Geodesic Deviation and Tidal Accelerations in Gravitational Waves of the Bianchi Universes

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The **Hamilton-Jacobi equation** for a test particle of mass m in a gravitational field with the metric tensor $g_{\alpha\beta}(x^\gamma)$ has the form:

$$g^{\alpha\beta}\frac{\partial S}{\partial x^\alpha}\frac{\partial S}{\partial x^\beta} = m^2c^2, \tag{1}$$

$$\alpha, \beta, \gamma ... = 0, ..., (n-1),$$

where the capital letter S denotes the test particle action function, n is the space dimension.
We will set the speed of light c to be equal to unity.

The Shapovalov wave spacetimes considered in this report admit the existence of "privileged" coordinate systems, where the complete integral of the Hamilton-Jacobi equation for test particle can be written in the "separated" form:

$$S = \phi_0(x^0, \lambda_k) + \phi_1(x^1, \lambda_k) + \phi_2(x^2, \lambda_k) + \phi_2(x^2, \lambda_k) + \phi_3(x^3, \lambda_k), \quad (2)$$

where λ_k are independent constant parameters.
Moreover, one of the separated variables is **null (wave) variable**, i.e. the space-time interval along this variable is zero.

The Shapovalov wave spacetimes 2

Thus, in the Shapovalov space-times, one can obtain the complete integral of the Hamilton-Jacobi equation for test particles. Since the test particles move in the gravitational field along geodesic lines, we get a complete integral for the geodesics.

There are three classes of Shapovalov space-times according to the number of commuting Killing vectors they admit, or, which is the same, according to the number of variables on which the metric in the privileged coordinate system depends. For example, type III Shapovalov spaces admit 3 commuting Killing vectors in "complete set".

The particle trajectory equations

The test particle coordinates x^α are functions of proper time τ on the base geodesic line along which the test particle moves, and is given in the Hamilton-Jacobi formalism by the particle trajectory equations in the form:

$$\frac{\partial S(x^\alpha, \lambda_k)}{\partial \lambda_j} = \sigma_j, \qquad \tau = S(x^\alpha, \lambda_k)\Big|_{m=1}, \tag{3}$$

$$\alpha, \beta, \gamma = 0, 1, 2, 3; \qquad i, j, k = 1, 2, 3;$$

where λ_k, σ_k are independent constant parameters determined by the initial (or boundary) data for the motion of a test particle along the base geodesic line, the variable τ is the proper time of the particle.

On the other hand, the presence of a complete integral for the equations of geodesics makes it possible to integrate **the equations of geodesic deviation**:

$$\frac{D^2\eta^\alpha}{d\tau^2} = R^\alpha{}_{\beta\gamma\delta} u^\beta u^\gamma \eta^\delta. \tag{4}$$

where τ is the proper time along the base geodesic line, $R^\alpha{}_{\beta\gamma\delta}$ is the Riemann curvature tensor, $u^\alpha = dx^\alpha/d\tau$ is the four-velocity vector along the base geodesic, and $\eta^\alpha(\tau)$ is the deviation vector of infinitely close geodesics.

Exact solution of the geodesic deviation equation for the model

We can construct the general solution of the geodesic deviation equation through the complete integral of the Hamilton-Jacobi equation.

The deviation vector η^α can be found as a solution of a linear algebraic system of equations on the trajectory of the "base" test particle

$$\eta^\alpha \frac{\partial u_\alpha(x^\beta, \lambda_j)}{\partial \lambda_i} + \rho_k \frac{\partial^2 S(x^\beta, \lambda_j)}{\partial \lambda_i \partial \lambda_k} = \vartheta_i, \tag{5}$$

$$u_\alpha(x^\beta, \lambda_i)\, \eta^\alpha = 0, \tag{6}$$

$$i, j, k = 1 \dots 3; \qquad \alpha, \beta, \gamma = 0 \dots 3,$$

where λ_i, ρ_i, ϑ_i are independent constant parameters.

