Roberge-Weiss Transition in the Lee-Yang approach

R. N. Rogalyov

NRC “Kurchatov Institute” - IHEP

18.07.2022

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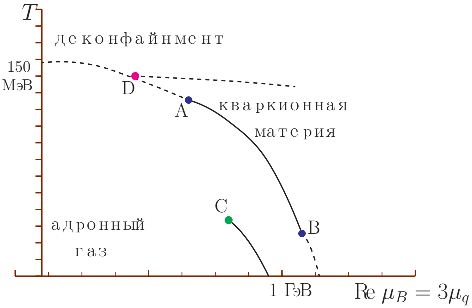
E-print 2103.07442
"Numerical Study of the Roberge-Weiss transition"

V.Bornyakov, N.Gerasimeniuk, V.Goy, A.Korneev, A.Molochkov, A.Nakamura, R.Rogalyov



$$\mu_B = \frac{T}{2n} \ln \left(\frac{\mathcal{P}_n}{\mathcal{P}_{-n}} \right)$$

- Dashed curves - crossover domains;
- AB - chiral phase transition;
- Line from C - phase transition "hadron gas–nuclear liquid".

- To the right of B - color superconductivity and all that;
- Quarkyonic matter - chirally symmetric state in confinement phase;
- Hypothesis on the critical end point A stems from chiral models.

$$Z_{GC}(\theta, T, V) = \sum_j \langle j| \exp\left(\frac{-\hat{H} + \mu\hat{N}}{T}\right)|j\rangle$$
$$= \int \mathbf{D}U e^{-S_G} (\det \mathcal{D}(\mu_q))^{N_f}$$

$$B(\theta) = \frac{1}{N_c}\frac{\partial \ln Z_{GC}}{\partial\theta} \qquad P(\theta, T) = -\frac{1}{V}\ln Z_{GC} \qquad \theta = \frac{\mu_q}{T}$$

Sign problem

$$\mathcal{D}(\mu_q) = \imath\gamma^\mu D_\mu + (\mu_q\gamma^0 + m)I$$

I is the unit matrix in color indices, and

$$\{\det \mathcal{D}(\mu_q)\}^* = \det\{\mathcal{D}(-\mu_q^*)\}$$

which is not real at real μ_q.

Grand canonical partition function

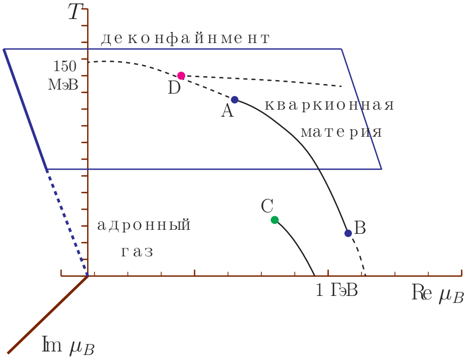
$$Z_{GC}(\theta, T, V) = \sum_j \langle j| \exp\left(\frac{-\hat{H} + \mu\hat{N}}{T}\right) |j\rangle$$

can be expanded in Laurent series in fugacity $\xi = e^{\theta}$ ($\theta = \mu/T = \theta_R + \imath\theta_I$):

$$Z_{GC}(\theta, T, V) \ = \ \sum_{n=-\infty}^{\infty} Z_C(n, T, V)\xi^n,$$

The inverse transformation has the form

$$Z_C\,(n, T, V) = \int_{-\pi}^{\pi} \frac{d\theta_I}{2\pi} e^{-in\theta_I} Z_{GC}(\theta, T, V)\Big|_{\theta_R=0} \ .$$



We study singularities of TD functions (quark density, pressure, entropy etc) at complex values of μ_q.

Roberge-Weiss hypothesis:

Space of states includes only colorless states at all temperatures and chemical potentials.

