Three loop spectral density and moments of photon polarization function in QED

Andrei Onishchenko
in collaboration with Roman Lee

BLTP JINR, SINP MSU, Budker INP

Dubna, 2022

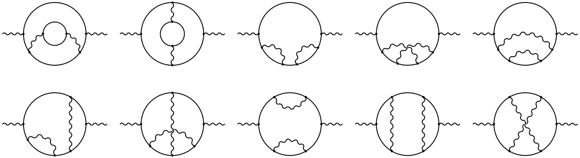
Outline

1. Polarization operator: calculational strategy
2. Master integrals: DE's and their solution
   - Polylogarithmic master integrals
   - Non-polylogarithmic master integrals
   - Iterated integrals
3. Results for spectral density
4. Asymptotics and cross-checks with moments
5. Conclusion

Polarization operator: calculational strategy

Photon polarization operator and its spectral density

$$\Pi(s) = \frac{1}{\pi}\int \frac{\rho(s')}{s'-s}$$



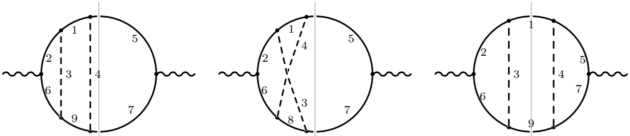
photon propagator has extra factor $\frac{1}{1+\Pi(s)}$

Strategy:

- IBP reduction. Construct differential equations for master integrals.
- Reduction of the differential equations to ϵ-form or regular basis.
- Fixing boundary conditions from threshold asymptotics.
- Constructing solution in terms of iterated integrals.

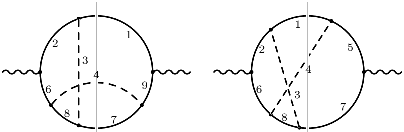
Polylogarithmic master integrals

(2, 0)-cuts



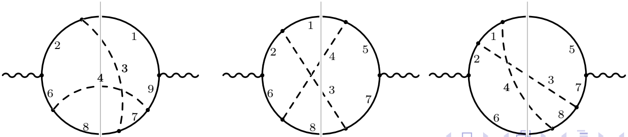
js[cut20,1,1,1,1,1,1,1,0,1] js[cut20,1,1,1,1,1,1,1,1,0] js[cut20a,1,1,1,1,1,1,1,0,1]

(2, 1)-cuts



js[cut21,1,1,1,1,0,1,1,1,1] js[cut21,1,1,1,1,1,1,1,1,0]

(2, 2)-cuts



Polylogarithmic master integrals

Solution via transformation to ϵ-form:

$$\mathrm{J}_{\mathrm{IBP}}(\beta) = e^{-3\epsilon\gamma_E} T(\beta)\mathrm{J}_{\mathrm{canonical}}(\beta)$$
$$= T(\beta)\mathrm{Pexp}\left[\epsilon\int_0^\beta S(t)dt\right] L \cdot \mathrm{c}e^{-3\epsilon\gamma_E},$$

where $\beta = \sqrt{1-4m^2/s}$, T is the transformation matrix to the canonical basis, and S is the matrix entering DE's for the canonical master integrals

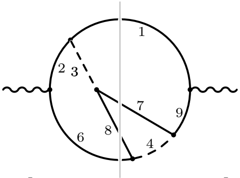
$$\partial_\beta \mathrm{J}_{\mathrm{canonical}}(\beta) = \epsilon S(\beta)\mathrm{J}_{\mathrm{canonical}}(\beta),$$

c is the column of the coefficients in threshold asymptotic expansions of the IBP master integrals, and L is a rational matrix depending on ϵ.

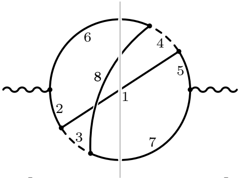
R.Lee, A.O. 2021

Non-polylogarithmic master integrals

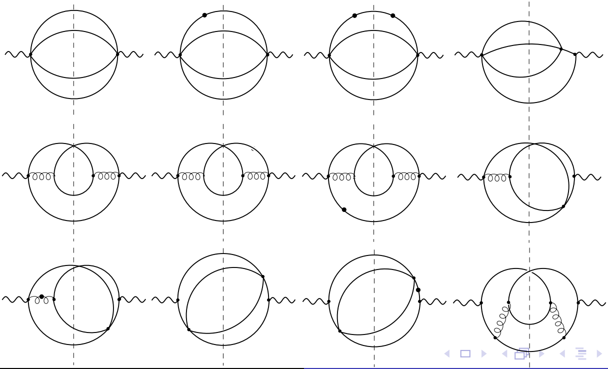
$(4,0)$-cuts



js[cut40,1,1,1,1,0,1,1,1,1]



js[cut40,1,1,1,1,1,1,1,1,0]