Shapovalov type III wave space-time metric in a privileged coordinate system depends on one wave variable and can be reduced to the following form:

$$ds^2 = 2dx^0dx^1 + g_{ab}(x^0)\Big(dx^a + g^a(x^0)\,dx^1\Big)\Big(dx^b + g^b(x^0)\,dx^1\Big), \qquad (7)$$

where indices a, b run through the values 2, 3. Thus, in the general case, the metric includes 5 arbitrary functions of the wave variable x^0.
Einstein's equations with cosmological constant Λ in vacuum

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} \qquad (8)$$

for the metric (7) lead to the following necessary conditions:

$$\Lambda = g^a = 0. \qquad (9)$$

The space-time metric for a gravitational wave (7) has type N according to Petrov's classification.

Bianchi type VII Shapovalov wave space metric in a privileged coordinate system can be represented as:

$$ds^2 = 2dx^0dx^1 - \frac{{x^0}^{2\omega}}{\gamma\left(\delta^2-1\right)}\Big[\left(1-\delta\cos\left(\theta-2\log x^0\right)\right)\,\left(dx^2\right)^2$$

$$-\,2\sin\left(\theta-2\log x^0\right)\,dx^2dx^3+\left(1+\delta\cos\left(\theta-2\log x^0\right)\right)\,\left(dx^3\right)^2\Big], \quad (10)$$

$$g=\det g_{ij}=\frac{-{x^0}^{4\omega}}{\gamma^2\left(1-\delta^2\right)}, \qquad \delta^2<1, \quad (11)$$

where x^0 is the wave (null) variable.
The constants γ, δ, θ and ω are independent parameters of the gravitational wave model.

The space with the metric (10) admits a covariantly constant vector K and, therefore, is a plane-wave space:

$$\nabla_\beta K_\alpha = 0 \quad \to \quad K_\alpha = \big(K_0, 0, 0, 0\big), \tag{12}$$

where K_0 is a constant.

A space with a metric (10) admits a spatial homogeneity group with Killing vectors $X_{(1)}$, $X_{(2)}$, and $X_{(3)}$, which can be chosen in the privileged coordinate system in the form

$$X^\alpha_{(1)} = \big(0, 0, 1, 0\big), \qquad X^\alpha_{(2)} = \big(0, 0, 0, 1\big), \tag{13}$$

$$X^\alpha_{(3)} = \big(-x^0,\, x^1,\, \omega\, x^2 - x^3,\, x^2 + \omega\, x^3\big). \tag{14}$$

The additional fourth Killing vector associated with the choice of the privileged coordinate system commutes with the vectors $X_{(1)}$ and $X_{(2)}$ and has the form in this coordinate system

$$X^\alpha_{(0)} = \big(0, 1, 0, 0\big). \tag{15}$$

The Killing vector $X_{(0)}$ is a null vector, because $g_{\alpha\beta} X^\alpha_{(0)} X^\beta_{(0)} = 0$.

The commutation relations for the Killing vectors have the form:

$$\left[X_{(0)}, X_{(1)}\right] = 0, \qquad \left[X_{(0)}, X_{(2)}\right] = 0, \qquad \left[X_{(0)}, X_{(3)}\right] = X_{(0)}, \tag{16}$$

$$\left[X_{(1)}, X_{(2)}\right] = 0, \tag{17}$$

$$\left[X_{(1)}, X_{(3)}\right] = \omega X_{(1)} + X_{(2)}, \tag{18}$$

$$\left[X_{(2)}, X_{(3)}\right] = -X_{(1)} + \omega X_{(2)}. \tag{19}$$

The Killing vectors $X_{(0)}$, $X_{(1)}$, $X_{(2)}$ generate a 3-dimensional abelian subgroup. The Killing vectors $X_{(1)}$, $X_{(2)}$ and $X_{(3)}$ generate a 3-dimensional subgroup of the spatial homogeneity of the type VII Bianchi model.

The restrictions from Einstein equations

From Einstein equations we obtain the following exhaustive restrictions on the parameters of the gravitational wave:

$$\delta^2 = \frac{\omega(\omega - 1)}{\omega^2 - \omega - 1}, \tag{20}$$

$$0 \leq \omega \leq 1, \qquad 0 \leq \delta^2 \leq 1/5. \tag{21}$$

Thus, three independent constant parameters of the gravitational wave remain in the model under consideration: the Bianchi type VII homogeneity subgroup parameter ω, the γ parameter related to the wave amplitude, and the angular parameter θ related to the wave phase.

