What Quantum Strings can tell us about Quantum Gravity

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Based on:

- Y. M. arXiv:2204.10205 [hep-th]
- J. Ambjørn, Y. M. MPLA 36, 2150136 (2021) [arXiv:2103.10259]
- Y. M. Nucl. Phys. B 967, 115398 (2021) [arXiv:2102.04753]
- Y. M. JHEP 07, 104 (2018) [arXiv:1802.07541]
- J. Ambjørn, Y. M. IJMPA 32, 1750187 (2017) [arXiv:1709.00995]
- J. Ambjørn, Y. M. Phys. Lett. B 770, 352 (2017) [arXiv:1703.05382]
- J. Ambjørn, Y. M. Phys. Lett. B 756, 142 (2016) [arXiv:1601.00540]
- J. Ambjørn, Y. M. Phys. Rev. D 93, 066007 (2016) [arXiv:1510.03390]

Two no-go theorems for string existence

inherited from 1980's

- Non-perturbative lattice regularization (by dynamical triangulation) scales to a continuum string for $d \leq 1$ but does not for $d > 1$ (same for hypercubic latticization of Nambu-Goto string in $d > 2$)

  Durhuus, Fröhlich, Jonsson (1984), Ambjørn, Durhuus (1987)

- Knizhnik-Polyakov-Zamolodchikov (1988), David (1988), Distler-Kawai (1989) string susceptibility index of (closed) Polyakov's string is not real for $1 < d < 25$

$$\gamma_{\text{str}} = (1-h)\frac{d-25-\sqrt{(d-1)(d-25)}}{12} + 2 \qquad \boxed{\text{genus } h}$$

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The presented solutions rely on subtleties in Quantum Field Theory enjoying diffeomorphism invariance: Strings(!) and Gravity(?) $\Longrightarrow$

1) Continuum limit is not as in Quantum Field Theory: Lilliputian
2) The Nambu-Goto and Polyakov strings differ quantumly:
   higher derivative terms $\sim \Lambda^{-2}$ in emergent action revive

2. Mean-field ground state of bosonic string

Nambu-Goto versus Polyakov strings

Polyakov string is quadratic in X^μ (independent metric tensor ρ_{ab})

$$\mathcal{S} = \frac{K_0}{2}\int \mathrm{d}^2\omega\,\sqrt{\rho}\rho^{ab}\partial_a X\cdot\partial_b X$$

Nambu-Goto string (plus Lagrange multiplier λ^{ab})

$$K_0\int \mathrm{d}^2\omega\,\sqrt{\det\partial_a X\cdot\partial_b X} = K_0\int \mathrm{d}^2\omega\,\sqrt{\rho} + \frac{K_0}{2}\int \mathrm{d}^2\omega\,\lambda^{ab}\left(\partial_a X\cdot\partial_b X - \rho_{ab}\right)$$

World-sheet parameters $\omega_1, \omega_2 \in \omega_L \times \omega_\beta$ rectangle

Closed bosonic string winding once around compactified dimension of length β, propagating (Euclidean) time L (cylinder or torus).
No tachyon if β is large enough

Usual classical ground state

$$X^\mu = X^\mu_{\mathrm{cl}}, \qquad [\rho_{ab}]_{\mathrm{cl}} = \partial_a X_{\mathrm{cl}}\cdot\partial_b X_{\mathrm{cl}}, \qquad \lambda^{ab}_{\mathrm{cl}} = \rho^{ab}_{\mathrm{cl}}\sqrt{\rho_{\mathrm{cl}}}$$

$\lambda^{ab}_{\mathrm{cl}} = \delta^{ab}$ in conformal gauge for $\omega_L = L$, $\omega_\beta = \beta$

The two string formulations are equivalent classically and at one loop

Induced (or emergent) action

Gaussian path integral over X^μ_{q} by splitting $X^\mu = X^\mu_{\mathsf{cl}} + X^\mu_{\mathsf{q}}$:

$$S_{\rm ind} = K_0 \int d^2\omega \sqrt{\rho} + \frac{K_0}{2} \int d^2\omega\, \lambda^{ab} \left(\partial_a X_{\rm cl} \cdot \partial_b X_{\rm cl} - \rho_{ab}\right)$$

$$+\frac{d}{2} {\rm tr} \log \mathcal{O}, \qquad\qquad \mathcal{O} = -\frac{1}{\sqrt{\rho}} \partial_a \lambda^{ab} \partial_b .$$

Operator $\mathcal{O}$ reproduces the Laplacian Δ for $\lambda^{ab} = \rho^{ab}\sqrt{\det \rho}$

Additional ghost determinant in the conformal gauge $\rho_{ab} = \rho\delta_{ab}$

$$-\frac{1}{2} {\rm tr} \log \left(-\Delta^b_a + \frac{1}{2} (\Delta^b_a \log \rho) \right)$$

Induced (or emergent) action coincides with effective action for smooth fields

2D determinants diverge and has to be regularized

Regularization of determinants

Proper-time regularization of the trace

$$\operatorname{tr}\log\mathcal{O}\big|_{\rm reg} = -\int_{a^2}^{\infty}\frac{d\tau}{\tau}\operatorname{tr}e^{-\tau\mathcal{O}}, \qquad \Lambda^2 = \frac{1}{4\pi a^2}$$

Pauli-Villars regularization of the trace Ambjørn, Y.M. (2017)

$$\det(\mathcal{O})\big|_{\rm reg} \equiv \frac{\det(\mathcal{O})\det(\mathcal{O}+2M^2)}{\det(\mathcal{O}+M^2)^2}$$

$$\operatorname{tr}\log\mathcal{O}\big|_{\rm reg} = -\int_{0}^{\infty}\frac{d\tau}{\tau}\operatorname{tr}e^{-\tau\mathcal{O}}\left(1-e^{-\tau M^2}\right)^2, \qquad \Lambda^2 = \frac{M^2}{2\pi}\log 2.$$

is convergent as finite regulator mass M and divergent as $M\to\infty$.