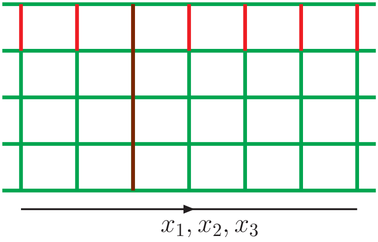
$\theta_B = N_c\theta_q = N_c\theta;$ $\boxed{B \in \mathbb{Z}}$

$$Z_{GC}(\theta_I) = Z_{GC}(\theta_I + 2\pi/N_c)$$

.

$$Z_C\,(n, T, V) = \int_0^{2\pi/N_c} \frac{d\theta_I}{2\pi} e^{-in\theta_I} Z_{GC}(\theta_I, T, V)\ ,$$

$\xi = e^{\theta N_c}$

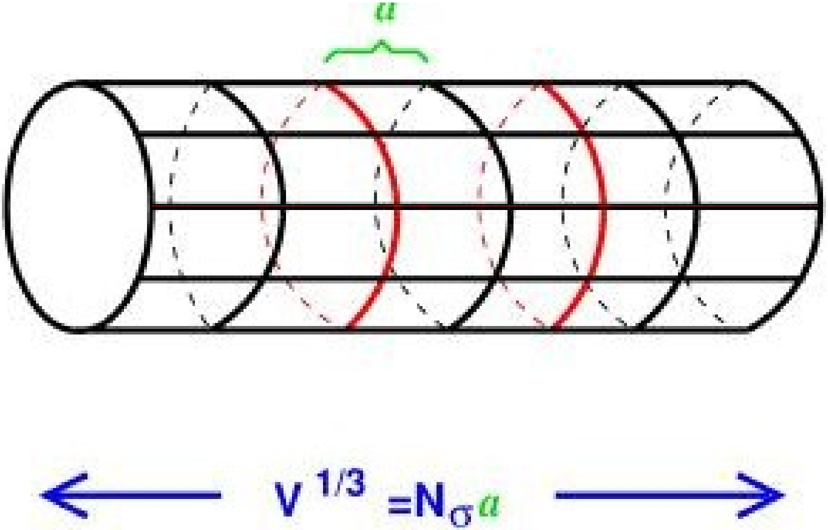


$$L(x_1, x_2) = P\exp\left(iga\oint A^c_\mu(z)\Gamma^c dz\right) \to Tr\prod_{j=1}^{N_\tau} U(x_4 + j, \vec{x}).$$

Center symmetry:

$\mathbb{Z}_N : U_{x,\mu} \to e^{2\imath\pi/N} U_{x,\mu}$

- $SU(2)$: $L(\vec{x}) \to -L(\vec{x})$
- $SU(3)$: $L(\vec{x}) \to \mathrm{diag}(e^{2\imath\pi/3}, e^{2\imath\pi/3}, e^{2\imath\pi/3})L(\vec{x})$


a
V 1/3 =Nσa

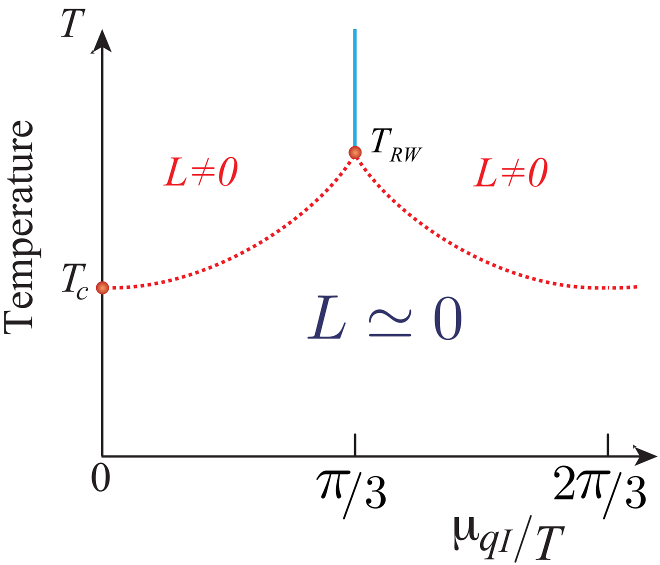
$$A_\mu \to A_\mu^\Lambda = (\Lambda Z)^\dagger A_\mu (\Lambda Z) + \frac{i}{g}\,(\Lambda Z)^\dagger \partial_\mu (\Lambda Z).$$

$$A_\mu \to A_\mu^\Lambda = \Lambda^\dagger A_\mu \Lambda + \frac{i}{g}\,\Lambda^\dagger \partial_\mu \Lambda.$$