The $(+,-,-,-)$ metric signature imposes additional restrictions on the choice of γ sign:

$$\gamma < 0. \tag{22}$$

We will look for the complete integral of the Hamilton-Jacobi equation in a separated form:

$$S(x^\alpha, \lambda_k) = \phi_0(x^0) + \sum_{k=1}^{3} \lambda_k x^k, \tag{23}$$

where the independent constant parameters λ_k are determined by the initial (or boundary) conditions for the motion of a test particle.
Then the Hamilton-Jacobi equation (1) gives for the function $\phi_0(x^0)$ ($\lambda_1 \neq 0$):

$$\begin{aligned} {\phi_0}' = \frac{1}{2\lambda_1} + \frac{\gamma {x^0}^{-2\omega}}{2\lambda_1} \Big[& -{\lambda_2}^2 - {\lambda_3}^2 \\ & - 2\delta\lambda_2\lambda_3 \sin\left(\theta - 2\log x^0\right) \\ & - \delta\left({\lambda_2}^2 - {\lambda_3}^2\right)\cos\left(\theta - 2\log x^0\right)\Big]. \end{aligned} \tag{24}$$

The integration of the equation (24) has a singularity at $\omega = 1/2$.

The complete integral for the case $\omega \neq 1/2$

We will assume that $\omega \neq 1/2$. Then the Hamilton-Jacobi equation gives for the function $\phi_0(x^0)$:

$$\phi_0(x^0) = \frac{x^0}{2\lambda_1} + \frac{{x^0}^{1-2\omega}\gamma({\lambda_2}^2 + {\lambda_3}^2)}{2\lambda_1(2\omega-1)} + \frac{\gamma\delta {x^0}^{1-2\omega}}{2\lambda_1\left(4\omega^2-4\omega+5\right)}\Big[$$
$$2\left({\lambda_2}^2 + \lambda_2\lambda_3(2\omega-1) - {\lambda_3}^2\right)\sin\left(\theta - 2\log x^0\right)$$
$$+\Big((2\omega-1)\left({\lambda_2}^2+{\lambda_3}^2\right) - 4\lambda_2\lambda_3\Big)\cos\left(\theta - 2\log x^0\right)\Big]. \qquad (25)$$

Thus, we have found the complete integral of the Hamilton-Jacobi equation of test particles $S(x^\alpha, \lambda_k)$.

We present the result of solving the trajectory equations (3) in the form of a test particle trajectory in a privileged coordinate system:

$$x^0(\tau) = \lambda_1 \tau, \tag{26}$$

$$x^1(\tau) = \frac{(\lambda_1\tau)^{1-2\omega}}{2{\lambda_1}^2(2\omega-1)}\Big((2\omega-1)(\lambda_1\tau)^{2\omega} + \gamma({\lambda_2}^2+{\lambda_3}^2)\Big)$$
$$+\frac{\gamma\delta(\lambda_1\tau)^{1-2\omega}}{2{\lambda_1}^2(4\omega^2-4\omega+5)}\Big[2\Big({\lambda_2}^2+\lambda_2\lambda_3(2\omega-1)-{\lambda_3}^2\Big)\sin\left(\theta-2\log\left(\lambda_1\tau\right)\right)$$
$$+\Big({\lambda_2}^2(2\omega-1)-4\lambda_2\lambda_3+{\lambda_3}^2(1-2\omega)\Big)\cos\left(\theta-2\log\left(\lambda_1\tau\right)\right)\Big], \tag{27}$$

$$x^2(\tau) = -\frac{\gamma(\lambda_1\tau)^{1-2\omega}}{\lambda_1(2\omega-1)(4\omega^2-4\omega+5)}\Big[\lambda_2\left(4\omega^2-4\omega+5\right)$$
$$+(2\omega-1)\delta(2\lambda_2+\lambda_3(2\omega-1))\sin\left(\theta-2\log\left(\lambda_1\tau\right)\right)$$
$$+(2\omega-1)\delta(\lambda_2(2\omega-1)-2\lambda_3)\cos\left(\theta-2\log\left(\lambda_1\tau\right)\right)\Big], \tag{28}$$

The trajectories of a test particle (Part 2)

$$x^3(\tau) = \frac{\gamma(\lambda_1\tau)^{1-2\omega}}{\lambda_1(2\omega-1)(4\omega^2-4\omega+5)}\Big[\lambda_3\left(-4\omega^2+4\omega-5\right)$$
$$-(2\omega-1)\delta(\lambda_2(2\omega-1)-2\lambda_3)\sin\left(\theta-2\log\left(\lambda_1\tau\right)\right)$$
$$+(2\omega-1)\delta(2\lambda_2+\lambda_3(2\omega-1))\cos\left(\theta-2\log\left(\lambda_1\tau\right)\right)\Big], \qquad (29)$$

where τ is the proper time of the test particle. In the process of integrating the equations (3), we set the constants σ_k equal to zero by choosing the origin of the variables x^α and the proper time τ.