For Pauli-Villars regularization beautiful diagrammatic technique and det's can be exactly computed for certain metrics by the Gel'fand-Yaglom technique to compare with the Seeley expansion

$$\left\langle\omega\right| e^{-\tau\mathcal{O}}\left|\omega\right\rangle = \frac{1}{4\pi\tau}\frac{1}{\sqrt{\det\lambda^{ab}}} + \frac{R}{24\pi} + \mathcal{O}(\tau)$$

which starts with the term $1/\tau$ in 2 dimensions. For $\tau\sim 1/\Lambda^2$ higher terms are suppressed as R/Λ^2 only for smooth fields but revive if not

Mean-field ground state

Ambjørn, Y.M. (2017)

For diagonal and constant $\lambda^{ab} = \bar{\lambda}\delta^{ab}$ and $\rho_{ab} = \bar{\rho}\delta_{ab}$

$$
\begin{aligned}
S_{\text{eff}} &= \frac{K_0}{2}\bar{\lambda}\left(\frac{L^2}{\omega_L^2} + \frac{\beta^2}{\omega_\beta^2}\right)\omega_L\omega_\beta + K_0(1-\bar{\lambda})\bar{\rho}\,\omega_L\omega_\beta \\
&\quad - \left(\frac{d}{2\bar{\lambda}} - 1\right)\Lambda^2\bar{\rho}\,\omega_L\omega_\beta - \frac{\pi(d-2)}{6}\frac{\omega_L}{\omega_\beta}
\end{aligned}
$$

Boundary terms omitted for $L \gg \beta$.
The minimum is reached at (quantum ground state)

$$
\begin{aligned}
\bar{\lambda} &= \frac{1}{2}\left(1 + \frac{\Lambda^2}{K_0} + \sqrt{\left(1 + \frac{\Lambda^2}{K_0}\right)^2 - \frac{2d\Lambda^2}{K_0}}\right) \\
\bar{\rho} &\propto \frac{\bar{\lambda}}{\sqrt{(1 + \frac{\Lambda^2}{K_0})^2 - \frac{2d\Lambda^2}{K_0}}} \\
\omega_\beta &= \frac{\omega_L}{L}\sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0\bar{\lambda}}} \\
S_{\text{mf}} &= K_0\bar{\lambda}L\sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0\bar{\lambda}}} \qquad \text{(Alvarez-Arvis)}
\end{aligned}
$$

Mean-field ground state (cont.)

The approximation describes a mean field which takes into account an infinite set of pertubative diagrams about the classical vacuum. Then λ^{ab} and ρ_{ab} do not fluctuate which becomes exact at large d.

It is like 2d $O(N)$ sigma-model at large N where the Lagrange multiplier does not fluctuate (summing the bubble graphs). The large-N vacuum is very closed to the physical vacuum even for $N = 3$.

The minimization over ω_β/ω_L is also needed at the saddle point.

The square root is well-defined for $d \geq 2$ if

$$K_0 > K_* = \left(d - 1 + \sqrt{d^2 - 2d}\right)\Lambda^2 \stackrel{d\to\infty}{\to} 2d\Lambda^2$$

Perturbation theory is recovered by expanding in $1/K_0 \sim \hbar$. Then $\bar{\lambda}$ ranges between 1 (classical) and (quantum) value

$$\bar{\lambda}_* = \frac{1}{2}\left(d - \sqrt{d^2 - 2d}\right) \stackrel{d\to\infty}{\to} \frac{1}{2}$$

3. Two scaling regimes: Gulliver's vs. Lilliputian

Lattice-like scaling limit (Gulliver's)

The ground state energy (Alvarez-Arvis)

$$E_0(\beta) = K_0\bar{\lambda}\sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0\bar{\lambda}}}$$

does not scale because $K_0 > K_* \sim \Lambda^2$ for $\bar{\lambda}$ to be real ($> \bar{\lambda}_*$). Let

$$\beta^2 > \beta^2_{\text{min}} = \frac{\pi(d-2)}{3K_*\bar{\lambda}_*}, \qquad \bar{\lambda}_* = \frac{1}{2}\left(d - \sqrt{d^2 - 2d}\right) \stackrel{d\to\infty}{\to} \frac{1}{2}$$

for not to have a tachyon. For the smallest possible value $\beta = \beta_{\text{min}}$

$$E_0(\beta) \propto \frac{K_0\bar{\lambda}}{\Lambda}\sqrt{\bar{\lambda} - \bar{\lambda}_*}$$

scales to m if

$$\bar{\lambda} - \bar{\lambda}_* \propto \frac{m^2}{\Lambda^2}, \qquad K_0 - K_* \propto \frac{m^4}{\Lambda^2}$$

The scaling does not exist for excited states (larger values of β) and thus is particle-like similar to lattice regularizations of a string, where only the lowest mass scales to finite, excitations scale to infinity