Non-polylogarithmic master integrals: ϵ regular basis

ϵ regular basis similar to ϵ finite basis

Chetyrkin, Faisst, Sturm, Tentyukov 2006

but we require master integrals $F_1, \ldots F_M$ satisfy in addition

1. The ϵ-expansion of each F_m starts from ϵ^0, i.e., $F_m = F_m^{(0)} + O(\epsilon)$.
2. The leading terms $F_1^{(0)}, \ldots, F_M^{(0)}$ are linearly independent (in the above sense).

If we have vanishing combination $F_{m_0}^{(0)} - \sum_{m<m_0} c_m F_m^{(0)} = 0$ then we redefine master integral in the basis as

$$F_{m_0} \to \frac{1}{\epsilon}[F_{m_0} - \sum_{m<m_0} c_m F_m] \, .$$

R.Lee, A.O. 2019

Non-polylogarithmic master integrals: ϵ regular basis

Masters from ϵ-regular basis $\mathrm{F}^{(0)} = (F_1^{(0)}, \ldots, F_{14}^{(0)})^\intercal$ satisfy

$$\partial_s \mathrm{F}^{(0)} = M_F \mathrm{F}^{(0)}$$

where

$$M_F = \begin{pmatrix}
\frac{3}{2(s-16)} & -\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{2(s-16)} & -\frac{1}{s} & -\frac{2}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{16s\beta^2} & \frac{s-16}{8s^2\beta^2} & -\frac{s+8}{2s^2\beta^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{3\beta}{2(s-16)} & -\frac{1}{s\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{s\beta^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{s\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2s\beta} & 0 & 0 & -\frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{s\beta} & 0 & -\frac{1}{s\beta} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{s^2\beta^2} & -\frac{3}{4s\beta} & -\frac{4}{s^2\beta^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{4s\beta} & 0 & \frac{1}{s\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{s\beta} & \frac{1}{s\beta} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{s} & 0 & 0 & \frac{1}{s\beta} & 0 & 0
\end{pmatrix}$$

R.Lee, A.O. 2019

Solution of elliptic block:

$$F_1^{(0)}(s) = (s-16) j_1|_{d=2} = \frac{16\pi(s-16)}{s}\left[K(1-k_-)K(k_+) - K(k_-)K(1-k_+)\right],$$

$$F_2^{(0)}(s) = \frac{3s}{2(s-16)} F_1^{(0)}(s) - s\frac{d}{ds}F_1^{(0)}(s),$$

$$F_3^{(0)}(s) = \frac{12s}{(s-16)^2} F_1^{(0)}(s) + \frac{s(s-64)}{4(s-16)}\frac{d}{ds}F_1^{(0)}(s) + \frac{s^2}{2}\frac{d^2}{ds^2}F_1^{(0)}(s)$$

Primo, Tancredi, 2017

We will write results in terms of iterated integrals

$$\mathrm{I}_{w_n,\ldots w_1} = I(w_n,\ldots w_1|\bar{s}) = \int\limits_{\bar{s}>s_n>\ldots>s_1>4} \prod_{k=1}^{n} ds_k\, w_k(s_k)\,,$$

where

$$l_0(\bar{s}) = \frac{1}{\bar{s}}, \quad l_1(\bar{s}) = \frac{1}{\bar{s}\beta}, \quad l_2(\bar{s}) = \frac{1}{\bar{s}-4}, \quad l_4(\bar{s}) = \frac{1}{\bar{s}-1}\,,$$

$$r_k(\bar{s}) = \frac{f(\bar{s})}{\bar{s}\beta^k}\theta(\bar{s}-16) \quad (k=0,1,2,3),$$

$$\tilde{r}_3(\bar{s}) = \frac{8\beta(\bar{s}+2)f(\bar{s})}{(\bar{s}-16)(\bar{s}-4)^2}\theta(\bar{s}-16)\,,$$

and

$$f(\bar{s}) = \frac{16(\bar{s}-16)}{\bar{s}}\left[\mathrm{K}(1-k_-)\mathrm{K}(k_+) - \mathrm{K}(k_-)\mathrm{K}(1-k_+)\right],$$

$$k_\pm = \frac{1}{2}\left[1 \pm \left(1-\frac{8}{\bar{s}}\right)\sqrt{1-\frac{16}{\bar{s}}} + \frac{16}{\bar{s}}\sqrt{1-\frac{4}{\bar{s}}}\right].$$