For $SU(3)$, as an example:

$$Z \in \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} e^{\frac{2i\pi}{3}} & 0 & 0 \\ 0 & e^{\frac{2i\pi}{3}} & 0 \\ 0 & 0 & e^{\frac{2i\pi}{3}} \end{pmatrix}, \begin{pmatrix} e^{\frac{4i\pi}{3}} & 0 & 0 \\ 0 & e^{\frac{4i\pi}{3}} & 0 \\ 0 & 0 & e^{\frac{4i\pi}{3}} \end{pmatrix} \right\}$$

Gauge transformation is the same on both sides!

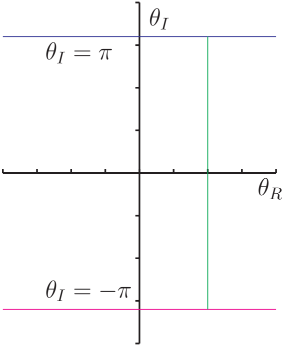

T
Temperature
T_c
T_RW
L≠0
L≠0
L ≃ 0
0
π/3
2π/3
μ_qI/T

Lee-Yang approach

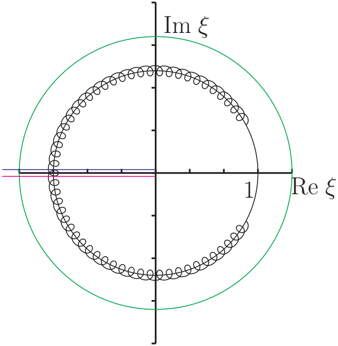
- Phase transition - nonanalyticity in TD functions:

  how does it emerge as $V \to \infty$?
- $Z_{GC}(\mu/T, T, V)$ is a polynomial in $\xi = e^{\theta}$
- Roots of such polynomial are arranged along some line in the ξ plane
  3mm]
- Such line becomes a cut in the Riemann surface of $Z_{GC}(\mu)$

Roots' location: Unit circle (Ising model)
Negative real semiaxis (high-T QCD)



Chemical potential plane



Fugacity plane

Wavy line - LYZ in the Ising model
at $T > T_c$

$N_s^3 \times 4$ lattices with $N_s = 16, 20, 40$ at $T/T_c = 1.35$, $m_\pi/m_\rho = 0.8$.

$$
\begin{aligned}
S &= S_g + S_q, \\
S_q &= \sum_{f=u,d} \sum_{x,y} \bar{\psi}_x^f \Delta_{x,y} \psi_y^f,
\end{aligned}
$$

$$
\begin{aligned}
\Delta_{x,y} = \; & \delta_{xy} - \kappa \sum_{i=1}^{3} \{(1-\gamma_i) U_{x,i} \delta_{x+\hat{i},y} (1+\gamma_i) U_{y,i}^\dagger \delta_{x,y+\hat{i}}\} \\
& -\kappa \{e^{a\mu_q} (1-\gamma_4) U_{x,4} \delta_{x+\hat{4},y} + e^{-a\mu_q} (1+\gamma_4) U_{y,4}^\dagger \delta_{x,y+\hat{4}}\} \\
& + \delta_{xy} c_{SW} \frac{\kappa}{\imath} \sum_{\mu<\nu} \sigma_{\mu\nu} P_{\mu\nu},
\end{aligned}
$$

where $\kappa = 0.136931$ is the hopping parameter.

$$S_g = -\beta \sum_{x,\mu\nu} \left(c_0 W^{1\times 1}_{\mu\nu}(x) + c_1 W^{1\times 2}_{\mu,\nu}(x) \right),$$

$\beta = 6/g^2 = 2.00$, $c_1 = -0.331$, $c_0 = 1 - 8c_1$.