For the solutions obtained, we find the 4-velocity of the test particle in the privileged coordinate system $u^\alpha(\tau) = Dx^\alpha/d\tau$:

$$u^0 = \lambda_1, \qquad x^0 = \lambda_1 \tau, \tag{30}$$

$$u^1 = \frac{1}{2\lambda_1} - \frac{{x^0}^{-2\omega}}{2\lambda_1}\Big[\gamma\delta\left({\lambda_2}^2 - {\lambda_3}^2\right)\cos\left(\theta - 2\log x^0\right) + 2\lambda_2\lambda_3\gamma\delta\sin\left(\theta - 2\log x^0\right) + \gamma({\lambda_2}^2 + {\lambda_3}^2)\Big], \tag{31}$$

$$u^2 = \gamma {x^0}^{-2\omega}\left[\lambda_2\delta\cos\left(\theta - 2\log x^0\right) + \lambda_3\delta\sin\left(\theta - 2\log x^0\right) + \lambda_2\right], \tag{32}$$

$$u^3 = \gamma {x^0}^{-2\omega}\left[\lambda_2\delta\sin\left(\theta - 2\log x^0\right) - \lambda_3\delta\cos\left(\theta - 2\log x^0\right) + \lambda_3\right], \tag{33}$$

We present the exact solution for the geodesic deviation vector $\eta^{\alpha}(\tau)$ in the privileged coordinate system ($\omega \neq 1/2$):

$$\eta^0(\tau) = \rho_1 \tau - \lambda_1 \Omega, \qquad x^0(\tau) = \lambda_1 \tau, \tag{34}$$

$$\eta^1(\tau) = \vartheta_1 - \frac{\gamma R_2 {x^0}^{1-2\omega}}{{\lambda_1}^3 (2\omega - 1)\left(4\omega^2 - 4\omega + 5\right)} \Big[\lambda_2 \left(4\omega^2 - 4\omega + 5\right)$$
$$+(2\omega - 1)\delta(2\lambda_2 + \lambda_3(2\omega - 1)) \sin\left(\theta - 2\log x^0\right)$$
$$+(2\omega - 1)\delta(\lambda_2(2\omega - 1) - 2\lambda_3) \cos\left(\theta - 2\log x^0\right)\Big]$$
$$+\frac{\gamma R_3 {x^0}^{1-2\omega}}{{\lambda_1}^3 (2\omega - 1)\left(4\omega^2 - 4\omega + 5\right)} \Big[\lambda_3 \left(-4\omega^2 + 4\omega - 5\right)$$
$$-(2\omega - 1)\delta(\lambda_2(2\omega - 1) - 2\lambda_3) \sin\left(\theta - 2\log x^0\right)$$
$$+(2\omega - 1)\delta(2\lambda_2 + \lambda_3(2\omega - 1)) \cos\left(\theta - 2\log x^0\right)\Big]$$

$$+\frac{{x^0}^{-2\omega}}{2{\lambda_1}^3}\Big[\gamma\delta\left({\lambda_2}^2-{\lambda_3}^2\right)\left({\lambda_1}^2\Omega-\rho_1 x^0\right)\cos\left(\theta-2\log x^0\right)$$
$$+2\lambda_2\lambda_3\gamma\delta\left({\lambda_1}^2\Omega-\rho_1 x^0\right)\sin\left(\theta-2\log x^0\right)-\rho_1{x^0}^{2\omega+1}$$
$$-{\lambda_1}^2\Omega{x^0}^{2\omega}+({\lambda_2}^2+{\lambda_3}^2)\gamma({\lambda_1}^2\Omega-\rho_1 x^0)\Big],\qquad (35)$$

