Analytical analysis of the origin of core or cusp matter density distributions in galaxies

S.A. Paston, A.D. Kapustin

Saint Petersburg State University

The mystery of dark matter:

The hypothesis of its existence provides a good fit to evidence in a wide range of scales –

from the galactic scale (rotation curves)
to the cosmological ones (the formation of cosmic structures and the prediction of the total mass of the matter in the Universe).

In a framework of the ΛCDM model – a modern standard model in cosmology – the properties of the dark matter are close to the non-relativistic dust-like matter.

However, attempts to detect dark matter (to detect its interaction with regular matter) are still unsuccessful!

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Mimetic gravity:

$$S = S^{\text{EH}}[g(...)] + S_m[g(...)], \qquad S^{\text{EH}} = -\frac{1}{2\varkappa}\int d^4x\sqrt{-g}\,R \quad (1)$$

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}(\partial_\alpha\varphi)(\partial_\beta\varphi) \qquad \left(\Rightarrow \quad g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) \equiv 1\right) \quad (2)$$

$$G_{\mu\nu} - \varkappa\, T_{\mu\nu} = \varkappa n(\partial_\mu\varphi)(\partial_\nu\varphi), \qquad D_\mu(ng^{\mu\nu}\partial_\nu\varphi) = 0,$$
$$n \equiv \frac{1}{\varkappa}\, g^{\mu\nu}(G_{\mu\nu} - \varkappa\, T_{\mu\nu}) \quad (3)$$

(A.H. Chamseddine, V. Mukhanov, JHEP, 2013:11 (2013), 135, arXiv:1308.5410)

Embeddig theory:

$$S = S^{\mathrm{EH}}[g(...)] + S_m[g(...)], \qquad S^{\mathrm{EH}} = -\frac{1}{2\varkappa}\int d^4x\sqrt{-g}\,R \quad (4)$$

$$g_{\mu\nu} = \eta_{ab}(\partial_\mu y^a)(\partial_\nu y^b) \quad (5)$$

(T. Regge, C. Teitelboim, *"General relativity a la string: a progress report"*, Proceedings of the First Marcel Grossmann Meeting (Trieste, Italy, 1975), 1977, p. 77, arXiv:1612.05256)

The simple geometric sense – a metric becomes the induced metric of a 4-dimensional surface in the ambient space which is described by the embedding function $y^a(x^\mu)$.

Equations of motion

$$D_\mu\Big((G^{\mu\nu} - \varkappa\, T^{\mu\nu})\,\partial_\nu y^a\Big) = 0 \qquad (6)$$

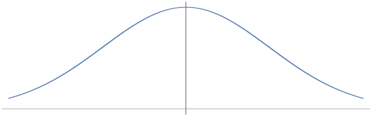
could be rewritten in the form of Einstein equations with an additional contribution $\tau^{\mu\nu}$ – EMT of some *fictitious* matter – of the dark matter:

$$G^{\mu\nu} = \varkappa\,(T^{\mu\nu} + \tau^{\mu\nu})\,, \qquad D_\mu\Big(\tau^{\mu\nu}\partial_\nu y^a\Big) = 0 \qquad (7)$$

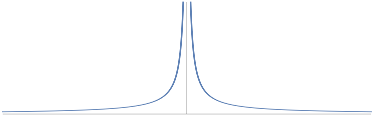
(M. Pavsic, *Class. Quant. Grav.*, 2 (1985), 869, arXiv:1403.6316)

Core-cusp problem:

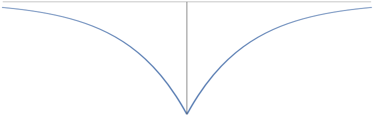
Observations generally show a smooth density distribution at the centers of galaxies – core:



while numerical simulations result in an increase $\rho(r) \sim r^{\alpha}, \quad \alpha \approx -1$ in density at the center – cusp:



which corresponds to the gravitational potential



The relation between density profile and particles distribution function

We consider a spherically symmetric and static on average distribution of particles.

Density $\rho(x)$ is related with the gravitational potential $\varphi(x)$ by:

$$\partial_k \partial_k \varphi(x) = 4\pi G \rho(x) \tag{8}$$

Finite motion can be closed and open as well:

It is defined by normalized energy $\varepsilon = E/m$ and angular momentum $\ell_k = L_k/m$ with addition of a vector τ_k and initial phase γ.
A motion of a single particle is given by the function $\hat{x}_i\,(t, \varepsilon, \ell_k, \tau_l, \gamma)$.