Thus we compute $B(\mu_B, T)$ and can restore the grand canonical partition function.

$$B(\theta) = \frac{1}{N_c} \frac{\partial (T \ln Z_{GC})}{\partial \mu_q} \implies Z_{GC}(\theta_I)|_{\theta_R=0} = \exp\left(N_c \int_0^{\theta_I} B(x)\, dx \right)$$

We use variables $Z_n = \dfrac{Z_C(n,T,V)}{Z_C(0,T,V)} \qquad (Z_n = Z_{-n})$

Partial probabilities: $\mathcal{P}_n = \dfrac{Z_C(n,T,V)}{Z_{GC}(\theta=0,T,V)} = \dfrac{Z_n}{1+2\sum_{k=1}^{\infty} Z_k}$.

Fit function at $T > T_{RW}$: $\quad B(\theta_I)\Big|_{\theta_R=0} = \imath \sum_{n=1}^{N_{param}} a_{2n-1}\theta_I^{2n-1}$

Problem: Given a_n, find Z_n

Solution: $\quad Z_C\,(n,T,V) = \displaystyle\int_0^{2\pi} \frac{d\theta_I}{2\pi} e^{-in\theta_I} Z_{GC}(\theta_I,T,V)$,

High-precision computations are needed

Problem of negative Z_n:

Z_n obtained by this method

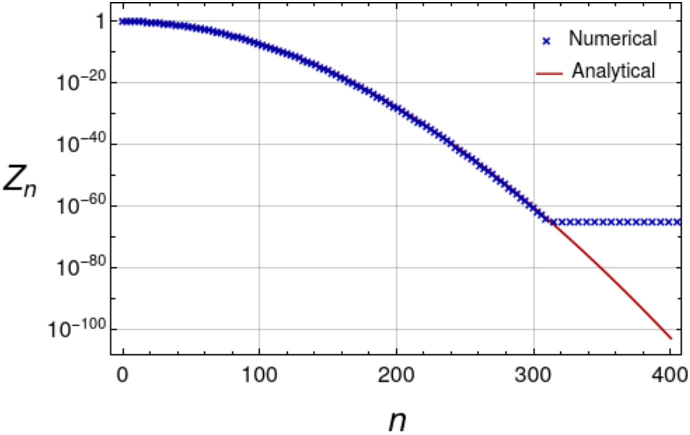
- have alternating sign strarting from some n:
  $\varrho = \frac{n}{VT^3} = \frac{nN_t^3}{N_s^3} \gtrsim 1.6$
- decrease in absolute value slowly

Solution: Use the asymptotic expansion!

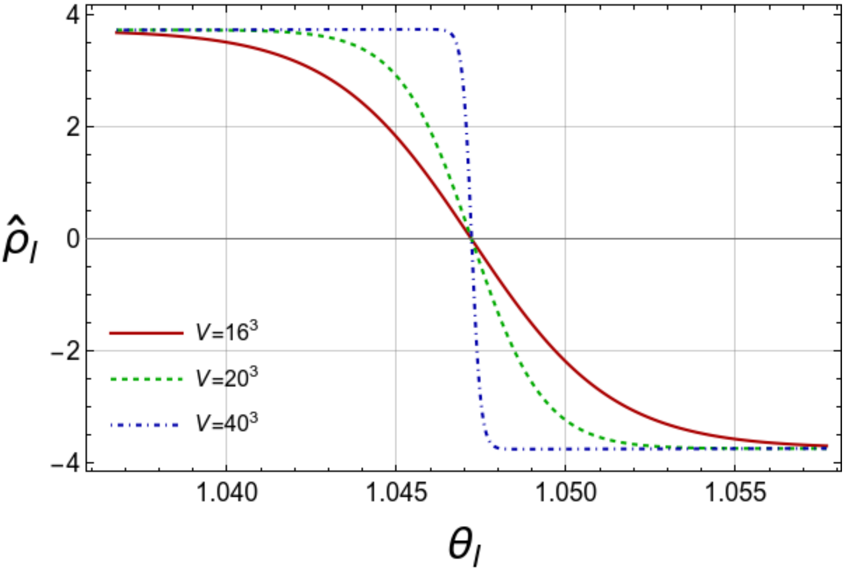
$$Z_{nA} = \frac{\int_{-\imath\pi/3}^{\imath\pi/3} d\theta e^{-\nu F_n(\theta)}}{\int_{-\imath\pi/3}^{\imath\pi/3} d\theta e^{-\nu F_0(\theta)}} \qquad \text{where} \qquad \nu = VT^3$$