$$\eta^2(\tau)=\vartheta_2-\frac{\gamma{x^0}^{-2\omega}\left({\lambda_1}^2\Omega-\rho_1 x^0\right)}{{\lambda_1}^2}\Big[\lambda_2+\lambda_2\delta\cos\left(\theta-2\log x^0\right)$$
$$+\lambda_3\delta\sin\left(\theta-2\log x^0\right)\Big]+\frac{\gamma R_2{x^0}^{1-2\omega}}{{\lambda_1}^2(2\omega-1)\left(4\omega^2-4\omega+5\right)}\Big[4\omega^2-4\omega+5$$
$$+2(2\omega-1)\delta\sin\left(\theta-2\log x^0\right)+(1-2\omega)^2\delta\cos\left(\theta-2\log x^0\right)\Big]$$
$$-\frac{\gamma R_3\delta{x^0}^{1-2\omega}\left[(1-2\omega)\sin\left(\theta-2\log x^0\right)+2\cos\left(\theta-2\log x^0\right)\right]}{{\lambda_1}^2\left(4\omega^2-4\omega+5\right)},\qquad (36)$$

The exact solution for the geodesic deviation vector (Part 3)

$$\eta^3(\tau) = \vartheta_3 + \frac{\gamma {x^0}^{-2\omega}\left({\lambda_1}^2\Omega - \rho_1 x^0\right)}{{\lambda_1}^2}\Big[-\lambda_2\delta\sin\left(\theta - 2\log x^0\right)$$

$$+\lambda_3\delta\cos\left(\theta - 2\log x^0\right) - \lambda_3\Big]$$

$$-\frac{\gamma R_2 \delta {x^0}^{1-2\omega}\left[(1-2\omega)\sin\left(\theta - 2\log x^0\right) + 2\cos\left(\theta - 2\log x^0\right)\right]}{{\lambda_1}^2\left(4\omega^2 - 4\omega + 5\right)}$$

$$-\frac{\gamma R_3 {x^0}^{1-2\omega}}{{\lambda_1}^2(2\omega-1)\left(4\omega^2 - 4\omega + 5\right)}\Big[-4\omega^2 + 4\omega - 5$$

$$+2(2\omega-1)\delta\sin\left(\theta - 2\log x^0\right) + (1-2\omega)^2\delta\cos\left(\theta - 2\log x^0\right)\Big], \qquad (37)$$

where, for brevity, the auxiliary notation R_2, R_3 and Ω is introduced:

$$R_2 = \lambda_2\rho_1 - \lambda_1\rho_2, \qquad R_3 = \lambda_3\rho_1 - \lambda_1\rho_3, \qquad (38)$$

$$\Omega = \lambda_1\vartheta_1 + \lambda_2\vartheta_2 + \lambda_3\vartheta_3. \qquad (39)$$

The transformation from the privileged coordinate system x^α to the synchronous frame $\tilde{x}^\alpha$ has the form $x^\alpha \to \tilde{x}^\alpha = (\tau, \lambda_1, \lambda_2, \lambda_3)$:

$$x^0 = \tilde{x}^1 \tau, \tag{40}$$

$$x^1 = \frac{\tau^2 (\tilde{x}^1 \tau)^{-2\omega-1}}{2(2\omega-1)(4\omega^2-4\omega+5)} \Big[\\ \left(4\omega^2-4\omega+5\right)\left((2\omega-1)(\tilde{x}^1\tau)^{2\omega}+\gamma({\tilde{x}^2}^2+{\tilde{x}^3}^2)\right) \\ +2\gamma(2\omega-1)\delta\Big({\tilde{x}^2}^2+\tilde{x}^2\tilde{x}^3(2\omega-1)-{\tilde{x}^3}^2\Big)\sin\left(\theta-2\log\left(\tilde{x}^1\tau\right)\right) \\ +\gamma(2\omega-1)\delta\Big({\tilde{x}^2}^2(2\omega-1)-4\tilde{x}^2\tilde{x}^3+{\tilde{x}^3}^2(1-2\omega)\Big)\cos\left(\theta-2\log\left(\tilde{x}^1\tau\right)\right)\Big], \tag{41}$$