In on-shell scheme for photon spectral density we get:

$$\rho(s) = \rho^{(1)}(s)\left(\frac{\alpha}{4\pi}\right) + \rho^{(2)}(s)\left(\frac{\alpha}{4\pi}\right)^2 + \rho^{(3)}(s)\left(\frac{\alpha}{4\pi}\right)^3 + \dots ,$$

where ($\bar{s} = \frac{s}{m^2}$, $\beta = \sqrt{1 - \frac{4}{\bar{s}}}$):

$$\rho^{(1)}(s) = \frac{4N\pi(2+\bar{s})\beta}{3\bar{s}} ,$$

$$\begin{aligned}\rho^{(2)}(s) = &\frac{16N\pi}{3}\Big\{\beta^2\left(2\mathrm{II}_{l_1,l_2} + 2\mathrm{II}_{l_2,l_1} + \mathrm{II}_{l_1,l_0} + \mathrm{II}_{l_0,l_1}\right) - \frac{\beta(2+\bar{s})}{\bar{s}}\left(\mathrm{II}_{l_0} + 2\mathrm{II}_{l_2}\right) \\ &- \frac{7 + 8\log 2 + \bar{s}(2 - (3 + \log 4)\bar{s})}{\bar{s}^2}\mathrm{II}_{l_1} \\ &+ \frac{4\pi^2(\bar{s}^2 - 4) - \beta\bar{s}(8(2+\bar{s})\log 2 - 3(6+\bar{s}))}{4\bar{s}^2}\Big\} .\end{aligned}$$

and N is the number of electron species

Photon spectral density in on-shell scheme

Three-loop contribution $\rho^{(3)}(s)$ can be naturally separated into two pieces corresponding to $2m$ and $4m$ cuts

$$\rho^{(3)}(s) = \rho^{(3)}_{2m}(s) + \theta(s-16m^2)\rho^{(3)}_{4m}(s)\,,$$

where

$$\begin{aligned}
\rho^{(3)}_{4m}(s) =& \frac{2\pi N^2}{9}\Big\{ -\frac{32}{27}c_{24}f''(\bar{s}) + \frac{4}{27}c_{32}f'(\bar{s}) - \frac{1}{9}c_{33}f(\bar{s}) + \frac{1}{3}c_9\left(2\mathrm{I}\mathrm{I}_{r_2} - \mathrm{I}\mathrm{I}_{l_1,\tilde{r}_3}\right) - \frac{2}{9}c_{28}\mathrm{I}\mathrm{I}_{\tilde{r}_3}\Big\} \\
&+ \frac{2\pi N}{9}\Big\{ -48c_1f''(\bar{s}) + 2c_{29}f'(\bar{s}) - \frac{1}{2}c_{31}f(\bar{s}) + 2c_7\left(\mathrm{I}\mathrm{I}_{r_1} - \mathrm{I}\mathrm{I}_{l_0,\tilde{r}_3}\right) + 3c_{14}\mathrm{I}\mathrm{I}_{\tilde{r}_3} \\
&+ c_{19}\left(\frac{3}{2}\mathrm{I}\mathrm{I}_{l_1,\tilde{r}_3} - 3\mathrm{I}\mathrm{I}_{r_2}\right) + c_8\left(2\mathrm{I}\mathrm{I}_{l_2,\tilde{r}_3} - 4\mathrm{I}\mathrm{I}_{r_3}\right) + c_{13}\left(2\mathrm{I}\mathrm{I}_{l_0,l_1,\tilde{r}_3} - 4\mathrm{I}\mathrm{I}_{l_0,r_2}\right) + 2c_{16}\mathrm{I}\mathrm{I}_{r_0} \\
&+ c_{26}\left(-2\mathrm{I}\mathrm{I}_{l_1,l_1,\tilde{r}_3} - \mathrm{I}\mathrm{I}_{l_1,r_0} + 4\mathrm{I}\mathrm{I}_{l_1,r_2}\right) + 2c_{18}\left(-\mathrm{I}\mathrm{I}_{l_1,l_0,\tilde{r}_3} + \mathrm{I}\mathrm{I}_{l_1,l_2,\tilde{r}_3} + \mathrm{I}\mathrm{I}_{l_1,r_1} - 2\mathrm{I}\mathrm{I}_{l_1,r_3}\right)\Big\}
\end{aligned}$$