Durhuus, Fröhlich, Jonsson (1984), Ambjørn, Durhuus (1987)

Lattice-like scaling limit (Gulliver's)

The ground state energy (Alvarez-Arvis)

$$E_N(\beta) = K_0\bar{\lambda}\sqrt{\beta^2 + \frac{1}{K_0\bar{\lambda}}\left(-\frac{\pi(d-2)}{3} + 8N\right)}$$

does not scale because $K_0 > K_* \sim \Lambda^2$ for $\bar{\lambda}$ to be real ($> \bar{\lambda}_*$). Let

$$\beta^2 > \beta^2_{\text{min}} = \frac{\pi(d-2)}{3K_*\bar{\lambda}_*}, \qquad \bar{\lambda}_* = \frac{1}{2}\left(d - \sqrt{d^2 - 2d}\right) \stackrel{d\to\infty}{\to} \frac{1}{2}$$

for not to have a tachyon. For the smallest possible value $\beta = \beta_{\text{min}}$

$$E_0(\beta) \propto \frac{K_0\bar{\lambda}}{\Lambda}\sqrt{\bar{\lambda} - \bar{\lambda}_*}$$

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Lilliputian string-like scaling limit

Let us "renormalize" the units of length

$$L_R = \sqrt{\frac{\bar\lambda}{\bar\lambda - \bar\lambda_*}}\, L, \qquad \beta_R = \sqrt{\frac{\bar\lambda}{\bar\lambda - \bar\lambda_*}}\, \beta$$

to obtain finite effective action

$$S_{\mathrm{mf}} = K_R\, L_R \sqrt{\beta_R^2 - \frac{\pi(d-2)}{3K_R}}, \qquad K_R = K_0(\bar\lambda - \bar\lambda_*)$$

The renormalized string tension K_R scales to finite

$$K_0 \to K_* + \frac{K_R^2}{2\Lambda^2\sqrt{d^2-2d}}, \qquad K_* = \left(d - 1 + \sqrt{d^2 - 2d}\right)\Lambda^2 \stackrel{d\to\infty}{\to} 2d\Lambda^2$$

reproducing the Alvarez-Arvis spectrum of continuum string

The average area is also finite

$$\langle Area \rangle = L_R \frac{\left(\beta_R^2 - \frac{\pi(d-2)}{6K_R}\right)}{\sqrt{\beta_R^2 - \frac{\pi(d-2)}{3K_R}}}$$

$\Rightarrow$ minimal area for large β_R and diverges if $\beta_R^2 \to \pi(d-2)/3K_R$

The Lilliputian world

Like for the zeta-function regularization except for nonlinearities, but

$$\text{length} = \sqrt{\frac{\bar\lambda - \bar\lambda_*}{\bar\lambda}}\,\text{length}_R \propto \frac{\sqrt{K_R}}{\Lambda}\,\text{length}_R$$

in target space which is of order of the cutoff ($\Rightarrow$ Lilliputian)

Nevertheless, the cutoff (in parameter space) $\boxed{\Delta\omega = 1/(\Lambda\sqrt[4]{g})}$ fixes maximal number of modes in the mode expansion to be

$$n_{\text{max}} \sim \Lambda\sqrt[4]{g}\omega_\beta$$

Classically $\quad \sqrt[4]{g}\omega_\beta = \beta \quad$ reproducing Brink-Nielsen (1973) but

Quantumly $\quad \sqrt[4]{g}\omega_\beta \propto \frac{\beta}{\sqrt{\bar\lambda - \bar\lambda_*}} = \frac{\sqrt{K_0}\,\beta}{\sqrt{K_R}} \quad$ is much larger

- Continuum because infinitely smaller distances can be probed (classical music can be played on the Lilliputian strings)

- Gulliver's tools are too coarse to resolve the Lilliputian world (this is why lattice string regularizations of 1980's never reproduce canonical quantization)

The KING of BROBDINGNAG, and GULLIVER.

—Vide. *Swift's Gulliver*: Voyage to Brobdingnag.

Pub. June 26th 1803. by H. Humphrey 27 St. James's Street

4. Instability of classical ground state

Semiclassical energy

Brink, Nielsen (1973)

Semiclassical (or one-loop) correction due to zero-point fluctuations

$$S_{1l} = \left[K_0 - \frac{(d-2)}{2}\Lambda^2\right] L\beta - \frac{\pi(d-2)L}{6\beta}$$

bulk term   Casimir energy

To make it finite, it is introduced the renormalized string tension

$$K_R = K_0 - \frac{(d-2)}{2}\Lambda^2$$

which is kept finite as $\Lambda \to \infty$. Then it is assumed it works order by order of the perturbative expansion about the classical ground state, so K_R can be made finite by fine tuning K_0.

We see however from the mean-field formula

$$S_{\text{mf}} = K_0 \bar{\lambda} L \sqrt{\beta^2 - \frac{\pi(d-2)}{3K_0\bar{\lambda}}}$$

that S_{mf} never vanishes with changing K_0 (except for $\beta = \beta_{\min}$). Thus the one-loop correction simply lowers for $d > 2$ the energy of the classical ground state which may indicate its instability.