The synchronous frame (Part 2)

$$x^2 = -\frac{\gamma\tau(\tilde{x}^1\tau)^{-2\omega}}{(2\omega-1)(4\omega^2-4\omega+5)}\Big[\tilde{x}^2\left(4\omega^2-4\omega+5\right)$$
$$+(2\omega-1)\delta(2\tilde{x}^2+\tilde{x}^3(2\omega-1))\sin\left(\theta-2\log\left(\tilde{x}^1\tau\right)\right)$$
$$+(2\omega-1)\delta(\tilde{x}^2(2\omega-1)-2\tilde{x}^3)\cos\left(\theta-2\log\left(\tilde{x}^1\tau\right)\right)\Big], \quad (42)$$

$$x^3 = \frac{\gamma\tau(\tilde{x}^1\tau)^{-2\omega}}{(2\omega-1)(4\omega^2-4\omega+5)}\Big[\tilde{x}^3\left(-4\omega^2+4\omega-5\right)$$
$$-(2\omega-1)\delta(\tilde{x}^2(2\omega-1)-2\tilde{x}^3)\sin\left(\theta-2\log\left(\tilde{x}^1\tau\right)\right)$$
$$+(2\omega-1)\delta(2\tilde{x}^2+\tilde{x}^3(2\omega-1))\cos\left(\theta-2\log\left(\tilde{x}^1\tau\right)\right)\Big]. \quad (43)$$

We can write down the form of the gravitational wave metric in the synchronous frame of reference:

$$ds^2 = d\tau^2 - dl^2 = d\tau^2 + \tilde{g}_{ij}(\tau, \tilde{x}^k)\, d\tilde{x}^i d\tilde{x}^j, \qquad i, j, k = 1, 2, 3; \tag{44}$$

where τ is the time variable (the observer's proper time on the base geodesic), dl is the spatial distance element, $\tilde{x}^k$ are the spatial coordinates. The components of the gravitational wave metric in the synchronous frame take the following form ($\omega \neq 1/2$):

$$\tilde{g}^{00} = 1, \qquad \tilde{g}^{0k} = 0, \qquad \tilde{g}^{1k} = -\frac{\tilde{x}^1 \tilde{x}^k}{\tau^2}, \tag{45}$$

$$\tilde{g}^{22} = \frac{(1-2\omega)^2\left(\omega^2-\omega-1\right)(\tau\tilde{x}^1)^{2\omega}}{5\gamma\tau^2}\Big[-4\omega^2+4\omega-5$$
$$+(2\omega+1)(2\omega-3)\delta\cos\Big(\theta-2\log(\tau\tilde{x}^1)\Big)$$
$$+4(2\omega-1)\delta\sin\Big(\theta-2\log(\tau\tilde{x}^1)\Big)\Big]-\frac{\tilde{x}^{2^2}}{\tau^2}, \tag{46}$$

$$\tilde{g}^{23} = \frac{(1-2\omega)^2\left(\omega^2-\omega-1\right)\delta(\tau\tilde{x}^1)^{2\omega}}{5\gamma\tau^2}\Big[4(1-2\omega)\cos\Big(\theta-2\log(\tau\tilde{x}^1)\Big)$$
$$+(2\omega+1)(2\omega-3)\sin\Big(\theta-2\log(\tau\tilde{x}^1)\Big)\Big]-\frac{\tilde{x}^2\tilde{x}^3}{\tau^2}, \tag{47}$$

$$\tilde{g}^{33} = -\frac{(1-2\omega)^2\left(\omega^2-\omega-1\right)(\tau\tilde{x}^1)^{2\omega}}{5\gamma\tau^2}\Big[4\omega^2-4\omega+5$$
$$+(2\omega+1)(2\omega-3)\delta\cos\Big(\theta-2\log(\tau\tilde{x}^1)\Big)$$
$$+4(2\omega-1)\delta\sin\Big(\theta-2\log(\tau\tilde{x}^1)\Big)\Big]-\frac{\tilde{x}^{3^2}}{\tau^2}, \tag{48}$$

where ω, γ and θ are independent parameters of the gravitational wave model.