R.Lee, A.O. 2022

Photon spectral density in on-shell scheme

$$
\begin{aligned}
\rho^{(3)}_{2m}(s) = &\frac{16\pi N^2}{3}\left\{ -\frac{4c_{25}}{27} + \frac{2}{27}c_{30}\mathrm{II}_{l_1} + \frac{4}{9}c_{27}\mathrm{II}_{l_1,l_1} - \frac{2}{3}c_9\mathrm{II}_{l_1,l_1,l_1}\right\} + \frac{16\pi N}{3}\left\{\frac{c_{45}}{12} + \frac{1}{2}c_{36}\mathrm{II}_{l_0}\right.\\
&- \frac{1}{24}c_{46}\mathrm{II}_{l_1} - c_{35}\mathrm{II}_{l_2} + \frac{1}{6}c_{37}\mathrm{II}_{l_0,l_1} - \frac{1}{4}c_{39}\mathrm{II}_{l_1,l_0} - c_{34}\mathrm{II}_{l_1,l_1} + \frac{1}{2}c_{38}\mathrm{II}_{l_1,l_2} + 4c_{43}\mathrm{II}_{l_2,l_1}\\
&+ 8c_2\left(\mathrm{II}_{l_0,l_0} + 2\left(\mathrm{II}_{l_0,l_2} + \mathrm{II}_{l_2,l_0} + 2\mathrm{II}_{l_2,l_2}\right)\right) + 2c_{15}\mathrm{II}_{l_0,l_0,l_1} - 2c_5\mathrm{II}_{l_0,l_1,l_1}\\
&+ 4c_3\left(\mathrm{II}_{l_0,l_1,l_0} + 2\mathrm{II}_{l_0,l_1,l_2}\right) - 2c_{42}\mathrm{II}_{l_1,l_0,l_1} + c_{44}\mathrm{II}_{l_1,l_1,l_1} + 4c_{40}\left(\mathrm{II}_{l_1,l_1,l_0} + 2\mathrm{II}_{l_1,l_1,l_2}\right)\\
&- 4c_{41}\mathrm{II}_{l_1,l_2,l_1} - 4c_6\mathrm{II}_{l_2,l_1,l_1} - 8c_{12}\left(\mathrm{II}_{l_1,l_0,l_0} + 2\mathrm{II}_{l_1,l_0,l_2} + 2\mathrm{II}_{l_1,l_2,l_0} + 4\mathrm{II}_{l_1,l_2,l_2}\right.\\
&\left.+\mathrm{II}_{l_2,l_0,l_1} - \mathrm{II}_{l_2,l_1,l_0} - 2\mathrm{II}_{l_2,l_1,l_2} - 2\mathrm{II}_{l_2,l_2,l_1}\right) - 8c_{17}\left(\mathrm{II}_{l_0,l_2,l_1} + \mathrm{II}_{l_0,l_1,l_1,l_1}\right)\\
&+ c_{21}\mathrm{II}_{l_1,l_1,l_0,l_1} + c_{11}\left(3\left(2\log(2)\mathrm{II}_{l_4,l_1} + \mathrm{II}_{l_4,l_1,l_0} + 2\mathrm{II}_{l_4,l_1,l_2}\right) - \mathrm{II}_{l_4,l_0,l_1}\right)\\
&+ c_{10}\left(3\log(2)\mathrm{II}_{l_0,l_4,l_1} - \frac{1}{2}\mathrm{II}_{l_0,l_4,l_0,l_1} + \frac{3}{2}\mathrm{II}_{l_0,l_4,l_1,l_0} + 3\mathrm{II}_{l_0,l_4,l_1,l_2}\right) - 2c_{22}\mathrm{II}_{l_1,l_2,l_1,l_1}\\
&+ 2c_4\left(\mathrm{II}_{l_1,l_1,l_1,l_0} + 2\mathrm{II}_{l_1,l_1,l_1,l_2}\right) + c_{26}\left(-18\log(2)\mathrm{II}_{l_1,l_4,l_1} - \mathrm{II}_{l_1,l_0,l_0,l_1} - 2\mathrm{II}_{l_1,l_0,l_1,l_0}\right.\\
&- 4\mathrm{II}_{l_1,l_0,l_1,l_2} + 4\mathrm{II}_{l_1,l_0,l_2,l_1} + 4\mathrm{II}_{l_1,l_1,l_0,l_0} + 8\mathrm{II}_{l_1,l_1,l_0,l_2} + 4\mathrm{II}_{l_1,l_1,l_1,l_1} + 8\mathrm{II}_{l_1,l_1,l_2,l_0}\\
&+ 16\mathrm{II}_{l_1,l_1,l_2,l_2} + 4\mathrm{II}_{l_1,l_2,l_0,l_1} - 4\mathrm{II}_{l_1,l_2,l_1,l_0} - 8\mathrm{II}_{l_1,l_2,l_1,l_2} - 8\mathrm{II}_{l_1,l_2,l_2,l_1} + 3\mathrm{II}_{l_1,l_4,l_0,l_1}\\
&\left.\left.-9\mathrm{II}_{l_1,l_4,l_1,l_0} - 18\mathrm{II}_{l_1,l_4,l_1,l_2}\right) - c_{23}\mathrm{II}_{l_1,l_0,l_1,l_1} + 2c_{20}\mathrm{II}_{l_1,l_1,l_2,l_1}\right\}
\end{aligned}
$$