Effective potential

To check stability of the ground state, add the source term like in QFT

$$S_{\text{src}} = \frac{K_0}{2} \int \mathrm{d}^2\omega \, j^{ab} \rho_{ab}$$

defining the field

$$\rho_{ab}(j) = -\frac{2}{K_0} \frac{\delta}{\delta j^{ab}} \log Z.$$

Minimizing for constant $j^{ab} = j\delta^{ab}$ we find Ambjørn, Y.M. (2017)

$$\bar{\lambda}(j) = \frac{1}{2}\left(1 + j + \frac{\Lambda^2}{K_0}\right) + \sqrt{\frac{1}{4}\left(1 + j + \frac{\Lambda^2}{K_0}\right)^2 - \frac{d\Lambda^2}{2K_0}}$$

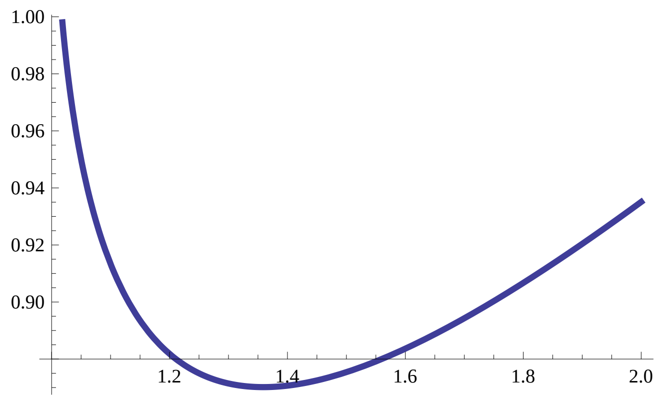
$$\bar{\rho}(j) = \frac{\bar{\lambda}(j)}{\sqrt{\left(1 + j + \frac{\Lambda^2}{K_0}\right)^2 - \frac{2d\Lambda^2}{K_0}}} \qquad \bar{\lambda}(\bar{\rho}) = \sqrt{\frac{d\Lambda^2}{2K_0}} \sqrt{\frac{\bar{\rho}}{\bar{\rho} - 1}}$$

in the mean field approximation for $\omega_L = L$ and $\omega_\beta = \beta \gg 1\sqrt{K_0}$,

Effective potential (cont.)

"Effective potential" is given by the Legendre transformation

$$\Gamma(\bar\rho) = -\frac{1}{K_0 L\beta}\log Z - j(\bar\rho)\bar\rho$$



In the mean-field approximation

$$\Gamma(\bar\rho) = \left(1+\frac{\Lambda^2}{K_0}\right)\bar\rho - \sqrt{\frac{2d\Lambda^2}{K_0}\bar\rho(\bar\rho-1)}$$

Classical vacuum $\bar\rho = 1$ is unstable and stable minimum occurs at

$$\bar\rho(0) = \bar\rho_{\mathrm{m.f.}} \qquad \text{if } K_0 > K_* \text{ (same value as before)}$$

Near the minimum (global stability)

$$\Gamma(\bar\rho) = \left[\left(1+\frac{\Lambda^2}{K_0}\right)^2 - \frac{2d\Lambda^2}{K_0}\right]^{1/2} + \frac{K_0}{2d\Lambda^2}\left[\left(1+\frac{\Lambda^2}{K_0}\right)^2 - \frac{2d\Lambda^2}{K_0}\right]^{3/2}[\bar\rho - \bar\rho(0)]^2$$

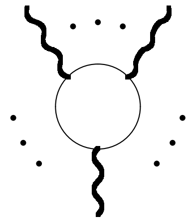
Nonlinearity $\Longrightarrow$ string susceptibility $\boxed{\gamma_{\mathrm{str}} = 1/2}$ for cylinder and torus
is quite different from $\boxed{\gamma_{\mathrm{str}} = 1}$ of KPZ-DDK

5. Fluctuations about mean field

Coleman-Weinberg potential

Integrating out X_q^μ we get (a part of) the effective action

$$\frac{d}{2}\,\mathrm{tr}\ln\left[-\frac{1}{\rho}\partial_a\lambda^{ab}\partial_b\right]_{\mathrm{reg}} \quad = \quad \sum_n \frac{1}{n}$$



wavy lines correspond to fluctuations $\delta\lambda^{ab}$ or $\delta\rho$ about ground state

$$\lambda^{ab}(\omega) = \bar\lambda\delta^{ab} + \delta\lambda^{ab}, \qquad \rho(\omega) = \bar\rho + \delta\rho$$

Same stability of wavy quadratic fluctuations about the mean field as about the classical ground state because of background independence. Positive definite quadratic form for imaginary $\delta\lambda^{ab}$ and real $\delta\rho$

Polyakov's book: typical $\delta\lambda \sim 1/\Lambda$ so λ^{ab} is localized and decouples. Thus only ρ fluctuates (stable fluctuations for $2 < d < 26$)

$$S^{(2)} = \frac{1}{16\pi b_0^2}\int\left[(\partial_a\varphi)^2 + 2\mu_0^2\,\mathrm{e}^{\varphi}\right], \qquad b_0^2 = \frac{6}{26-d}$$

The private life occurs at distances $\sim \Lambda^{-1}$ but is observable Y.M. (2021)

Path integrating over λ^{ab}

Simplified quadratic action $(\lambda^{z\bar{z}} = 0)$

$$\mathcal{S}^{(2)} = \int \left[\frac{1}{4\pi b_0^2} \partial\varphi\bar{\partial}\varphi + \nu \left(\lambda^{zz} \nabla\partial\varphi + \lambda^{\bar{z}\bar{z}} \bar{\nabla}\bar{\partial}\varphi \right) - d\Lambda^2 \bar{\rho}\, \mathrm{e}^{\varphi} \lambda^{zz} \lambda^{\bar{z}\bar{z}} \right]$$

