FeynGrav
Feynman rules for gravity implmented in FeynCalc

Boris Latosh

Joint Institute for Nuclear Research
State University Dubna

Based on arXiv:2201.06812
Classical and Quantum Gravity, 2022

Motivation

Feynman rules for gravity are hard to obtain!

- DeWitt, Phys.Rev. 160 (1967) 1113;
  Phys.Rev. 162 (1967) 1195;
- 't Hooft, Veltman,
  Ann.Inst.H.Poincare Phys.Theor.A 20 (1974) 69;
- Goroff, Sagnotti, Phys.Lett.B 160 (1985) 81;
- Sannan, Phys.Rev.D 34 (1986) 1749;
- Prinz, Class.Quant.Grav. 38 (2021) 21, 215003;

3-**graviton term contains 171 terms**
4-**graviton term contains 2850 terms**

Motivation

$$
\begin{aligned}
V_{\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3)=\mathrm{sym}[&-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma})-\tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma})+\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma})\\
&+P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma})+2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma})-P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma})\\
&+P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta})+P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta})+2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma})\\
&+2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha})-2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]\ ,
\end{aligned}
$$

$$
\begin{aligned}
V_{\mu\alpha,\nu\beta,\sigma\gamma,\rho\lambda}(k_1,k_2,k_3,k_4)=\mathrm{sym}[&-\tfrac{1}{4}P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}\eta_{\rho\lambda})-\tfrac{1}{4}P_{12}(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}\eta_{\rho\lambda})-\tfrac{1}{2}P_6(k_{1\nu}k_{2\mu}\eta_{\alpha\beta}\eta_{\sigma\gamma}\eta_{\rho\lambda})\\
&+\tfrac{1}{4}P_6(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}\eta_{\rho\lambda})+\tfrac{1}{2}P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\rho}\eta_{\gamma\lambda})+\tfrac{1}{2}P_{12}(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\rho}\eta_{\gamma\lambda})\\
&+P_6(k_{1\nu}k_{2\mu}\eta_{\alpha\beta}\eta_{\sigma\rho}\eta_{\gamma\lambda})-\tfrac{1}{2}P_6(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\rho}\eta_{\gamma\lambda})+\tfrac{1}{2}P_{24}(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}\eta_{\rho\lambda})\\
&+\tfrac{1}{2}P_{24}(k_{1\nu}k_{1\beta}\eta_{\mu\sigma}\eta_{\alpha\gamma}\eta_{\rho\lambda})+\tfrac{1}{2}P_{12}(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\rho\lambda})+P_{24}(k_{1\nu}k_{2\sigma}\eta_{\beta\mu}\eta_{\alpha\gamma}\eta_{\rho\lambda})\\
&-P_{12}(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu}\eta_{\rho\lambda})+P_{12}(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}\eta_{\rho\lambda})+P_{12}(k_{1\nu}k_{1\sigma}\eta_{\beta\gamma}\eta_{\mu\alpha}\eta_{\rho\lambda})\\
&-P_{24}(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\beta\sigma}\eta_{\gamma\rho}\eta_{\lambda\nu})-2P_{12}(k_{1\nu}k_{1\beta}\eta_{\alpha\sigma}\eta_{\gamma\rho}\eta_{\lambda\mu})-2P_{12}(k_{1\sigma}k_{2\gamma}\eta_{\alpha\rho}\eta_{\lambda\nu}\eta_{\beta\mu})\\
&-2P_{24}(k_{1\nu}k_{2\sigma}\eta_{\beta\rho}\eta_{\lambda\mu}\eta_{\alpha\gamma})-2P_{12}(k_{1\sigma}k_{2\rho}\eta_{\gamma\nu}\eta_{\beta\mu}\eta_{\alpha\lambda})+2P_6(k_1\cdot k_2\eta_{\alpha\sigma}\eta_{\gamma\nu}\eta_{\beta\rho}\eta_{\lambda\mu})\\
&-2P_{12}(k_{1\nu}k_{1\sigma}\eta_{\mu\alpha}\eta_{\beta\rho}\eta_{\lambda\gamma})-P_{12}(k_1\cdot k_2\eta_{\mu\sigma}\eta_{\alpha\gamma}\eta_{\nu\rho}\eta_{\beta\lambda})-2P_{12}(k_{1\nu}k_{1\sigma}\eta_{\beta\gamma}\eta_{\mu\rho}\eta_{\alpha\lambda})\\
&-P_{12}(k_{1\sigma}k_{2\rho}\eta_{\gamma\lambda}\eta_{\mu\nu}\eta_{\alpha\beta})-2P_{24}(k_{1\nu}k_{2\sigma}\eta_{\beta\mu}\eta_{\alpha\rho}\eta_{\lambda\gamma})-2P_{12}(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\rho}\eta_{\lambda\alpha})\\
&+4P_6(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\rho}\eta_{\lambda\mu})]\ .
\end{aligned}
$$

Sannan, Phys.Rev.D 34 (1986) 1749

Motivation

$$\mathfrak{G}_n^{\mu_1\nu_1|\dots|\mu_n\nu_n}\left(p_1^\sigma,\dots,p_n^\sigma\right)=\frac{\mathrm{i}}{2^n}\sum_{\mu_i\leftrightarrow\nu_i}\sum_{s\in S_n}\mathfrak{g}_n^{\mu_{s(1)}\nu_{s(1)}|\dots|\mu_{s(n)}\nu_{s(n)}}\left(p_{s(1)}^\sigma,\dots,p_{s(n)}^\sigma\right)$$