and

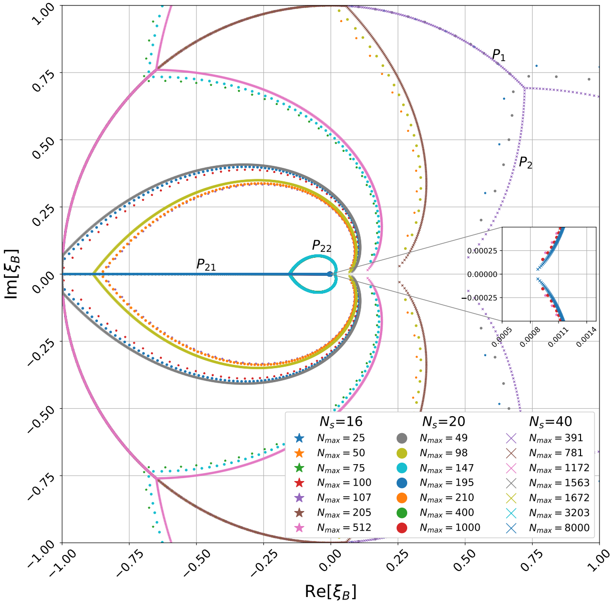
$$F_n(\theta) = -\varrho\theta + \frac{1}{2}a_1\theta^2 + \frac{1}{4}a_3\theta^4$$



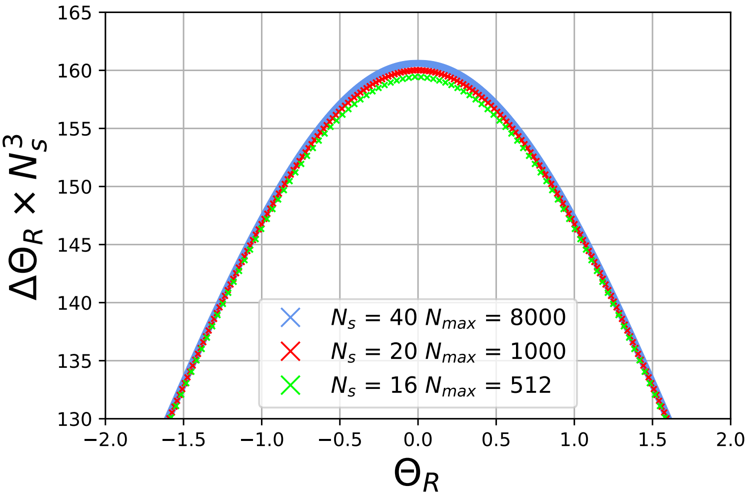
Numerical
Analytical
1
10−20
10−40
10−60
10−80
10−100
Zn
0
100
200
300
400
n



4
2
0
−2
−4
$\hat{\rho}_I$
V=16³
V=20³
V=40³
1.040
1.045
1.050
1.055
θ_I



P_1
P_2
P_{22}
P_{21}
Im[ξ_B]
Re[ξ_B]
N_s=16
N_{max} = 25
N_{max} = 50
N_{max} = 75
N_{max} = 100
N_{max} = 107
N_{max} = 205
N_{max} = 512
N_s=20
N_{max} = 49
N_{max} = 98
N_{max} = 147
N_{max} = 195
N_{max} = 210
N_{max} = 400
N_{max} = 1000
N_s=40
N_{max} = 391
N_{max} = 781
N_{max} = 1172
N_{max} = 1563
N_{max} = 1672
N_{max} = 3203
N_{max} = 8000



165
160
155
150
145
140
135
130
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
$\Delta\Theta_R \times N_s^3$
Θ_R
N_s = 40 N_{max} = 8000
N_s = 20 N_{max} = 1000
N_s = 16 N_{max} = 512

Conclusions:

- In the infinite-volume limit, there appears a discontinuity in the dependence of $\hat{\rho}$ on θ_I
- High values of ϱ_{max} to obtain correct results for LYZ
- The RW phase transition can be described in the Lee-Yang approach