Integrating out λ^{zz} and $\lambda^{\bar{z}\bar{z}}$

$$\mathcal{S}^{(2)} \;=\; \int \left[\frac{1}{4\pi b_0^2} \partial\varphi\bar{\partial}\varphi + \frac{\nu^2}{d\Lambda^2 \bar{\rho}} \mathrm{e}^{-\varphi} (\nabla\partial\varphi)(\bar{\nabla}\bar{\partial}\varphi) \right]$$

Integrating by parts (only these two terms are independent)

$$\mathcal{S}^{(2)} = \frac{1}{4\pi b_0^2} \int \left\{ \partial\varphi\bar{\partial}\varphi + 4\varepsilon\, \mathrm{e}^{-\varphi} \left[(\partial\bar{\partial}\varphi)^2 + \partial\varphi\bar{\partial}\varphi\partial\bar{\partial}\varphi \right] \right\}, \qquad \varepsilon = \frac{\pi \nu^2 b_0^2}{d\Lambda^2 \bar{\rho}}$$

modulo boundary terms

The first additional term appears for Polyakov's string from the Seeley expansion of the heat kernel but the second does not

Higher-derivative action

Integrating over X^μ, ghosts, regulators Y^μ, $\bar{Y}^\mu$, Z^μ and λ^{ab}

$$\mathcal{S} = \frac{1}{16\pi b_0^2}\int \sqrt{g}\left[-R\frac{1}{\Delta}R + 2m_0^2 + a^2 R\left(R + G g^{ab}\,\partial_a \frac{1}{\Delta}R\,\partial_b\frac{1}{\Delta}R\right)\right]$$

curvature squared R^2 (Polyakov) + nonlocal $G \neq 0$ (Nambu-Goto) or the beyond Liouville action

$$\mathcal{S} = \frac{1}{4\pi b_0^2}\int [\partial\varphi\bar{\partial}\varphi + \frac{\mu_0^2}{2}\,\mathrm{e}^{\varphi} + 4\varepsilon\,\mathrm{e}^{-\varphi}(\partial\bar{\partial}\varphi)^2 - 4G\varepsilon\,\mathrm{e}^{-\varphi}\partial\varphi\bar{\partial}\varphi\partial\bar{\partial}\varphi]$$

in conformal gauge $\rho_{ab} = \delta_{ab}\bar{\rho}\,\mathrm{e}^{\varphi}$ with worldsheet cutoff $\varepsilon = a^2/\bar{\rho}$ and $\mu_0^2 = m_0^2\bar{\rho}$

Classically higher-derivative terms vanish for smooth $\varepsilon R \ll 1$ while quartic derivative provides UV cutoff but also interaction with coupling ε $\Rightarrow$ uncertainties $\varepsilon \times \varepsilon^{-1}$ so they revive quantumly $\Rightarrow$ produce anomalies (yet higher terms do not change – universality)

Smallness of ε is compensated by change of the metric (shift of φ)

6. CFT á la KPZ-DDK

Review of KPZ-DDK

Knizhnik-Polyakov-Zamolodchikov (1988), David (1988), Distler-Kawai (1989)

Liouville action in fiducial (or background) metric $\hat{g}_{ab}$

$$S_L = \frac{1}{8\pi b^2} \int \sqrt{\hat{g}} \left(\frac{1}{2} \hat{g}^{ab} \partial_a \varphi \partial_b \varphi + q \hat{R} \varphi \right) + \mu^2 \int \sqrt{\hat{g}}\, \mathrm{e}^{\alpha\varphi}$$

$$b^2 = b_0^2 + \mathcal{O}(b_0^4), \quad q = 1 + \mathcal{O}(b_0^2), \quad \alpha = 1 + \mathcal{O}(b_0^2), \quad b_0^2 = \frac{6}{26-d}$$

are "renormalized" parameters of the effective action.
Energy-momentum pseudotensor

$$T_{zz} = \text{matter} + \text{ghosts} - \frac{1}{4b^2} \left(\partial_z \varphi \partial_z \varphi - 2q \partial_z^2 \varphi \right)$$

Background independence: the total central charge

$$d - 26 + 1 + 6\frac{q^2}{b^2} = 0$$

and the conformal weight

$$\text{weight}\,(\mathrm{e}^{\alpha\varphi}) = q\alpha - b^2\alpha^2 = 1$$

$$\Longrightarrow \qquad \alpha b = \sqrt{\frac{25-d}{24}} - \sqrt{\frac{1-d}{24}}, \qquad q = \alpha^{-1} + b^2\alpha$$

