(t) &= \mathrm{Reg}_{0,1} \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} K_6 * L_6^i + \mathrm{Reg}_{0,1} \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} K_4 * L_7^i , \\
I^{(3)}_{1,2}(t) &= \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_7 * L_8^i \left[K_9 * L_4^j\right]_1^x \\
&\quad + \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_7 * L_8^i \left[K_8 * L_9^j\right]_1^x ,
\end{aligned}
$$

where K_i and L_i are depth one eMPLs.

Since the integrals $I_1^{(2)}(t)$ and $I_1^{(3)}(t)$ contribute to $\hat{J}_{1,2,2}$ with the corresponding factors, it is convenient to present also

$$\left(\frac{t^2}{2}\right)^{\varepsilon} I_1^{(2)}(t) = \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_4 * \hat{L}_5^i \left[K_5 * L_4^j\right]_1^x ,$$

$$\left(\frac{t}{2}\right)^{\varepsilon} I_{1,1}^{(3)}(t) = \mathrm{Reg}_{0,1} \sum_{i=0}^{\infty} \frac{\epsilon^{i}}{i!} K_6 * \hat{L}_6^i + \mathrm{Reg}_{0,1} \sum_{i=0}^{\infty} \frac{\epsilon^{i}}{i!} K_4 * \hat{L}_7^i ,$$

$$\left(\frac{t}{2}\right)^{\varepsilon} I_{1,2}^{(3)}(t) = \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_7 * \hat{L}_8^i \left[K_9 * L_4^j\right]_1^x + \mathrm{Reg}_{0,1} \sum_{i,j=0}^{\infty} \frac{\epsilon^{i+j}}{i!j!} K_7 * \hat{L}_8^i \left[K_8 * L_9^j\right]_1^x ,$$

where,

$$\hat{L}_5 = L_5 + 2\log t - \log 2, \quad \hat{L}_k = L_k + \log t - \log 2 \quad (k = 6, 7, 8) .$$

5.4.2. Results for L_i and K_i

Here we provide the definitions for the eMPLs expressions

$$L_1 = E_4\begin{pmatrix} 1 & \\ & ;x \\ -1 & \end{pmatrix} - 2E_4\begin{pmatrix} 1 & \\ & ;x \\ 0 & \end{pmatrix} + E_4\begin{pmatrix} 1 & \\ & ;x \\ 1 & \end{pmatrix},$$

$$L_2 = \sum_{i=1}^{4} E_4\begin{pmatrix} 1 & \\ & ;x \\ a_i & \end{pmatrix} - 2E_4\begin{pmatrix} 1 & \\ & ;x \\ -1 & \end{pmatrix} - 2E_4\begin{pmatrix} 1 & \\ & ;x \\ 1 & \end{pmatrix} + \log\left(t^2+1\right) - 2\log(t),$$

$$L_3 = -\sum_{i=1}^{4} E_4\begin{pmatrix} 1 & \\ & ;x \\ a_i & \end{pmatrix} + 3E_4\begin{pmatrix} 1 & \\ & ;x \\ -1 & \end{pmatrix} - 2E_4\begin{pmatrix} 1 & \\ & ;x \\ 0 & \end{pmatrix} + 3E_4\begin{pmatrix} 1 & \\ & ;x \\ 1 & \end{pmatrix}$$
$$- \log\left(t^2+1\right) + 2\log(t),$$

$$L_4 = \sum_{i=1}^{4} E_4\begin{pmatrix} 1 & \\ & ;x \\ a_i & \end{pmatrix} - E_4\begin{pmatrix} 1 & \\ & ;x \\ -1 & \end{pmatrix} - E_4\begin{pmatrix} 1 & \\ & ;x \\ 1 & \end{pmatrix} + \log\left(t^2+1\right) - \log(t),$$

$$L_5 = -\sum_{i=1}^{4} E_4\begin{pmatrix} 1 & \\ & ;x \\ a_i & \end{pmatrix} + 3E_4\begin{pmatrix} 1 & \\ & ;x \\ -1 & \end{pmatrix} - 2E_4\begin{pmatrix} 1 & \\ & ;x \\ 0 & \end{pmatrix} + 3E_4\begin{pmatrix} 1 & \\ & ;x \\ 1 & \end{pmatrix}$$
$$- \log\left(t^2+1\right),$$

$$L_6 = 2E_4\begin{pmatrix} 1 & \\ & ;x \\ -1 & \end{pmatrix} - 2E_4\begin{pmatrix} 1 & \\ & ;x \\ 0 & \end{pmatrix} + 2E_4\begin{pmatrix} 1 & \\ & ;x \\ 1 & \end{pmatrix},$$

$$L_7 = -\frac{1}{2}\sum_{i=1}^{4} E_4\begin{pmatrix} 1 & \\ & ;x \\ a_i & \end{pmatrix} + 2E_4\begin{pmatrix} 1 & \\ & ;x \\ -1 & \end{pmatrix} - 2E_4\begin{pmatrix} 1 & \\ & ;x \\ 0 & \end{pmatrix} + 2E_4\begin{pmatrix} 1 & \\ & ;x \\ 1 & \end{pmatrix}$$