Prinz, Class.Quant.Grav. 38 (2021) 21, 215003

$$
\mathfrak{g}_n^{\mu_1\nu_1|\cdots|\mu_n\nu_n}\left(p_1^{\sigma},\cdots,p_n^{\sigma}\right) =
$$

$$
\frac{(-\kappa)^{n-2}}{2}\sum_{m_1+m_2=n}\Bigg\{\sum_{i=0}^{m_1-1}\left(\hat{\delta}^{\mu}_{\mu_0}\hat{\delta}^{\rho}_{\nu_{i+1}}\prod_{a=0}^{i}\hat{\eta}^{\mu_a\nu_{a+1}}\right)\left(\hat{\delta}^{\nu}_{\mu_i}\hat{\delta}^{\sigma}_{\nu_{m_1}}\prod_{b=i}^{m_1-1}\hat{\eta}^{\mu_b\nu_{b+1}}\right)
$$

$$
\times\,\delta_{m_1\neq n}\Big[p_\mu^{m_1}p_\nu^{m_1}\hat{\delta}_\rho^{\mu_{m_1}}\hat{\delta}_\sigma^{\nu_{m_1}} - p_\mu^{m_1}p_\rho^{m_1}\hat{\delta}_\nu^{\mu_{m_1}}\hat{\delta}_\sigma^{\nu_{m_1}}\Big] - \sum_{j+k+l=m_1-2}\left(\hat{\delta}^{\mu}_{\mu_0}\hat{\delta}^{\rho}_{\nu_{j+1}}\prod_{a=0}^{j}\hat{\eta}^{\mu_a\nu_{a+j}}\right)\left(\hat{\delta}^{\nu}_{\mu_j}\hat{\delta}^{\sigma}_{\nu_{j+k+1}}\prod_{b=j}^{j+k}\hat{\eta}^{\mu_b\nu_{b+1}}\right)
$$

$$
\times\left(\hat{\delta}^{\kappa}_{\mu_{j+k}}\hat{\delta}^{\lambda}_{\nu_{m_1-1}}\prod_{c=j+k}^{m_1-2}\hat{\eta}^{\mu_c\nu_{c+1}}\right)\times\Bigg(\delta_{m_1\neq n}\bigg[\left(p_\mu^{n-1}\hat{\delta}_\rho^{\mu_{n-1}}\hat{\delta}_\kappa^{\nu_{n-1}}\right)\left(\frac{1}{2}p_\lambda^{n}\hat{\delta}_\nu^{\mu_n}\hat{\delta}_\sigma^{\nu_n} - p_\nu^{n}\hat{\delta}_\lambda^{\mu_n}\hat{\delta}_\sigma^{\nu_n}\right)
$$

$$
+\frac{1}{2}\left(p_\nu^{n-1}\hat{\delta}_\mu^{\mu_{n-1}}\hat{\delta}_\kappa^{\nu_{n-1}}\right)\left(p_\sigma^{n}\hat{\delta}_\rho^{\mu_n}\hat{\delta}_\lambda^{\nu_n}\right)\bigg] + \left(p_\kappa^{n-1}\hat{\delta}_\mu^{\mu_{n-1}}\hat{\delta}_\rho^{\nu_{n-1}}\right)\left(\frac{1}{2}p_\nu^{n}\hat{\delta}_\sigma^{\mu_n}\hat{\delta}_\lambda^{\nu_n} - \frac{1}{4}p_\lambda^{n}\hat{\delta}_\nu^{\mu_n}\hat{\delta}_\sigma^{\nu_n}\right)
$$

$$
-\left(p_\nu^{n-1}\hat{\delta}_\mu^{\mu_{n-1}}\hat{\delta}_\kappa^{\nu_{n-1}}\right)\left(\frac{1}{2}p_\rho^{n}\hat{\delta}_\sigma^{\mu_n}\hat{\delta}_\lambda^{\nu_n} - \frac{1}{4}p_\sigma^{n}\hat{\delta}_\rho^{\mu_n}\hat{\delta}_\lambda^{\nu_n}\right)\Bigg)\Bigg\}\times\Bigg\{\sum_{\substack{i+j+k+l=m_2\\ i\geq j\geq k\geq l\geq 0}}\sum_{p=0}^{j-k}\sum_{q=0}^{k-l}\sum_{r=0}^{q}\sum_{s=0}^{l}\sum_{t=0}^{s}\sum_{u=0}^{t}\sum_{v=0}^{u}
$$

$$
\binom{\frac{1}{2}}{i}\binom{i}{j}\binom{j}{k}\binom{k}{l}\binom{j-k}{p}\binom{k-l}{q}\binom{q}{r}\binom{l}{s}\binom{s}{t}\binom{t}{u}\binom{u}{v}\times(-1)^{p+q-r+s-t+v}\,2^{-j+l+r+s+2t-3u+v}\,3^{-k+q-r+s-t+u}
$$

$$
\times\left(\prod_{a=m_1+1}^{m_1+a}\hat{\eta}^{\mu_a\nu_a}\right)\left(\prod_{b=m_1+a+1}^{m_1+a+b}\hat{\eta}^{\mu_b\mu_{b+b}}\hat{\eta}^{\nu_b\nu_{b+b}}\right)\left(\prod_{c=m_1+a+2b+1}^{m_1+a+2b+c}\hat{\eta}^{\mu_c\nu_{c+c}}\hat{\eta}^{\mu_{c+c}\nu_{c+2c}}\hat{\eta}^{\mu_{c+2c}\nu_c}\right)
$$

$$
\times\left(\prod_{d=m_1+a+2b+3c+1}^{m_1+a+2b+3c+d}\hat{\eta}^{\mu_d\nu_{d+d}}\hat{\eta}^{\mu_{d+d}\nu_{d+2d}}\hat{\eta}^{\mu_{d+2d}\nu_{d+3d}}\hat{\eta}^{\mu_{d+3d}\nu_d}\right)\Bigg\}
$$

Motivation

Goals:

- Find an analytical framework to deal with the Feynman rules for gravity
- Create an implementation within FeynCalc
- Use the implementation for amplitudes calculation

Analytical approach

Assumptions

- Effective theory approach
- Perturbative gravity
- Gravity described by general relativity
- Matter is minimally coupled to gravity
- Matter with $s = 0$; $s = \frac{1}{2}$; $s = 1, m = 0$
- Matter without supersymmetry

Perturbative expansion

Perturbative expansion

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}$$

Infinite series

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa\, h^{\mu\nu} + \kappa^2\, h^{\mu\sigma} h_\sigma{}^{\nu} + \cdots$$

General relativity

$$\mathcal{A}_{s=2,m=0} = \int d^4x \sqrt{-g} \left[-\frac{2}{\kappa^2} R \right], \; \kappa^2 = 32\pi G$$

Gauge fixing term

$$\mathcal{A}_{\text{gf}} = \int d^4x \left(\partial_\mu\, h^{\mu\nu} - \frac{1}{2} \partial^\mu h \right)^2$$

Perturbative expansion

Actions

$$\mathcal{A}_{s=0} = \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\,\nabla_\mu\phi\,\nabla_\nu\phi - \frac{m^2}{2}\phi^2\right]$$

$$\mathcal{A}_{s=1/2} = \int d^4x\sqrt{-g}\left[\frac{i}{2}\left(\overline{\psi}\gamma^\mu \mathfrak{e}_\mu{}^\nu\nabla_\nu\psi - \nabla_\nu\overline{\psi}\gamma^\mu\mathfrak{e}_\mu{}^\nu\psi\right) - m\,\overline{\psi}\,\psi\right]$$

$$\mathcal{A}_{s=1,m=0} = \int d^4x\sqrt{-g}\left[-\frac{1}{4}g^{\mu\nu}g^{\alpha\beta}\,F_{\mu\alpha}\,F_{\nu\beta}\right]$$

$$\mathcal{A}_{s=2,m=0} = \int d^4x\sqrt{-g}\left[-\frac{2}{\kappa^2}R\right]$$

Reparametrization

Surface terms

$$\sqrt{-g}\,R = \frac{1}{4}\sqrt{-g}\,g^{\mu\nu}g^{\alpha\beta}g^{\rho\sigma}$$
$$\times\left[\partial_\mu g_{\alpha\beta}\partial_\nu g_{\rho\sigma} - \partial_\mu g_{\alpha\rho}\partial_\nu g_{\beta\sigma} + 2\,\partial_\mu g_{\beta\rho}\partial_\alpha g_{\nu\sigma} - 2\,\partial_\mu g_{\nu\alpha}\partial_\beta g_{\rho\sigma}\right]$$
$$+\ \text{full divergence}$$

Einstein parametrization

$$\mathcal{A}_{s=2,m=0} = \int d^4x\sqrt{-g}\,g^{\mu\nu}g^{\alpha\beta}g^{\rho\sigma}$$
$$\times\left[-\frac{1}{2}\left(\partial_\mu h_{\alpha\beta}\partial_\nu h_{\rho\sigma} - \partial_\mu h_{\alpha\rho}\partial_\nu h_{\beta\sigma} + 2\,\partial_\mu h_{\beta\rho}\partial_\alpha h_{\nu\sigma} - 2\,\partial_\mu h_{\nu\alpha}\partial_\beta h_{\rho\sigma}\right)\right]$$

Reparametrization

Factorization

$$\mathcal{A}_{s=0} = \int d^4x \left[\sqrt{-g}\, g^{\mu\nu} \left(\frac{1}{2}\nabla_\mu\phi\, \nabla_\nu\phi\right) + \sqrt{-g}\, \frac{m^2}{2}\phi^2\right]$$

$$\mathcal{A}_{s=1/2} = \int d^4x \left[\sqrt{-g}\, \mathfrak{e}_\mu{}^\nu \left(\frac{i}{2}\left(\overline{\psi}\gamma^\mu\nabla_\nu\psi - \nabla_\nu\overline{\psi}\gamma^\mu\psi\right)\right) + \sqrt{-g}\, m\,\overline{\psi}\,\psi\right]$$

$$\mathcal{A}_{s=1,m=0} = \int d^4x \sqrt{-g}\, g^{\mu\nu}\, g^{\alpha\beta} \left[-\frac{1}{4} F_{\mu\alpha}\, F_{\nu\beta}\right]$$

$$\mathcal{A}_{s=2,m=0} = \int d^4x\, \sqrt{-g}\, g^{\mu\nu} g^{\alpha\beta} g^{\rho\sigma}$$

$$\times\left[-\frac{1}{2}\left(\partial_\mu h_{\alpha\beta}\partial_\nu h_{\rho\sigma} - \partial_\mu h_{\alpha\rho}\partial_\nu h_{\beta\sigma} + 2\,\partial_\mu h_{\beta\rho}\partial_\alpha h_{\nu\sigma} - 2\,\partial_\mu h_{\nu\alpha}\partial_\beta h_{\rho\sigma}\right)\right]$$

$\mathcal{T}$ tensors

Scalar field with m=0

$$\frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi \rightarrow \frac{1}{2}\Big(i(p_1)_\mu\phi(p_1)\Big)\Big(i(p_2)_\nu\phi(p_2)\Big) = -\frac{1}{2}(p_1)_\mu(p_2)_\nu\,\phi(p_1)\phi(p_2)$$

$$= -\,\frac{1}{2}\left[\frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha})\right](p_1)^\alpha(p_2)^\beta\,\phi(p_1)\phi(p_2)$$

$$= \mathcal{T}^{(\phi)}_{\mu\nu}(p_1,p_2)\,\phi(p_1)\phi(p_2)$$

with

$$\mathcal{T}^{(\phi)}_{\mu\nu}(p_1,p_2) = -\frac{1}{2}I_{\mu\nu\alpha\beta}\,(p_1)^\alpha\,(p_2)^\beta$$

$$I_{\mu\nu\alpha\beta} = \frac{1}{2}\left[\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}\right]$$