The distribution function of particles f:

$$dN = f\left(\varepsilon, \ell_k, \tau_l, \gamma\right)\, d\varepsilon\, d^3\ell\, d\tau\, d\gamma \tag{9}$$

Then the density can be written as:

$$\rho(x_i) = m \int d\varepsilon\, d^3\ell\, d\tau\, d\gamma\, f\left(\varepsilon, \ell_k, \tau_l, \gamma\right) \delta\left(x_i - \hat{x}_i\left(t, \varepsilon, \ell_k, \tau_l, \gamma\right)\right) \tag{10}$$

Due to spherical symmetry and stationarity:

$$\rho(r) = \frac{m}{4\pi r^2 T} \int d\varepsilon\, d^3\ell\, d\tau\, d\gamma\; f\left(\varepsilon, \ell_k, \tau_l, \gamma\right) \times$$

$$\times \int\limits_{S_r} d^2x \int\limits_0^T dt\, \delta\left(x_i - \hat{x}_i\left(t, \varepsilon, \ell_k, \tau_l, \gamma\right)\right) =$$

$$= \frac{m}{4\pi r^2 T} \int d\varepsilon\, d^3\ell\, d\tau\, d\gamma\; f\left(\varepsilon, \ell_k, \tau_l, \gamma\right) \frac{n}{|v_r|} \tag{11}$$

In the spherically symmetric field, the energy

$$E = \frac{m}{2}\left(v_r^2 + v_\tau^2\right) + m\varphi(r) \tag{12}$$

is conserved as well as the angular momentum. Therefore,

$$|v_r| = \sqrt{2\varepsilon - 2\varphi(r) - \frac{\ell^2}{r^2}}, \quad \text{since} \quad L = mrv_\tau. \tag{13}$$

At large T

$$n(\varepsilon, \ell, r) \approx \frac{2T}{\hat{T}(\varepsilon, \ell)}\theta\left(2\varepsilon - 2\varphi(r) - \frac{\ell^2}{r^2}\right), \tag{14}$$

which leads to

$$\rho(r) = \frac{m}{2\pi r^2}\int d\varepsilon\, d^3\ell\, \frac{f(\varepsilon, \ell_k)\,\theta\left(2\varepsilon - 2\varphi(r) - \frac{\ell^2}{r^2}\right)}{\hat{T}(\varepsilon, \ell)\sqrt{2\varepsilon - 2\varphi(r) - \frac{\ell^2}{r^2}}}, \tag{15}$$

where

$$f(\varepsilon, \ell_k) = \int d\tau\, d\gamma\, f(\varepsilon, \ell_k, \tau_l, \gamma) \tag{16}$$

Integrating over all directions of the vector ℓ_k, we obtain:

$$\rho(r) = \frac{m}{2\pi r^2} \int d\varepsilon \int_0^\infty d\ell \, \frac{\hat{f}(\varepsilon,\ell)\,\theta\left(2\varepsilon - 2\varphi(r) - \frac{\ell^2}{r^2}\right)}{\hat{T}(\varepsilon,\ell)\sqrt{2\varepsilon - 2\varphi(r) - \frac{\ell^2}{r^2}}} \qquad (17)$$

Let's discuss the behavior of the $\rho(r)$ at $r \to 0$.

The only contribution comes from the area $\ell \leqslant r\sqrt{2\varepsilon - 2\varphi(r)}$, so the asymptotic of the $\rho(r)$ at $r \to 0$ is defined by the asymptotic at $\ell \to 0$ of the functions in the integral.

Function $\hat{T}(\varepsilon,\ell)$ has finite limit, while for $\hat{f}(\varepsilon,\ell)$ several cases are possible:

Case 1) $\hat{f}(\varepsilon,0) \neq 0$ for some ε

Case 2) $\hat{f}(\varepsilon,\ell) \approx \hat{f}'(\varepsilon,0)\ell$ for all ε

Performing a change of variables $\ell = r\tilde{\ell}$ in the integral, we obtain:

Case 1)

$$\rho(r) = \frac{m}{2\pi r} \int d\varepsilon \, d\tilde{\ell} \, \frac{\hat{f}(\varepsilon, 0) \, \theta\left(2\varepsilon - 2\varphi(r) - \tilde{\ell}^2\right)}{\hat{T}(\varepsilon, 0) \sqrt{2\varepsilon - 2\varphi(r) - \tilde{\ell}^2}} = \frac{m}{4r} \int d\varepsilon \, \frac{\hat{f}(\varepsilon, 0)}{\hat{T}(\varepsilon, 0)} \quad (18)$$

which gives cusp-profile.

Case 2)

$$\rho(r) = \frac{m}{2\pi} \int d\varepsilon \, d\tilde{\ell} \, \frac{\hat{f}'(\varepsilon, 0) \tilde{\ell} \, \theta\left(2\varepsilon - 2\varphi(r) - \tilde{\ell}^2\right)}{\hat{T}(\varepsilon, 0) \sqrt{2\varepsilon - 2\varphi(r) - \tilde{\ell}^2}} =$$

$$= \frac{m}{2\pi} \int d\varepsilon \, \frac{\hat{f}'(\varepsilon, 0)}{\hat{T}(\varepsilon, 0)} \sqrt{2\varepsilon - 2\varphi(r)} \quad (19)$$

which gives core-profile.

Asymptotic of the distribution function at $\ell \to 0$

Situation A) – a formation of the static structure takes place in a preliminary given spherically symmetric potential.