Energy-momentum tensor

For minimal coupling to gravity Gibbons, Pope, Solodukhin (2019) at G=0

$$-4b_0^2 T_{ab}^{(\min)} = \partial_a\varphi\partial_b\varphi - \frac{1}{2}g_{ab}\partial^c\varphi\partial_c\varphi - \mu_0^2 g_{ab} - \varepsilon\partial_a\varphi\partial_b\Delta\varphi - \varepsilon\partial_a\Delta\varphi\partial_b\varphi$$
$$+\varepsilon g_{ab}\partial^c\varphi\partial_c\Delta\varphi + \frac{\varepsilon}{2}g_{ab}(\Delta\varphi)^2 - G\varepsilon\partial_a\varphi\partial_b\varphi\Delta\varphi + G\frac{\varepsilon}{2}\partial_a\varphi\partial_b(\partial^c\varphi\partial_c\varphi)$$
$$+G\frac{\varepsilon}{2}\partial_a(\partial^c\varphi\partial_c\varphi)\partial_b\varphi - G\frac{\varepsilon}{2}g_{ab}\partial^c\varphi\partial_c(\partial^d\varphi\partial_d\varphi)$$

For diffeomorphism invariant action

$$-4b_0^2 T_{ab} = -4b_0^2 T_{ab}^{(\min)} - 2(\partial_a\partial_b - g_{ab}\partial^c\partial_c)(\varphi - \varepsilon\Delta\varphi + G\frac{\varepsilon}{2}g^{ab}\partial_a\varphi\partial_b\varphi)$$
$$+2G\varepsilon(\partial_a\partial_b - g_{ab}\partial^c\partial_c)\frac{1}{\Delta}\partial^d(\partial_d\varphi\Delta\varphi)$$

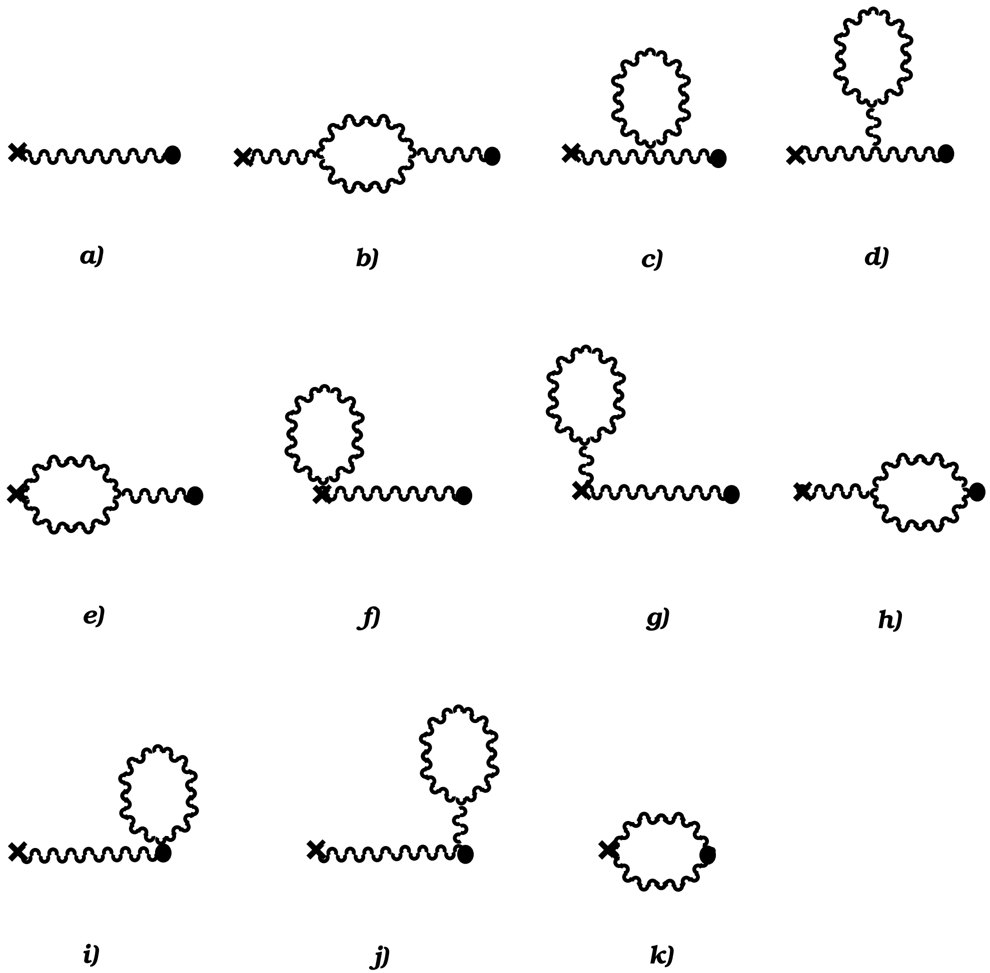
It is conserved and traceless (!) thanks to diffeomorphism invariance

T_{zz} component (in two dimensions) Kawai, Nakayama (1993) at G=0

$$4b_0^2 T_{zz} = (\partial\varphi)^2 - 2\varepsilon\partial\varphi\partial\Delta\varphi - 2\partial^2(\varphi - \varepsilon\Delta\varphi) - G\varepsilon(\partial\varphi)^2\Delta\varphi$$
$$+4G\varepsilon\partial\varphi\partial(\mathrm{e}^{-\varphi}\partial\varphi\bar\partial\varphi) - 4G\varepsilon\partial^2(\mathrm{e}^{-\varphi}\partial\varphi\bar\partial\varphi) + G\varepsilon\partial(\partial\varphi\Delta\varphi)$$
$$+G\varepsilon\frac{1}{\bar\partial}\partial^2(\bar\partial\varphi\Delta\varphi)$$

DDK for the beyond Liouville action

One-loop operator products $T_{zz}(z)\, e^{\alpha\varphi(0)}$ and $T_{zz}(z)T_{zz}(0)$