$\mathcal{T}$ tensors

Gravitational field

$$\partial_\mu h_{\alpha\beta}\partial_\nu h_{\rho\sigma} - \partial_\mu h_{\alpha\rho}\partial_\nu h_{\beta\sigma} + 2\,\partial_\mu h_{\beta\rho}\partial_\alpha h_{\nu\sigma} - 2\,\partial_\mu h_{\nu\alpha}\partial_\beta h_{\rho\sigma}$$
$$\to (p_1)^{\lambda_1}(p_2)^{\lambda_2}\, h^{\rho_1\sigma_1}_{(p_1)}\, h^{\rho_2\sigma_2}_{(p_2)}\, \mathcal{T}^{(h)}_{\mu\nu\alpha\beta\rho\sigma|\lambda_1\lambda_2\rho_1\sigma_1\rho_2\sigma_2}$$

$$\mathcal{T}^{(h)}_{\mu\nu\alpha\beta\rho\sigma|\lambda_1\lambda_2\rho_1\sigma_1\rho_2\sigma_2} = -\,\eta_{\mu\lambda_1}\eta_{\nu\lambda_2} I_{\alpha\beta\rho_1\sigma_1} I_{\rho\sigma\rho_2\sigma_2} + \eta_{\mu\lambda_1}\eta_{\nu\lambda_2} I_{\alpha\rho\rho_1\sigma_1} I_{\beta\sigma\rho_2\sigma_2}$$
$$-\,2\,\eta_{\mu\lambda_1}\eta_{\alpha\lambda_2} I_{\beta\rho\rho_1\sigma_1} I_{\nu\sigma\rho_2\sigma_2} + 2\,\eta_{\mu\lambda_1}\eta_{\beta\lambda_2} I_{\nu\alpha\rho_1\sigma_1} I_{\rho\sigma\rho_2\sigma_2}$$

$\mathbb{I}$ tensors

$g^{\mu\nu}$ expansion

$$
\begin{aligned}
g^{\mu\nu} =& \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\sigma} h_\sigma{}^\nu - \kappa^3 h^{\mu\sigma_1} h_{\sigma_1}{}^{\sigma_2} h_{\sigma_2}{}^\nu + \cdots \\
=& \sum_{n=0}^{\infty} (-\kappa)^n \, I^{\mu\nu\rho_1\sigma_1\cdots\rho_n\sigma_n} \, h_{\rho_1\sigma_1} \cdots h_{\rho_n\sigma_n}
\end{aligned}
$$

$$
\begin{aligned}
I^{\mu\nu} =& \eta^{\mu\nu} \\
I^{\mu_1\nu_1\mu_2\nu_2} =& \frac{1}{2} \left(\eta^{\nu_1\mu_2} \eta^{\nu_2\mu_1} + \eta^{\mu_1\mu_2} \eta^{\nu_2\nu_1} \right) \\
&\vdots \\
I^{\mu_1\nu_1\cdots\mu_n\nu_n} =& \frac{1}{2^n} \left(\eta^{\nu_1\mu_2} \eta^{\nu_2\mu_3} \cdots \eta^{\nu_n\mu_1} + \cdots \right)
\end{aligned}
$$

$\mathbb{C}$ tensors

$\sqrt{-g}$ expansion

$$\sqrt{-g} = \sum_{n=0}^{\infty} \kappa^n \, \mathcal{C}^{\mu_1\nu_1\cdots\mu_n\nu_n} \, h_{\mu_1\nu_1} \cdots h_{\mu_n\nu_n}$$

$$\mathcal{C}^{\mu_1\nu_1\cdots\mu_n\nu_n} = \text{Symm}\left[\sum_{m=1}^{n} \frac{(-1)^{n+m}}{m!\,2^m} \sum_{k_1+\cdots+k_m=n} \frac{1}{k_1 \cdots k_m} \Big(I_{(k_1)} \cdots I_{(k_m)} \Big)^{\mu_1\nu_1\cdots\mu_n\nu_n} \right]$$