The deviation vector $\eta^\alpha(\tau)$ in the synchronous frame:

$$\tilde{\eta}^0 = 0, \tag{49}$$

$$\tilde{\eta}^1(\tau) = \rho_1 - \frac{\lambda_1 \Omega}{\tau}, \tag{50}$$

$$\begin{aligned}\tilde{\eta}^2 = {} & \frac{(1-2\omega)\left(1+\omega-\omega^2\right)\lambda_1\vartheta_2(\lambda_1\tau)^{2\omega-1}}{5\gamma}\Big[2(1-2\omega)\delta\sin\Big(\theta-2\log(\lambda_1\tau)\Big) \\ & -(1-2\omega)^2\delta\cos\Big(\theta-2\log(\lambda_1\tau)\Big)+4\omega^2-4\omega+5\Big] \\ & -\frac{(1-2\omega)^2\left(1+\omega-\omega^2\right)\delta\lambda_1\vartheta_3(\lambda_1\tau)^{2\omega-1}}{5\gamma}\Big[2\cos\Big(\theta-2\log(\lambda_1\tau)\Big) \\ & +(1-2\omega)\sin\Big(\theta-2\log(\lambda_1\tau)\Big)\Big]-\frac{\lambda_2\Omega}{\tau}+\rho_2, \end{aligned} \tag{51}$$

$$\tilde{\eta}^3(\tau) = -\frac{(1-2\omega)^2\left(1+\omega-\omega^2\right)\delta\lambda_1\vartheta_2(\lambda_1\tau)^{2\omega-1}}{5\gamma}\Big[2\cos\Big(\theta - 2\log(\lambda_1\tau)\Big)$$
$$+ (1-2\omega)\sin\Big(\theta - 2\log(\lambda_1\tau)\Big)\Big]$$
$$+\frac{(1-2\omega)\left(1+\omega-\omega^2\right)\lambda_1\vartheta_3(\lambda_1\tau)^{2\omega-1}}{5\gamma}\Big[2(2\omega-1)\delta\sin\Big(\theta - 2\log(\lambda_1\tau)\Big)$$
$$+(1-2\omega)^2\delta\cos\Big(\theta - 2\log(\lambda_1\tau)\Big) + 4\omega^2 - 4\omega + 5\Big] - \frac{\lambda_3\Omega}{\tau} + \rho_3. \quad (52)$$

Here the parameters ω, γ and θ determine the gravitational wave model, the constants λ_k, ρ_k and ϑ_k are given by the initial (or boundary) conditions for the velocities and mutual positions of particles on neighboring geodesics.

The tidal accelerations in the synchronous frame $\tilde{A}^\alpha = D^2\tilde{\eta}^\alpha/d^2\tau$:

$$\tilde{A}^0 = \tilde{A}^1 = 0, \tag{53}$$

$$\begin{aligned}
\tilde{A}^2(\tau) = \frac{(2\omega-1)(\omega^2-\omega-1)\delta\vartheta_2(\lambda_1\tau)^{2\omega}}{\gamma\tau^3}\Big[2\cos\Big(\theta-2\log(\lambda_1\tau)\Big) \\
+ (1-2\omega)\sin\Big(\theta-2\log(\lambda_1\tau)\Big)\Big] \\
+ \frac{(2\omega-1)(\omega^2-\omega-1)\delta\vartheta_3(\lambda_1\tau)^{2\omega}}{\gamma\tau^3}\Big[2\sin\Big(\theta-2\log(\lambda_1\tau)\Big) \\
+ (2\omega-1)\cos\Big(\theta-2\log(\lambda_1\tau)\Big)\Big] \\
+\frac{R_2\left(\omega^2-\omega-1\right)\delta}{\lambda_1\tau^2}\Big[(1-2\omega)\sin\Big(\theta-2\log(\lambda_1\tau)\Big)+2\cos\Big(\theta-2\log(\lambda_1\tau)\Big)\Big] \\
+\frac{R_3}{\lambda_1\tau^2}\Big[2\left(\omega^2-\omega-1\right)\delta\sin\Big(\theta-2\log(\lambda_1\tau)\Big) \\
+(2\omega-1)(\omega^2-\omega-1)\delta\cos\Big(\theta-2\log(\lambda_1\tau)\Big)+\omega(2\omega-1)(1-\omega)\Big],
\end{aligned} \tag{54}$$