Possible realisation – the spherically symmetric potential is already created by the dark matter and we consider a formation of a static structure of the regular matter inside this potential.

Spherical symmetry $\Longrightarrow$ the angular momentum is conserved $\Longrightarrow$ the distribution function $f(\varepsilon, \ell_k)$ doesn't change with time and it is enough to find it at the initial moment.

Using the distribution function $\chi(x_i, v_k)$ of particles over its coordinates x_i and velocities v_k, we have

$$f(\varepsilon, \ell_k) = \int d^3x\, d^3v\, \chi(x_i, v_i)\delta(\ell_i - \epsilon_{ikl} x_k v_l)\delta\left(\varepsilon - \frac{v^2}{2} - \varphi(x_i)\right) \quad (20)$$

If we assume spherical symmetry and take $\ell_k = (\ell, 0, 0)$, we find

$$f(\varepsilon, \ell_k) = \frac{1}{\ell} \int dx_2\, dx_3\, dv_2\, dv_3 \left[\chi(x_i, v_i)\delta\left(\varepsilon - \frac{v^2}{2} - \varphi(x_i)\right)\right]\Bigg|_{x_1 = v_1 = 0} \times$$
$$\times \delta(\ell - x_2 v_3 + x_3 v_2) \quad (21)$$

Here coefficient at $1/\ell$ has a finite limit at $\ell \to 0$, which is true even without spherical symmetry.

As a result for

$$\hat{f}(\varepsilon,\ell) = \int\limits_{S_\ell} d^2\ell\, f\,(\varepsilon,\ell_k) \qquad (22)$$

at $\ell \to 0$ we have $\hat{f}(\varepsilon,\ell) \approx C(\varepsilon)\ell$, i. e. *Case 2)* takes place, and hence, the core-profile will arise.

Situation B) – the gravitational potential forms simultaneously with the static structure and significant deviations from spherical symmetry are possible in this process.

Possible realisation – we consider a formation of the static structure of the dark matter particles. Such setup corresponds to the numerical simulations.

Due to the lack of spherical symmetry, the previously mentioned description of particles' trajectories in terms of parameters $\varepsilon, \ell_k, \tau_l, \gamma$ will no longer be exact, but approximate, and not all of these parameters will be conserved with time.

In particular, the normalized angular momentum ℓ_k will be changing, and hence the distribution functions $f(\varepsilon, \ell_i)$ and $\hat{f}(\varepsilon, \ell)$ will change with time.

For a single particle

$$\dot{\ell}_i = \epsilon_{ikl} x_k a_l \quad \Rightarrow \quad \dot{\ell} = \frac{d}{dt}\sqrt{\ell_i \ell_i} = \varepsilon_{ikl}\frac{\ell_i}{\ell} x_k a_l, \tag{23}$$

where a_l – is the acceleration, caused by attraction to scattered matter concentrations.
Quantity $\dot{\ell}$ can be positive and negative as well.

For the distribution function $\hat{f}(\varepsilon, \ell)$ we can write a usual continuity equation

$$\frac{d}{dt}\hat{f}(\varepsilon, \ell) = -\frac{d}{d\ell}\left(\hat{f}(\varepsilon, \ell)\bar{\dot{\ell}}\right), \qquad (24)$$

where $\bar{\dot{\ell}}$ – is the value of $\dot{\ell}$ averaged over all particles with given ℓ. Here $\hat{f}(\ell)\bar{\dot{\ell}}$ – is a particles "flow" in terms of the module of angular momentum.

Without spherical symmetry, $\bar{\dot{\ell}}$ doesn't have to be zero and can sometimes be negative in the general case. This means an "inflow" of particles to the point $\ell = 0$ in the space of the module of angular momentum.

This leads to $\hat{f}(\varepsilon, 0) \neq 0$, i. e.
Case 1) take place, and hence, the cusp-profile will arise.

Results

1. We managed to relate the type of arising density profile with the asymptotic at zero of the distribution function $\hat{f}(\varepsilon, \ell)$ of particles over the module of the normalized angular momentum:

$\hat{f}(\varepsilon, 0) \neq 0$ – cusp-profile

$\hat{f}(\varepsilon, 0) = 0$ – core-profile

2. If the formation of a static structure (of a regular matter, for example) goes in a given (for example – by the dark matter) spherically symmetric potential well, then the core-profile arises. Moreover, if specific properties of dark matter prevent the appearance of $\hat{f}(\varepsilon, 0) \neq 0$, then the core-profile will also arise for dark matter and we obtain a good agreement with observations.

3. We also managed to provide supporting arguments for the appearance of $\hat{f}(\varepsilon, 0) \neq 0$ in the case of simultaneous formation of the static particles' distribution and the potential well, which leads to the appearance of the cusp-profile.
This result is consistent with the results of numerical simulations.

For details, see the work:

A.D. Kapustin, S.A. Paston, arXiv:2207.04288