R.Lee, A.O. 2022

and c_i coefficients are rational functions in β:

$$c_1 = -\frac{29}{8(\beta+1)} + \frac{15}{4(\beta+1)^2} + \frac{29}{8(\beta-1)} + \frac{15}{4(\beta-1)^2} - \frac{1}{4}\,,\ c_2 = \frac{3\beta}{4} - \frac{\beta^3}{4}\,,$$
$$c_3 = -\frac{\beta^4}{4} + \frac{\beta^2}{2} - \frac{1}{4}\,,\ c_4 = -\frac{3\beta^4}{4} + \frac{7\beta^2}{2} - \frac{11}{4}\,,\ c_5 = -\frac{69\beta^3}{4} + \frac{113\beta}{4} + \frac{3}{\beta}\,,$$
$$c_6 = -\frac{15\beta^3}{2} + \frac{35\beta}{4} + \frac{3}{4\beta}\,,\ c_7 = -\frac{15\beta^3}{4} + 8\beta + \frac{3}{4\beta}\,,\ c_8 = -\frac{9\beta^3}{4} + \frac{7\beta}{2} + \frac{3}{4\beta}\,,$$
$$c_9 = -\frac{5\beta^4}{4} + \frac{5\beta^2}{2} + \frac{3}{4}\,,\ c_{10} = -\frac{3\beta^4}{4} + \frac{\beta^2}{2} + \frac{17}{4}\,,\ c_{11} = -\frac{\beta^4}{2} + \frac{11\beta^2}{4} + \frac{23}{4}\,,$$
$$c_{12} = -\frac{\beta^4}{4} + \frac{\beta^2}{2} + \frac{3}{4}\,,\ c_{13} = -\frac{\beta^4}{4} + \frac{\beta^2}{2} + \frac{7}{4}\,,\ c_{14} = -\frac{\beta^3}{4} + \frac{3\beta}{4} + \frac{2}{\beta}\,,$$
$$c_{15} = -\frac{\beta^4}{4} + \beta^2 + \frac{5}{4}\,,\ c_{16} = \frac{9}{4} - \frac{\beta^4}{4}\,,\ c_{17} = \frac{5}{4} - \frac{\beta^4}{4}\,,\ c_{18} = -\frac{\beta^4}{4} + \frac{3\beta^2}{4} - \frac{3}{2}\,,$$
$$c_{19} = -\frac{5\beta^4}{4} + \frac{9\beta^2}{2} - \frac{5}{4}\,,\ c_{20} = -\frac{7\beta^4}{4} + \frac{13\beta^2}{2} - \frac{35}{4}\,,\ c_{21} = -\frac{9\beta^4}{4} + 8\beta^2 - \frac{47}{4}\,,\ldots$$