Diagrams a) to j) contribute $q\alpha$, diagrams k) contributes $-b^2\alpha^2$ to the conformal weight of $e^{\alpha\varphi(0)}$: $\quad 1 = q\alpha - b^2\alpha$ as before

One-loop central charge

Diagrams a) to j) contribute $6q^2/b^2$ to the central charge as usual.
Diagram k) contributes usual 1 to the central charge but
the nonlocal term in T_{zz} revives

$$2 \cdot \frac{1}{16} \left\langle 2G\varepsilon\partial^3 \frac{1}{\bar\partial} \bar\partial\varphi(z)\bar\partial\varphi(z) \left[\partial\varphi(0)\partial\varphi(0) - 8\varepsilon\partial\varphi(0)\partial^2\bar\partial\varphi(0)\right] \right\rangle$$
$$\to -\frac{G\pi}{2}\partial^3 \frac{1}{\bar\partial}\delta^{(2)}(z) = 3G\frac{1}{z^4}$$

modulo subtleties with conformal Ward identities for $G \neq 0$

The second DDK equation is modified (assuming one loop is exact)

$$-\frac{6}{b_0^2} + \frac{6q^2}{b^2} + 1 + 6Gq = 0 \quad \Rightarrow \quad \alpha b = \sqrt{\frac{25 - d - 6Gq}{24}} - \sqrt{\frac{1 - d + 6Gq}{24}}$$

Tremendous cancellations occur thanks to diffeomorphism invariance

7. Algebraic check of DDK

Salieri:

"I checked the harmony with algebra.
Then finally proficient in the science,
I risked the rare delights of creativity."

A. Pushkin, *Mozart and Salieri*

Pauli-Villars' regularization

Pauli-Villars' regulators: Grassmann Y, $\bar{Y}$ (M^2) and normal Z $(2M^2)$

$$\mathcal{S}_{\text{reg.}} = \frac{1}{16\pi b_0^2}\int\sqrt{g}\left[g^{ab}\partial_a Y\partial_b Y + M^2Y^2 + \varepsilon(\Delta Y)^2 + G\varepsilon g^{ab}\partial_a Y\partial_b Y R\right]$$

or in conformal gauge

$$\mathcal{S}_{\text{reg.}} = \frac{1}{4\pi b_0^2}\int\left[\partial Y\bar{\partial}Y + \frac{M^2}{4}\,\mathrm{e}^{\varphi}Y^2 + 4\varepsilon\,\mathrm{e}^{-\varphi}(\partial\bar{\partial}Y)^2 - 4G\varepsilon\,\mathrm{e}^{-\varphi}\partial Y\bar{\partial}Y\partial\bar{\partial}\varphi\right]$$

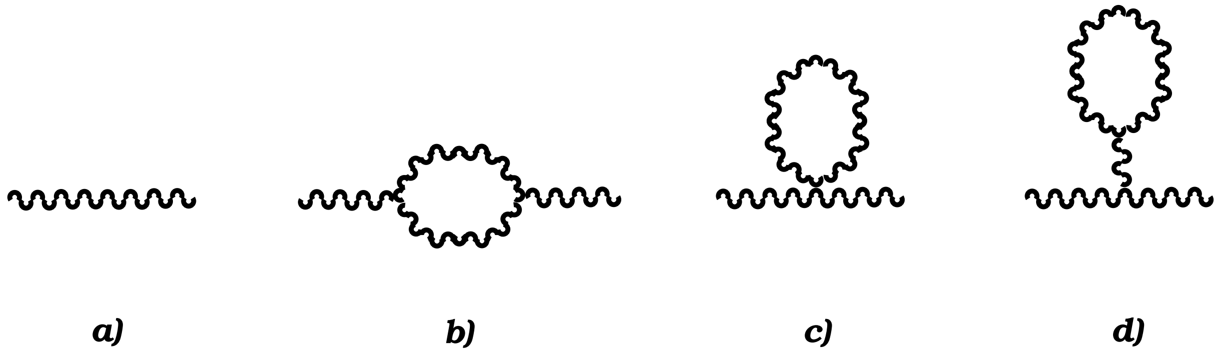
Conserved and traceless (!) energy-momentum tensor

$$\begin{aligned}-4b_0^2T_{ab}^{(\text{reg})} = {} & \partial_a Y\partial_b Y - \frac{1}{2}g_{ab}\partial^c Y\partial_c Y - \frac{M^2}{2}g_{ab}Y^2 - \varepsilon\partial_a Y\partial_b\Delta Y \\ & -\varepsilon\partial_a\Delta Y\partial_b Y + \varepsilon g_{ab}\partial^c Y\partial_c\Delta Y + \frac{\varepsilon}{2}g_{ab}(\Delta Y)^2 - G\varepsilon\partial_a Y\partial_b Y\Delta\varphi \\ & +G\frac{\varepsilon}{2}\partial_a\varphi\partial_b(\partial^c Y\partial_c Y) + G\frac{\varepsilon}{2}\partial_a(\partial^c Y\partial_c Y)\partial_b\varphi - G\frac{\varepsilon}{2}g_{ab}\partial^c\varphi\partial_c(\partial^d Y\partial_d Y) \\ & -G\varepsilon(\partial_a\partial_b - g_{ab}\partial^c\partial_c)(\partial^c Y\partial_c Y).\end{aligned}$$