Symmetrization is performed with respect to index pairs

$$(\mu_i, \nu_i) \leftrightarrow (\mu_j, \nu_j)$$

$\mathbb{I}$ tensor examples

$$
\begin{aligned}
I^{\mu\nu} =& \eta^{\mu\nu} \\
I^{\mu\nu\alpha\beta} =& \frac{1}{2}\left[\eta^{\nu\alpha}\eta^{\beta\mu} + \eta^{\nu\beta}\eta^{\alpha\mu}\right] \\
I^{\mu\nu\alpha\beta\rho\sigma} =& \frac{1}{8}\Big[\eta^{\alpha\sigma}\eta^{\beta\nu}\eta^{\mu\rho} + \eta^{\alpha\nu}\eta^{\beta\sigma}\eta^{\mu\rho} + \eta^{\alpha\rho}\eta^{\beta\nu}\eta^{\mu\sigma} + \eta^{\alpha\nu}\eta^{\beta\rho}\eta^{\mu\sigma} \\
& + \eta^{\alpha\sigma}\eta^{\beta\mu}\eta^{\nu\rho} + \eta^{\alpha\mu}\eta^{\beta\sigma}\eta^{\nu\rho} + \eta^{\alpha\rho}\eta^{\beta\mu}\eta^{\nu\sigma} + \eta^{\alpha\mu}\eta^{\beta\rho}\eta^{\nu\sigma}\Big] \\
I^{\mu\nu\alpha\beta\rho\sigma\lambda\tau} =& \frac{1}{16}\Big[\eta^{\alpha\sigma}\eta^{\beta\nu}\eta^{\lambda\rho}\eta^{\mu\tau} + \eta^{\alpha\nu}\eta^{\beta\sigma}\eta^{\lambda\rho}\eta^{\mu\tau} + \eta^{\alpha\rho}\eta^{\beta\nu}\eta^{\lambda\sigma}\eta^{\mu\tau} \\
& + \eta^{\alpha\nu}\eta^{\beta\rho}\eta^{\lambda\sigma}\eta^{\mu\tau} + \eta^{\alpha\sigma}\eta^{\beta\mu}\eta^{\lambda\rho}\eta^{\nu\tau} + \eta^{\alpha\mu}\eta^{\beta\sigma}\eta^{\lambda\rho}\eta^{\nu\tau} \\
& + \eta^{\alpha\rho}\eta^{\beta\mu}\eta^{\lambda\sigma}\eta^{\nu\tau} + \eta^{\alpha\mu}\eta^{\beta\rho}\eta^{\lambda\sigma}\eta^{\nu\tau} + \eta^{\alpha\sigma}\eta^{\beta\nu}\eta^{\lambda\mu}\eta^{\rho\tau} \\
& + \eta^{\alpha\nu}\eta^{\beta\sigma}\eta^{\lambda\mu}\eta^{\rho\tau} + \eta^{\alpha\sigma}\eta^{\beta\mu}\eta^{\lambda\nu}\eta^{\rho\tau} + \eta^{\alpha\mu}\eta^{\beta\sigma}\eta^{\lambda\nu}\eta^{\rho\tau} \\
& + \eta^{\alpha\rho}\eta^{\beta\nu}\eta^{\lambda\mu}\eta^{\sigma\tau} + \eta^{\alpha\nu}\eta^{\beta\rho}\eta^{\lambda\mu}\eta^{\sigma\tau} + \eta^{\alpha\rho}\eta^{\beta\mu}\eta^{\lambda\nu}\eta^{\sigma\tau} \\
& + \eta^{\alpha\mu}\eta^{\beta\rho}\eta^{\lambda\nu}\eta^{\sigma\tau}\Big]
\end{aligned}
$$

$\mathbb{C}$ tensor examples

$$
\begin{aligned}
\mathcal{C}_{\mu\nu} =& \frac{1}{2}\eta_{\mu\nu} \\
\mathcal{C}_{\mu\nu\alpha\beta} =& \frac{1}{8}\big[-\eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\beta}\eta^{\mu\nu}\big] \\
\mathcal{C}_{\mu\nu\alpha\beta\rho\sigma} =& \frac{1}{48}\big[-\eta^{\alpha\sigma}\eta^{\beta\rho}\eta^{\mu\nu} - \eta^{\alpha\rho}\eta^{\beta\sigma}\eta^{\mu\nu} + \eta^{\alpha\beta}\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\alpha\sigma}\eta^{\beta\nu}\eta^{\mu\rho} \\
& + \eta^{\alpha\nu}\eta^{\beta\sigma}\eta^{\mu\rho} + \eta^{\alpha\rho}\eta^{\beta\nu}\eta^{\mu\sigma} + \eta^{\alpha\nu}\eta^{\beta\rho}\eta^{\mu\sigma} + \eta^{\alpha\sigma}\eta^{\beta\mu}\eta^{\nu\rho} \\
& + \eta^{\alpha\mu}\eta^{\beta\sigma}\eta^{\nu\rho} - \eta^{\alpha\beta}\eta^{\mu\sigma}\eta^{\nu\rho} + \eta^{\alpha\rho}\eta^{\beta\mu}\eta^{\nu\sigma} + \eta^{\alpha\mu}\eta^{\beta\rho}\eta^{\nu\sigma} \\
& - \eta^{\alpha\beta}\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\alpha\nu}\eta^{\beta\mu}\eta^{\rho\sigma} - \eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\sigma}\big]
\end{aligned}
$$

Other $\mathbb{C}$ tensors

Expansions

$$g^{\mu\nu} = \sum_{n=0}^{\infty} (-\kappa)^n I^{\mu\nu\rho_1\cdots\sigma_n} h_{\rho_1\sigma_1} \cdots h_{\rho_n\sigma_n}$$

$$\sqrt{-g} = \sum_{n=0}^{\infty} \kappa^n \mathcal{C}^{\mu_1\nu_1\cdots\mu_n\nu_n} h_{\mu_1\nu_1} \cdots h_{\mu_n\nu_n}$$

produce the other expansions

$$\sqrt{-g}g^{\mu\nu} = \sum_{n=0}^{\infty} \kappa^n \mathcal{C}_I^{\mu\nu|\rho_1\cdots\sigma_n} h_{\rho_1\sigma_1} \cdots h_{\rho_n\sigma_n}$$

$$\sqrt{-g}g^{\mu\nu}g^{\alpha\beta} = \sum_{n=0}^{\infty} \kappa^n \mathcal{C}_{II}^{\mu\nu\alpha\beta|\rho_1\cdots\sigma_n} h_{\rho_1\sigma_1} \cdots h_{\rho_n\sigma_n}$$

$$\sqrt{-g}g^{\mu\nu}g^{\alpha\beta}g^{\rho\sigma} = \sum_{n=0}^{\infty} \kappa^n \mathcal{C}_{III}^{\mu\nu\alpha\beta\rho\sigma|\rho_1\cdots\sigma_n} h_{\rho_1\sigma_1} \cdots h_{\rho_n\sigma_n}$$