$$-\frac{1}{2}\log\left(t^2+1\right),$$

$$L_8 = -\sum_{i=1}^{4} E_4\left(\begin{matrix}1\\ a_i\end{matrix};x\right) + 3E_4\left(\begin{matrix}1\\ -1\end{matrix};x\right) - 2E_4\left(\begin{matrix}1\\ 0\end{matrix};x\right) + 3E_4\left(\begin{matrix}1\\ 1\end{matrix};x\right)$$

$$-\log\left(t^2+1\right)+\log(t),$$

$$L_9 = \frac{1}{2}\sum_{i=1}^{4} E_4\left(\begin{matrix}1\\ a_i\end{matrix};x\right) - E_4\left(\begin{matrix}1\\ -1\end{matrix};x\right) - E_4\left(\begin{matrix}1\\ 1\end{matrix};x\right) + \frac{1}{2}\log\left(t^2+1\right) - \log(t).$$

while the primitives of the relevant integration kernels are defined as,

$$K_1 = E_4\left(\begin{matrix}-1\\ -1\end{matrix};x\right) - E_4\left(\begin{matrix}-1\\ 1\end{matrix};x\right) - E_4\left(\begin{matrix}1\\ -1\end{matrix};x\right),$$

$$K_2 = E_4\left(\begin{matrix}-1\\ -1\end{matrix};x\right) - E_4\left(\begin{matrix}-1\\ 1\end{matrix};x\right) - E_4\left(\begin{matrix}1\\ 1\end{matrix};x\right),$$

$$K_3 = \sum_{i=1}^{4} E_4\left(\begin{matrix}1\\ a_i\end{matrix};x\right) + 2E_4\left(\begin{matrix}-1\\ -1\end{matrix};x\right) + 2E_4\left(\begin{matrix}-1\\ 1\end{matrix};x\right) - 2E_4\left(\begin{matrix}1\\ -1\end{matrix};x\right),$$

$$K_4 = E_4\left(\begin{matrix}-1\\ -1\end{matrix};x\right) - E_4\left(\begin{matrix}-1\\ 1\end{matrix};x\right) - E_4\left(\begin{matrix}1\\ 1\end{matrix};x\right),$$

$$K_5 = -2E_4\left(\begin{matrix}-1\\ \infty\end{matrix};x\right),$$

$$K_6 = E_4\left(\begin{matrix}1\\ \\ 1\end{matrix};x\right) - E_4\left(\begin{matrix}1\\ \\ -1\end{matrix};x\right),$$

$$K_7 = E_4\left(\begin{matrix}-1\\ \\ -1\end{matrix};x\right) - E_4\left(\begin{matrix}-1\\ \\ 1\end{matrix};x\right) - E_4\left(\begin{matrix}1\\ \\ 1\end{matrix};x\right),$$

$$K_8 = -\sum_{i=1}^{4} E_4\left(\begin{matrix}1\\ \\ a_i\end{matrix};x\right) - 2E_4\left(\begin{matrix}1\\ \\ -1\end{matrix};x\right) - 2E_4\left(\begin{matrix}1\\ \\ 1\end{matrix};x\right),$$

$$K_9 = 2E_4\left(\begin{matrix}-1\\ \\ -1\end{matrix};x\right) + 2E_4\left(\begin{matrix}-1\\ \\ 1\end{matrix};x\right) + 2E_4\left(\begin{matrix}1\\ \\ 1\end{matrix};x\right).$$

We see that the integral $J_{1,2,2}$ in the form of the elliptic integrals is finite since $K_1 = K_4 + K_6$ and,

$$I_{1,1}^{(1)}(t) - \left(\frac{t}{2}\right)^{\varepsilon} I_{1,1}^{(3)}(t) = \mathrm{Reg}_{0,1} \sum_{i=1}^{\infty} \frac{\epsilon^i}{i!} K_1 * (L_1^i - \hat{L}_6^i)$$
$$+\mathrm{Reg}_{0,1} \sum_{i=1}^{\infty} \frac{\epsilon^i}{i!} K_2 * (L_1^i - \hat{L}_7^i) \sim O(\varepsilon).$$

6. Conclusions

In this paper we studied a family of sunrise integrals with two different internal masses and pseudo-threshold kinematics in dimensional regularisation. These integrals admit a closed-form solution in terms of hypergeometric functions (Kalmykov,2008)
and we use this representation as the starting point of our analysis.

- In particular, we show that all corresponding hypergeometric functions can be represented in terms of one- and two-fold integral representations.
- In each ε-order, these representations can be represented as multiple integrals containing the elliptic kernel and Goncharov's MPLs in their integrands (see (Besuglov)). In the first two ε-orders under consideration, there are only logarithms and dilogarithms.
- Moreover, integral representations make it possible to represent them as a combination of eMPLs, in the case of y_4, which were obtained using the procedure (Brodel,2017).

In the case of y_3, we observed an appearence of a term, which is not an element of the Broedel et al basis..

So, in the case, the Broedel et al basis is not full. Some (nonelliptic) term should be added.

!!! So, it is possible that in the case of Elliptic Polylogarithms these two representations (based on y_4 and y_3), are not completely interchangeable, as they were in the case of elliptic integrals. !!!

!!! But it needs additional investigations !!!