$\Rightarrow$ conformal invariance expected to be maintained quantumly

$$\begin{aligned}-4b_0^2T_{zz}^{(\text{reg})} &= \partial Y\partial Y - 2\varepsilon\partial Y\partial\Delta Y - G\varepsilon\partial Y\partial Y\Delta\varphi + 4G\varepsilon\partial\varphi\partial(\,\mathrm{e}^{-\varphi}\partial Y\bar{\partial}Y) \\ & \quad -4G\varepsilon\partial^2(\,\mathrm{e}^{-\varphi}\partial Y\bar{\partial}Y)\end{aligned}$$

One-loop propagator



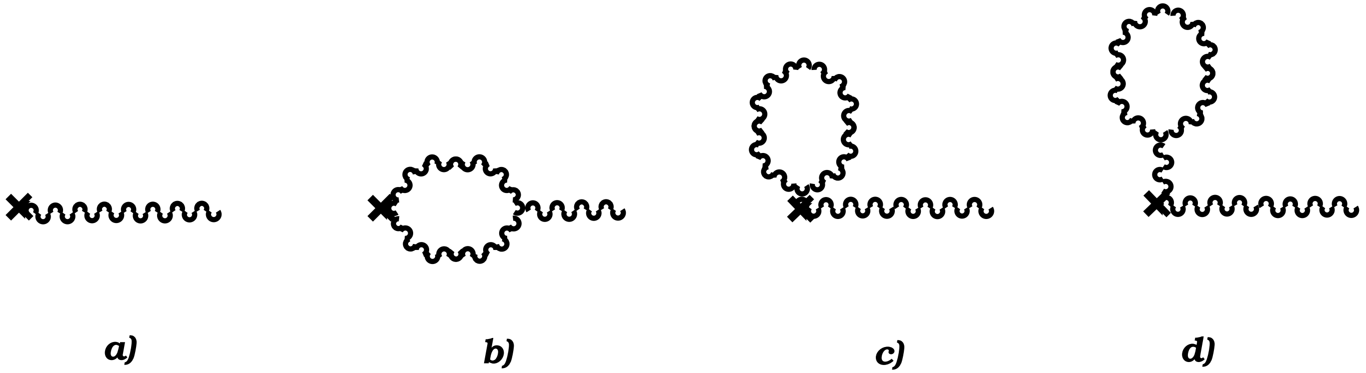
$$
\begin{aligned}
\text{b)} \;=\; & -\frac{1}{4}\int \frac{\mathrm{d}^2 k}{(2\pi)^2}\Bigg\{ \frac{\varepsilon^2 k^2 (k-p)^2}{(1+\varepsilon k^2)[1+\varepsilon(k-p)^2]} \\
& -2\frac{(\varepsilon k^2 (k-p)^2 - M^2)^2}{(k^2+M^2+\varepsilon k^4)[(k-p)^2+M^2+\varepsilon(k-p)^4]} \\
& +\frac{(\varepsilon k^2 (k-p)^2 - 2M^2)^2}{(k^2+2M^2+\varepsilon k^4)[(k-p)^2+2M^2+\varepsilon(k-p)^4]}\Bigg\}|\varphi(p)|^2 \\
\rightarrow \; & \text{b)}|_{\text{reg div}} - \frac{p^2}{96\pi}|\varphi(p)|^2
\end{aligned}
$$

One-loop renormalization of b^2 where $A(\varepsilon M^2) \sim \varepsilon M^2 =$ tadpole d)

$$
\frac{1}{b^2} = \frac{1}{b_0^2} - \left(\frac{1}{6} - 4 + A + 2G\int \mathrm{d}k^2 \frac{\varepsilon}{(1+\varepsilon k^2)} - \frac{1}{2}GA\right) + \mathcal{O}(b_0^2)
$$

One-loop renormalization of T_{zz}

One-loop renormalization of $T_{zz}^{(1)}$



$$\frac{q}{b^2} = \frac{1}{b_0^2} - \frac{1}{6} + 2 - \frac{1}{2}A - \frac{1}{2}G - G\int \mathrm{d}k^2 \frac{\varepsilon}{(1+\varepsilon k^2)} + \frac{1}{4}GA$$

or, multiplying by b^2,

$$\frac{q^2}{b^2} = \left(\frac{q}{b^2}\right)^2 \times b^2 = \frac{1}{b_0^2} - \frac{1}{6} - G + \mathcal{O}(b_0^2)$$

This precisely confirms the above shift of the central charge by $6G$ obtained by conformal field theory technique of DDK

$$-\frac{6}{b_0^2} + \frac{6q^2}{b^2} + 1 + 6G + \mathcal{O}(b_0^2) = 0$$

Conclusion

- Classical (perturbative) ground state is stable only for $d < 2$. For $2 < d < 26$ the mean-field ground state is stable instead
- Lilliputian strings for $d > 2$ versus Gulliver's strings for $d \leq 2$
- Higher-derivative terms in the beyond Liouville action for φ revive, telling the Nambu-Goto and Polyakov strings apart
- 2D conformal invariance is maintained by fluctuations in spite of ε but the central charge of φ gets additional $6G$ at one loop
- All that is specific to the theory with diffeomorphism invariance

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- All that is specific to the theory with diffeomorphism invariance

Final remark:

Large-d strings = bubble diagrams like $O(N)$ sigma model but
Large-d gravity = planar diagrams like Yang-Mills Strominger (1981)