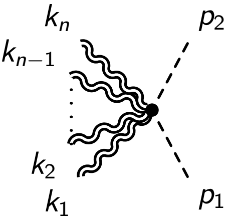
Interaction structure

Scalar field with $m = 0$

$$\mathcal{A}_{s=0,m=0} = \int d^4x \sqrt{-g}\, g^{\mu\nu} \left[\frac{1}{2} \nabla_\mu \phi \, \nabla_\nu \phi \right]$$

$$= \int \sum_{n=0}^{\infty} \left[\frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \prod_{i=1}^{n} \frac{d^4 k_i}{(2\pi)^4} (2\pi)^4 \delta \left(p_1 + p_2 + \sum_{i=1}^{n} k_i \right) \right.$$

$$\left. \times\, \kappa^n \, \mathcal{C}_I^{\mu\nu|\rho_1\cdots\sigma_n} \, h^{(k_1)}_{\rho_1\sigma_1} \cdots h^{(k_n)}_{\rho_n\sigma_n} \, \mathcal{T}^{(\phi)}_{\mu\nu}(p_1, p_2) \, \phi(p_1) \, \phi(p_2) \right]$$



Interaction structure

Gravity

$$\mathcal{A}_{s=2,m=0} = \int \sum_{n=0}^{\infty} \left[\frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} \prod_{i=1}^{n} \frac{d^4 k_i}{(2\pi)^4} (2\pi)^4 \delta \left(p_1 + p_2 + \sum_{i=1}^{n} k_i \right) \right.$$

$$\left. \times\, \kappa^n \mathcal{C}_{III}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3|\rho_1\cdots\sigma_n}\, h^{(k_1)}_{\rho_1\sigma_1} \cdots h^{(k_n)}_{\rho_n\sigma_n}\;\; \mathcal{T}^{(h)}_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3|\lambda_1\lambda_2\alpha_1\beta_1\alpha_2\beta_2}\, (p_1)^{\lambda_1} (p_2)^{\lambda_2}\, h^{\alpha_1\beta_1}_{(p_1)}\, h^{\alpha_2\beta_2}_{(p_2)} \right]$$

Rules for graviton interactions

$$(\mathfrak{V}^{(n)})^{\rho_1\sigma_1\cdots\rho_n\sigma_n}(p_1, \cdots, p_n)$$

$$= i\, \kappa^n \left[\mathcal{C}_{III}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3|\rho_3\cdots\sigma_n}\;\; \mathcal{T}^{(h)}_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3|\lambda_1\lambda_2}{}^{\rho_1\sigma_1\rho_2\sigma_2}\, (p_1)^{\lambda_1} (p_2)^{\lambda_2} + \cdots \right].$$

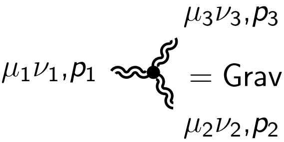
The expression is symmetrized with respect to <u>all</u> momenta.

Implementation for FeynCalc

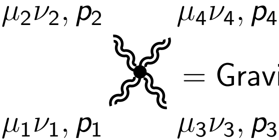
FeynGrav

- FeynGrav implemented within FeynCalc
- Expressions are taken from a pre-generated library
- User can access commands generating Feynman rules
- Expressions for graviton propagator are implemented
- Nieuwenhuizen operators and gauge projectors are implemented

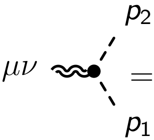
Commands



$$= \text{GravitonVertex}[\mu_1, \nu_1, p_1, \mu_2, \nu_2, p_2, \mu_3, \nu_3, p_3]$$



$$= \text{GravitonVertex}[\mu_1, \nu_1, p_1, \mu_2, \nu_2, p_2, \mu_3, \nu_3, p_3, \mu_4, \nu_4, p_4]$$



$$= \text{GravitonScalarVertex}[\mu, \nu, p_1, p_2]$$

Graviton propagator

$$G_{\mu\nu\alpha\beta}(k) = i\,\frac{\frac{1}{2}\left[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right]}{k^2}$$

$$\begin{array}{cl}
\text{"GravitonPropagator"} & : \; 1/k^2 \leftrightarrow \mathrm{FAD}[k] \\
\text{"GravitonPropagatorAlternative"} & : \; 1/k^2 \leftrightarrow 1/\,\mathrm{SPD}[k,k] \\
\text{"GravitonPropagatorTop"} & = \frac{1}{2}\left[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right]
\end{array}$$

Nieuwenhuizen operators

$$\Theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \qquad \omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}$$

$$P^1_{\mu\nu\alpha\beta} = \frac{1}{2}(\Theta_{\mu\alpha}\omega_{\nu\beta} + \Theta_{\mu\beta}\omega_{\nu\alpha} + \Theta_{\nu\beta}\omega_{\mu\alpha} + \Theta_{\nu\alpha}\omega_{\mu\beta})$$

$$P^2_{\mu\nu\alpha\beta} = \frac{1}{2}(\Theta_{\mu\alpha}\Theta_{\nu\beta} + \Theta_{\mu\beta}\Theta_{\nu\alpha}) - \frac{1}{3}\Theta_{\mu\nu}\Theta_{\alpha\beta}$$

$$P^0_{\mu\nu\alpha\beta} = \frac{1}{3}\Theta_{\mu\nu}\Theta_{\alpha\beta}$$

$$\overline{P}^0_{\mu\nu\alpha\beta} = \omega_{\mu\nu}\omega_{\alpha\beta}$$

$$\overline{\overline{P}}^0_{\mu\nu\alpha\beta} = \Theta_{\mu\nu}\omega_{\alpha\beta} + \Theta_{\alpha\beta}\omega_{\mu\nu}$$