Asymptotics of spectral densities

It is not hard to determine asymptotics of spectral densities

$$
\begin{aligned}
\rho^{(1)}_{thr}(s) &= \frac{2}{3}N\pi\beta(3-\beta^2)\,,\\
\rho^{(1)}_{high}(s) &= \frac{4N\pi}{3}\left\{1-\frac{6}{\bar{s}^2}-\frac{8}{\bar{s}^3}-\frac{18}{\bar{s}^4}\right\}+O\left(\frac{1}{\bar{s}^5}\right)\,,\\
\rho^{(2)}_{thr}(s) &= 4N\pi\left\{\pi^2-8\beta+\frac{2\pi^2\beta^2}{3}-\frac{\pi^2\beta^4}{3}\right.\\
&\left.+\frac{4}{9}\beta^3(-37+36\log 2+24\log\beta)\right\}+O\left(\beta^5\right)\,,\\
\rho^{(2)}_{high}(s) &= 4N\pi\left\{1+\frac{12}{\bar{s}}+\frac{2(5+12\log\bar{s})}{\bar{s}^2}\right.\\
&\left.+\frac{16(-47+87\log\bar{s})}{27\bar{s}^3}+\frac{-983+1218\log\bar{s}}{9\bar{s}^4}\right\}+O\left(\frac{1}{\bar{s}^5}\right)
\end{aligned}
$$

$$\rho^{(3)}_{2m,thr}(s) = \frac{32N^2\pi}{9}\left\{4(11-\pi^2)\beta - \frac{1}{9}(245-24\pi^2)\beta^3\right\}$$
$$+ \frac{8N\pi}{9}\left\{\frac{3\pi^4}{\beta} - 72\pi^2 + (351 - 70\pi^2 + 5\pi^4 + 48\pi^2\log 2 - 24\pi^2\log\beta - 36\zeta_3)\beta\right.$$
$$+ 2(-43\pi^2 + 24\pi^2\log 2 + 48\pi^2\log\beta)\beta^2$$
$$+ (1411 + 51\pi^2 + \pi^4 - 1152\log 2 - 42\pi^2\log 2 - (768 - 20\pi^2)\log\beta + 117\zeta_3)\beta^3$$
$$\left. + \frac{8}{75}(-947\pi^2 + 480\pi^2\log 2 + 960\pi^2\log\beta)\beta^4\right\} + O\left(\beta^5\right) ,$$

$$\rho^{(3)}_{4m,thr}(s) = \frac{\pi^2N^2(\bar{s}-16)^{9/2}}{516096}\left\{1 - \frac{629(\bar{s}-16)}{2640} + \frac{10243(\bar{s}-16)^2}{274560} - \frac{7973(\bar{s}-16)^3}{1647360}\right\}$$
$$+ \frac{\pi^2N(\bar{s}-16)^{9/2}}{5160960}\left\{1 - \frac{7(\bar{s}-16)}{48} + \frac{307(\bar{s}-16)^2}{27456} - \frac{193(\bar{s}-16)^3}{658944}\right\} + O\left(\bar{s}-16\right)^{17/2} ,$$

$$\rho^{(3)}_{2m,high}(s) = \frac{16N^2\pi}{81}\left\{766 - 66\pi^2 - 265\log\bar{s} + 12\pi^2\log\bar{s} + 57\log^2\bar{s} - 6\log^3\bar{s}\right.$$
$$\left. + \frac{2}{\bar{s}}(-65 + 24\pi^2 - 216\log\bar{s} + 108\log^2\bar{s})\right\} + \frac{2N\pi}{135}\left\{16065 - 900\pi^2 + 76\pi^4\right.$$
$$- 1440\pi^2\log 2 - 1440\zeta_3 + \log\bar{s}(-2340 + 360\pi^2 - 1440\zeta_3) + \frac{1}{\bar{s}}(-35100 + 3600\pi^2$$
$$- 114\pi^4 - 11520\zeta_3 + \log\bar{s}(-7200 + 240\pi^2 + 1440\zeta^3) + (-7200 + 60\pi^2)\log^2\bar{s}$$
$$\left. + 240\log^3\bar{s} - 30\log^4\bar{s})\right\} + O\left(\frac{1}{\bar{s}^2}\right) ,$$