The tidal accelerations in the synchronous frame (Part 2)

$$
\begin{aligned}
\tilde{A}^3(\tau) = \frac{(2\omega-1)(\omega^2-\omega-1)\delta\vartheta_2(\lambda_1\tau)^{2\omega}}{\gamma\tau^3}\Big[2\sin\Big(\theta-2\log(\lambda_1\tau)\Big) \\
+ (2\omega-1)\cos\Big(\theta-2\log(\lambda_1\tau)\Big)\Big] \\
- \frac{(2\omega-1)(\omega^2-\omega-1)\delta\vartheta_3(\lambda_1\tau)^{2\omega}}{\gamma\tau^3}\Big[2\cos\Big(\theta-2\log(\lambda_1\tau)\Big) \\
+ (1-2\omega)\sin\Big(\theta-2\log(\lambda_1\tau)\Big)\Big] \\
+ \frac{R_2}{\lambda_1\tau^2}\Big[\omega(2\omega-1)(\omega-1) + 2\left(\omega^2-\omega-1\right)\delta\sin\Big(\theta-2\log(\lambda_1\tau)\Big) \\
+ (2\omega-1)(\omega^2-\omega-1)\delta\cos\Big(\theta-2\log(\lambda_1\tau)\Big)\Big] \\
- \frac{R_3\left(\omega^2-\omega-1\right)\delta}{\lambda_1\tau^2}\Big[2\cos\Big(\theta-2\log(\lambda_1\tau)\Big) \\
+ (1-2\omega)\sin\Big(\theta-2\log(\lambda_1\tau)\Big)\Big].
\end{aligned}
\tag{55}
$$

The Lienard-Wiechert potentials A^α for the Minkowski spacetime can be written as:

$$A^\alpha(\tau, \vec{r}) = e \, \frac{u^\alpha}{R_\beta u^\beta}, \tag{56}$$

where e is the charge of the particle, u^α is the 4-velocity of the charge at time τ', $R^\beta = \{c(\tau - \tau'), \vec{r} - \vec{r}'\}$ is the difference of the observer's 4-vector at time τ and the 4-vector of the charge at the time τ', c is the speed of light, $\vec{r}$ is the spatial radius vector of the charge, and the time τ' and the components of the radius vector charges $\vec{r}'$ are related to τ and $\vec{r}$ by the following relation:

$$R_\alpha R^\alpha = 0. \tag{57}$$

In the weak field approximation we can apply the Lienard-Wiechert potential formula for an approximate, qualitative study of the problem, but taking into account the specifics of the exact solution we obtained for the deviation vector η^α. The charge vector $x^\alpha(\tau)$ and the vector R^α in the synchronous reference frame, one can take the expressions

$$\tilde{x}^\alpha(\tau) = \left\{c\tau,\, \vec{\eta}(\tau) + \vec{\lambda}\,\right\}, \qquad R^\alpha = \left\{c(\tau - \tau'), -\vec{\eta}(\tau')\right\}, \tag{58}$$

$$\vec{\eta} = \left\{\tilde{\eta}^1, \tilde{\eta}^2, \tilde{\eta}^3\,\right\}, \qquad \vec{\lambda} = \{\lambda_1, \lambda_2, \lambda_3\,\}\,. \tag{59}$$

In the expression for the Lienard-Wiechert potentials $A^\alpha(\tau)$ the values on the right are taken at the moment time τ', which is related to the time τ in the synchronous reference frame by the relation

$$c^2(\tau - \tau')^2 + \tilde{g}^{kl}\,\tilde{\eta}_k\tilde{\eta}_l\,\Big|_{x^\alpha = \{c\tau', \vec{\lambda}\}} = 0, \tag{60}$$

where $k, l = 1, 2, 3$ and the gravitational wave metric $\tilde{g}^{\alpha\beta}$ in the synchronous frame (45)-(48).

Conclusion

- An **exact solution for the trajectories of test particles** in a gravitational wave is obtained for Bianchi type VII cosmological models in a privileged coordinate system;
- An **exact solution of the geodesic deviation equations** in a gravitational wave is obtained for Bianchi type VII cosmological models in a privileged coordinate system and in synchronous frame;
- An explicit form for **the wave metric in a synchronous frame** is obtained;
- An explicit form of **the tidal acceleration in a gravitational wave** is obtained for a privileged coordinate system and for a synchronous frame.

The exact models obtained describe the primordial gravitational waves of the Universe and can be used to calculate the secondary physical effects that arise during the passage of a wave. The approach presented in the report can be used both in the general theory of relativity and in modified theories of gravity.

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https://arxiv.org/abs/2206.15234

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https://rscf.ru/project/22-21-00265/