Nieuwenhuizen, Nucl.Phys.B 60 (1973) 478

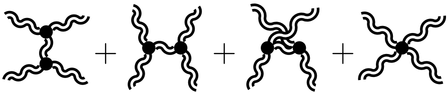
$2 \to 2$ on-shell graviton scattering

Sannan, Phys.Rev.D 34 (1986) 1749

- $\eta_{\mu\nu} = \mathrm{diag}(- + + \cdots +)$
- $\kappa^2 = 32\pi G$
- $1 + 2 \to 3 + 4$
- $k^\mu e_{\mu\nu} = e^\mu{}_\mu = 0$

$$
\begin{aligned}
k_1^\mu &= (k, 0, 0, k) & \epsilon_1^{(\pm)\mu} &= \mp\tfrac{1}{\sqrt{2}}(0, 1, \pm i, 0)\\
k_2^\mu &= (k, 0, 0, -k) & \epsilon_2^{(\pm)\mu} &= \mp\tfrac{1}{\sqrt{2}}(0, -1, \pm i, 0)\\
k_3^\mu &= (-k, -k\sin\theta, 0, -k\cos\theta) & \epsilon_3^{(\pm)\mu} &= \mp\tfrac{1}{\sqrt{2}}(0, -\cos\theta, \pm i, \sin\theta)\\
k_4^\mu &= (-k, k\sin\theta, 0, k\cos\theta) & \epsilon_4^{(\pm)\mu} &= \mp\tfrac{1}{\sqrt{2}}(0, \cos\theta, \pm i, -\sin\theta)
\end{aligned}
$$

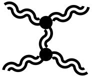
$2 \to 2$ on-shell graviton scattering



$$T_{2,2;2,2} = -\frac{\kappa^2}{4}\frac{s^4}{s\,t\,u}; \quad T_{2,-2;2,-2} = -\frac{\kappa^2}{4}\frac{u^4}{s\,t\,u}; \quad T_{2,-2;-2,2} = -\frac{\kappa^2}{4}\frac{t^4}{s\,t\,u};$$

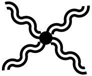
$$T_{2,2;-2,-2} = T_{2,2;2,-2} = 0$$

$2 \to 2$ on-shell graviton scattering

 $= \mathcal{M}_s$

$$= \text{GravitonVertex}[\mu_1, \nu_1, p_1, \mu_2, \nu_2, p_2, \alpha_1, \beta_1, -p_1 - p_2]$$
$$\times \text{GravitonPropagatorAlternative}[\alpha_1, \beta_1, \alpha_2, \beta_2, p_1 + p_2]$$
$$\times \text{GravitonVertex}[\mu_3, \nu_3, p_3, \mu_4, \nu_4, p_4, \alpha_2, \beta_2, p_1 + p_2]$$

Timing approximately 3 minutes

 $= \mathcal{M}_4$

$$= \text{GravitonVertex}[\mu_1, \nu_1, p_1, \mu_2, \nu_2, p_2, \mu_3, \nu_3, p_3, \mu_4, \nu_4, p_4]$$

Timing approximately 15 seconds

$2 \to 2$ on-shell graviton scattering

Amplitudes evaluated in FeynGrav (about 6 minutes)

$$T_{2,2;2,2} = i\,\frac{\kappa^2}{4}\,\frac{s^4}{s\,t\,u} \quad T_{2,-2;2,-2} = i\,\frac{\kappa^2}{4}\,\frac{u^4}{s\,t\,u} \quad T_{2,-2;-2,2} = i\,\frac{\kappa^2}{4}\,\frac{t^4}{s\,t\,u}$$

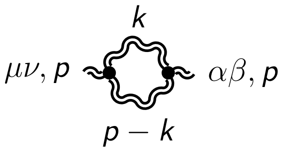
$$T_{2,2;-2,-2} = T_{2,2;2,-2} = 0$$

Amplitudes evaluated analytically

$$T_{2,2;2,2} = -\frac{\kappa^2}{4}\,\frac{s^4}{s\,t\,u} \quad T_{2,-2;2,-2} = -\frac{\kappa^2}{4}\,\frac{u^4}{s\,t\,u} \quad T_{2,-2;-2,2} = -\frac{\kappa^2}{4}\,\frac{t^4}{s\,t\,u}$$

$$T_{2,2;-2,-2} = T_{2,2;2,-2} = 0$$

Graviton polarization operator



$$= \text{GravitonVertex}[\mu,\nu,p,\alpha_1,\beta_1,-k,\rho_1,\sigma_1,-(p-k)]$$
$$\times\ \text{GravitonVertex}[\alpha_2,\beta_2,k,\rho_2,\sigma_2,p-k,\alpha,\beta,-p]$$
$$\times\ \text{GravitonPropagator}[\alpha_1,\beta_1,\alpha_2,\beta_2,k]$$
$$\times\ \text{GravitonPropagator}[\rho_1,\sigma_1,\rho_2,\sigma_2,p-k]$$

Graviton polarization operator

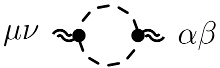
Timing approximately 1 minute

$$
\begin{aligned}
= i\pi^2 B_0(p^2,0,0)\,\frac{\kappa^2}{128}\,p^4 \Bigg[&-\frac{4\left(d^4-10d^3-137d^2+458d+176\right)}{d^2-1}\,P^2_{\mu\nu\alpha\beta} \\
&+\frac{2\left(3d^6-60d^5+433d^4-808d^3-1172d^2+1988d+3896\right)}{d^2-1}\,P^0_{\mu\nu\alpha\beta} \\
&+\frac{16(d-2)(3d-11)}{d-1}\,P^1_{\mu\nu\alpha\beta}+8(d-2)(d-3)\,\overline{P^0}_{\mu\nu\alpha\beta} \\
&+\frac{4(d-2)\left(d^3-12d^2+44d-46\right)}{d-1}\,\overline{\overline{P^0}}_{\mu\nu\alpha\beta}\Bigg]
\end{aligned}
$$