$$\rho^{(3)}_{4m,high}(s) = \frac{8N^2\pi}{81}\left\{ -1829 + 132\pi^2 + 216\zeta_3 + (584 - 24\pi^2)\log\bar{s} - 114\log^2\bar{s} + 12\log^3\bar{s}\right.$$
$$\left. + \frac{1}{\bar{s}}(-1144 - 96\pi^2 + 1512\log\bar{s} - 432\log^2\bar{s})\right\} + \frac{4N\pi}{135}\left\{ -8100 + 450\pi^2\right.$$
$$- 38\pi^4 + 720\pi^2\log 2 + 720\zeta_3 + (1170 - 180\pi^2 + 720\zeta_3)\log\bar{s} + \frac{1}{\bar{s}}(18360$$
$$- 1800\pi^2 + 57\pi^4 + 5760\zeta_3 - (6120 + 120\pi^2 + 720\zeta_3)\log\bar{s}$$
$$\left. + (3600 - 30\pi^2)\log^2\bar{s} - 120\log^3\bar{s} + 15\log^4\bar{s})\right\} + O\left(\frac{1}{\bar{s}^2}\right) .$$

Cross-checks with moments

The moments of photon correlation function are given by:

$$\Pi(\bar{s}) = \sum_{n>0} M_n \left(\frac{\bar{s}}{4}\right)^n ,$$

where

$$M_n = M_n^{(1)} \left(\frac{\alpha}{4\pi}\right) + M_n^{(2)} \left(\frac{\alpha}{4\pi}\right)^2 + M_n^{(3)} \left(\frac{\alpha}{4\pi}\right)^3 + \dots$$

and

$$M_1^{(1)} = \frac{16N}{15}, \qquad M_1^{(2)} = \frac{1312N}{81},$$

$$M_2^{(1)} = \frac{16N}{35}, \qquad M_2^{(2)} = \frac{7184N}{675},$$

$$M_3^{(1)} = \frac{256N}{945}, \qquad M_3^{(2)} = \frac{3998656N}{496125},$$

$$M_4^{(1)} = \frac{128N}{693}, \qquad M_4^{(2)} = \frac{831776N}{127575},$$

$$M_5^{(1)} = \frac{2048N}{15015}, \qquad M_5^{(2)} = \frac{6918163456N}{1260653625}.$$

Cross-checks with moments

$$M_1^{(3)} = N^2\left(\frac{406\zeta_3}{27} - \frac{45628}{729} + \frac{256\pi^2}{45}\right) + N\left(\frac{22781\zeta_3}{108} - \frac{8687}{54} + \frac{32\pi^2}{3} - \frac{256}{15}\pi^2\log(2)\right),$$

$$M_2^{(3)} = N^2\left(\frac{14203\zeta_3}{1152} - \frac{1520789}{25920} + \frac{512\pi^2}{105}\right) + N\left(\frac{4857587\zeta_3}{2880} - \frac{223404289}{116640} + \frac{64\pi^2}{7} - \frac{512}{35}\pi^2\log(2)\right),$$

$$M_3^{(3)} = N^2\left(\frac{12355\zeta_3}{864} - \frac{83936527}{1458000} + \frac{4096\pi^2}{945}\right) + N\left(\frac{33067024499\zeta_3}{3225600} - \frac{885937890461}{72576000} + \frac{512\pi^2}{63} - \frac{4096}{315}\pi^2\log(2)\right),$$

$$M_4^{(3)} = N^2\left(\frac{2522821\zeta_3}{147456} - \frac{129586264289}{2239488000} + \frac{8192\pi^2}{2079}\right) + N\left(\frac{1507351507033\zeta_3}{25804800} - \frac{269240669884818833}{3840721920000} + \frac{5120\pi^2}{693} - \frac{8192}{693}\pi^2\log(2)\right),$$

$$M_5^{(3)} = N^2\left(\frac{1239683\zeta_3}{61440} - \frac{512847330943}{8692992000} + \frac{32768\pi^2}{9009}\right) + N\left(\frac{939939943788973\zeta_3}{2980454400} - \frac{360248170450504167133}{950578675200000} + \frac{20480\pi^2}{3003} - \frac{32768\pi^2\log(2)}{3003}\right).$$

P.Baikov, D. Broadhurst 1995, R.Lee, A.O. 2022

Conclusion

TODO:

- Further develop techniques for master integrals with elliptics
- Obtain photon spectral density in the case of QCD
- Further application of the presented techniques to physically interesting problems

Thank you for your attention!