Graviton polarization operator

Scalars

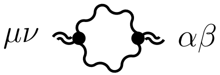
Timing approximately 3 seconds



$$=i\pi^2 B_0(p^2,0,0)\,\frac{\kappa^2}{16}p^4\,\frac{1}{d^2-1}\left[2\,P^2_{\mu\nu\alpha\beta}+\left(3d^2-6d-4\right)\,P^0_{\mu\nu\alpha\beta}\right]$$

Graviton polarization operator

Vectors
Timing approximately 4 seconds

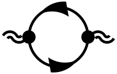


$$=i\pi^2 B_0(p^2,0,0)\,\frac{\kappa^2}{16}\,p^4\,\frac{1}{d^2-1}$$
$$\times\left[2\left(2d^2-3d-8\right)\,P^2_{\mu\nu\alpha\beta}+(d-4)\left(3d^2-8d-8\right)P^0_{\mu\nu\alpha\beta}\right]$$

Graviton polarization operator

Fermions

Timing approximately 2 seconds

$\mu\nu$  $\alpha\beta = i\pi^2 B_0(p^2, 0, 0)\,\frac{\kappa^2}{16}\,p^4\,\frac{2-d}{d-1}\,P^1_{\mu\nu\alpha\beta}$

Gravitational polarization operators

 $i\pi^2\, B_0(p^2, 0, 0)\, \frac{\kappa^2}{32}(d-4)(2-d)\, p^4$

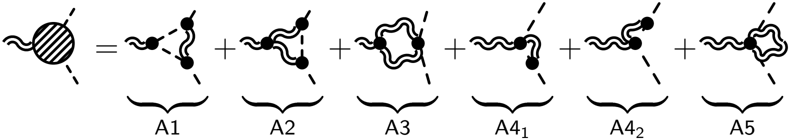
λ_1  λ_2 $= i\pi^2 B_0(p^2, 0, 0)\, \frac{\kappa^2}{32}\, \frac{(d-2)(d^2-14d+32)}{d-1}\, p^4\, \Theta_{\lambda_1\lambda_2}(p)$

 $= i\pi^2\, B_0(p^2, 0, 0)\, \frac{\kappa^2}{128}(3-2d)(d-2)\, p^2\, \gamma \cdot p$

Timing for each polarization operator is smaller then 2 seconds.

Graviton-scalar structure functions

Scalar-graviton interaction at one-loop



Timing approximately 2 minutes

Graviton-scalar structure functions

$$
\begin{aligned}
&(\mathcal{M}_{\text{1-loop}})_{\mu\nu} \\
=&\eta_{\mu\nu}\Big[B_0(p_1,0,0)\,(F_3)^{B_0}_{p_1} + B_0(p_2,0,0)\,(F_3)^{B_0}_{p_2} + B_0(k,0,0)(F_3)^{B_0}_{k} + C_0(p_1,p_2,k,0,0,0)(F_3)^{C_0}\Big] \\
&+B_0(p_1,0,0)\Big[(F_{1,1})^{B_0}_{p_1}p_{1\mu}p_{1\nu} + (F_{1,2})^{B_0}_{p_1}p_{2\mu}p_{2\nu}\Big] + B_0(p_2,0,0)\Big[(F_{1,1})^{B_0}_{p_2}p_{1\mu}p_{1\nu} + (F_{1,2})^{B_0}_{p_2}p_{2\mu}p_{2\nu}\Big] \\
&+B_0(p_1,0,0)\Big[(F_{2,1})^{B_0}_{p_1}p_{1\mu}p_{2\nu} + (F_{2,2})^{B_0}_{p_1}p_{2\mu}p_{1\nu}\Big] + B_0(p_2,0,0)\Big[(F_{2,1})^{B_0}_{p_2}p_{1\mu}p_{2\nu} + (F_{2,2})^{B_0}_{p_2}p_{2\mu}p_{2\nu}\Big] \\
&+B_0(k,0,0)\Big[(F_{1,1})^{B_0}_{k}p_{1\mu}p_{1\nu} + (F_{1,2})^{B_0}_{k}p_{2\mu}p_{2\nu}\Big] + B_0(k,0,0)\Big[(F_{2,1})^{B_0}_{k}p_{1\mu}p_{2\nu} + (F_{2,2})^{B_0}_{k}p_{2\mu}p_{1\nu}\Big] \\
&+C_0(p_1,p_2,k,0,0,0)\Big[(F_{1,1})^{C_0}p_{1\mu}p_{1\nu} + (F_{1,2})^{C_0}p_{2\mu}p_{2\nu} + (F_{2,1})^{C_0}p_{1\mu}p_{2\nu} + (F_{2,2})^{C_0}p_{2\mu}p_{1\nu}\Big]
\end{aligned}
$$

Graviton-scalar structure functions

$$
\begin{aligned}
&\left(\mathcal{M}_{\text{1-loop}}\right)_{\mu\nu}\Bigg|_{\text{on-shell}}^{d=4} \\
&= -\frac{\kappa^3}{8} k^4 \, i\, \pi^2 \, C_0(0,0,k^2,0,0,0) \left[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right] p_1^\alpha \, p_2^\beta \\
&\quad + \frac{\kappa^3}{8} k^4 \, i\, \pi^2 \, B_0(k^2,0,0) \, \Theta_{\mu\nu}(k) \\
&\quad - \frac{7\,\kappa^3}{24} k^2 \, i\, \pi^2 \, B_0(k^2,0,0) \left[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right] p_1^\alpha \, p_2^\beta
\end{aligned}
$$

New non-minimal UV-finite interaction $G_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi$ is generated

Latosh, Eur.Phys.J.C 80 (2020) 9, 845

Conclusions

Results

- Framework to operate with Feynman rules is constructed
- Feynman rules for gravity are implemented in FeynCalc
- The framework reproduces the known results
- Polarization operators are found
- Structure of scalar-tensor gravit at one-loop is studied

Thank you for attention!