Solutions of pentagon identity and multidimensional orthogonal polynomials

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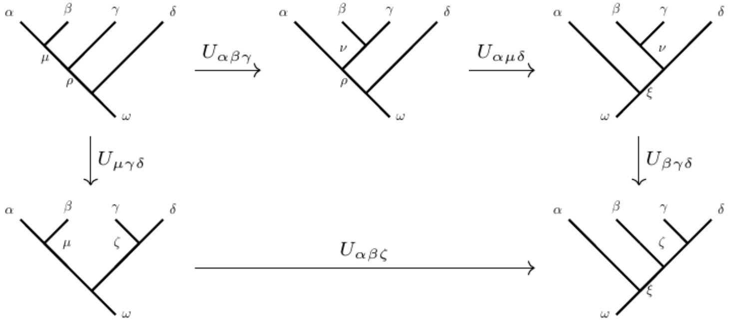
Pentagon identity



$$\sum_{\nu,l,m,k} D_\nu \begin{Bmatrix} \alpha & \beta & \mu \\ \gamma & \rho & \nu \end{Bmatrix}_{lm}^{r_1 r_2} \begin{Bmatrix} \alpha & \nu & \rho \\ \delta & \omega & \xi \end{Bmatrix}_{kr_4}^{ms_1} \begin{Bmatrix} \beta & \gamma & \nu \\ \delta & \xi & \zeta \end{Bmatrix}_{s_2 r_3}^{lk} =$$

$$= \sum_t \begin{Bmatrix} \mu & \gamma & \rho \\ \delta & \omega & \zeta \end{Bmatrix}_{s_2 t}^{r_2 s_1} \begin{Bmatrix} \alpha & \beta & \mu \\ \zeta & \omega & \xi \end{Bmatrix}_{r_3 r_4}^{r_1 t}$$

Quantum algebra $U_q(sl_N)$

Let $(c_{ij})_{1\leq i,j\leq N-1}$ be the Cartan matrix. The quantized universal enveloping algebra $U_q(sl_N)$ is defined by the generators $E_i, F_i, K_i = q^{H_i}, K_i^{-1}$ and the relations:

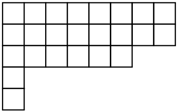
1. $K_iK_j = K_jK_i = 1, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1$
2. $K_iE_jK_i^{-1} = q^{c_{ij}}E_j, \quad K_iF_jK_i^{-1} = q^{-c_{ij}}F_j$
3. $E_iF_j - F_jE_i = \delta_{ij}\dfrac{K_i - K_i^{-1}}{q - q^{-1}}$
4. $\displaystyle\sum_{s=0}^{1-c_{ij}}(-1)^s \left[\begin{array}{c} 1-c_{ij} \\ s \end{array}\right] E_i^{1-c_{ij}-s}E_jE_i^s = 0, \quad i \neq j$

$$\left[\begin{array}{c} n \\ k \end{array}\right] = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Representations and Multiplicity space

Finite-dimensional representations of $U_q(sl_N)$ enumerated by Young diagrams.

$$\lambda = [r_1 \geq r_2 \geq \ldots \geq r_{N-1}] =$$



Let us consider the tensor product of 2 f.-d. irreps $V_\mu \otimes V_\nu$ and decompose it into irreps:

$$V_\mu \otimes V_\nu = \bigoplus_\rho M^\rho_{\mu\nu} \otimes V_\rho \tag{1}$$

$$\text{for } U_q(sl_2) \quad V_{s_1} \otimes V_{s_2} = V_{|s_1-s_2|} \oplus \ldots \oplus V_{s_1+s_2} \tag{2}$$

Here $M^\rho_{\mu\nu}$ is the multiplicity space, i.e. the vector space of highest weight ρ in the product, whose dimension $m = \dim(M^\rho_{\mu\nu})$ is equal to the number of V_ρ in the decomposition. If $m = 1$ is one-dimensional, the representation is called **multiplicity-free**.

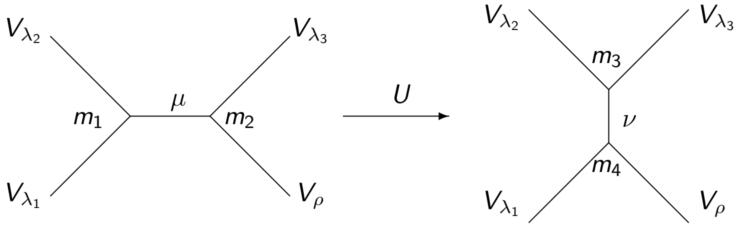
Two bases

Let us consider three representations $V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3}$.
Associativity of tensor product implies that $(V_{\lambda_1} \otimes V_{\lambda_2}) \otimes V_{\lambda_3}$ is isomorphic to $V_{\lambda_1} \otimes (V_{\lambda_2} \otimes V_{\lambda_3})$:

$$(V_{\lambda_1} \otimes V_{\lambda_2}) \otimes V_{\lambda_3} = \left(\bigoplus_{\mu} M_{\mu}^{\lambda_1 \lambda_2} \otimes V_{\mu} \right) \otimes V_{\lambda_3} = \bigoplus_{\mu,\rho} M_{\mu}^{\lambda_1 \lambda_2} \otimes M_{\rho}^{\mu \lambda_3} \otimes V_{\rho} \quad (3)$$

$$V_{\lambda_1} \otimes (V_{\lambda_2} \otimes V_{\lambda_3}) = V_{\lambda_1} \otimes \left(\bigoplus_{\nu} M_{\nu}^{\lambda_2 \lambda_3} \otimes V_{\nu} \right) = \bigoplus_{\rho,\nu} M_{\rho}^{\lambda_1 \nu} \otimes M_{\nu}^{\lambda_2 \lambda_3} \otimes V_{\rho} \quad (4)$$

Graphically one can depict these bases as follows



6j-symbols

The rotation matrix of one basis (3) into another (4) is called a matrix of 6j-symbols or Racah-Wigner matrix:

$$U\begin{bmatrix}\lambda_1 & \lambda_2\\ \lambda_3 & \rho\end{bmatrix}^{m_3 m_4}_{m_1 m_2} : \bigoplus_{\mu} M^{\lambda_1\lambda_2}_{\mu}\otimes M^{\mu\lambda_3}_{\rho} \to \bigoplus_{\nu} M^{\lambda_1\nu}_{\rho}\otimes M^{\lambda_2\lambda_3}_{\nu} \quad (5)$$

$$\begin{Bmatrix}\lambda_1 & \lambda_2 & \mu\\ \lambda_3 & \rho & \nu\end{Bmatrix}^{m_3 m_4}_{m_1 m_2} = \frac{1}{\sqrt{D_\mu\cdot D_\nu}}\, U_{\mu,\nu}\begin{bmatrix}\lambda_1 & \lambda_2\\ \lambda_3 & \rho\end{bmatrix}^{m_3 m_4}_{m_1 m_2} \quad (6)$$

6j-symbols: Applications

6j-symbols are used in the generalized theory of angular momentum to describe complex systems: atoms, nuclei, molecules, hadrons. Partial list of applications:

1. nuclear physics
2. QCD (e.g., Landau-Pomeranchuk-Migdal effect)
3. condensed matter (e.g., ultracold alkaline-earth atoms)
4. conformal field theories (fusion matrix)
5. 3d quantum gravity
6. integrable systems
7. knot theory (Reshetikhin-Turaev invariants)
8. invariant of 3-manifolds (Turaev-Viro invariants)
9. topological quantum computer
10. special functions (e.g., orthogonal polynomials)

Symmetries of 6j-symbols: orthogonality

Values and various properties of quantum 6j-symbols are well-known for $U_q(sl_2)$ [*A.N.Kirillov and N.Y.Reshetikhin, 1989*], but much less is known for $N > 2$. Nevertheless, some properties and relations for them are known for an arbitrary N. Let us briefly discuss them [*C.R.Lienert and P.H.Butler, 1992*].

- Racah matrices are orthogonal:

$$\sum_{\lambda_{12}} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_{123} & \lambda_{23} \end{Bmatrix} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_{123} & \lambda'_{23} \end{Bmatrix} D_{\lambda_{12}} = \frac{\delta_{\lambda_{23},\lambda'_{23}}}{\sqrt{D_{\lambda_{23}} \cdot D_{\lambda'_{23}}}} \tag{7}$$

Symmetries of 6j-symbols: tetrahedron I

Tetrahedral symmetries are the symmetries between 6-j symbols that are generated by permutations of rows and columns

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_{123} & \lambda_{23} \end{Bmatrix} = \begin{Bmatrix} \overline{\lambda_3} & \overline{\lambda_2} & \overline{\lambda_{23}} \\ \overline{\lambda_1} & \overline{\lambda_{123}} & \overline{\lambda_{12}} \end{Bmatrix} = \tag{8}$$

$$= \begin{Bmatrix} \lambda_3 & \overline{\lambda_{123}} & \overline{\lambda_{12}} \\ \lambda_1 & \overline{\lambda_2} & \overline{\lambda_{23}} \end{Bmatrix} = \begin{Bmatrix} \lambda_2 & \overline{\lambda_{12}} & \overline{\lambda_1} \\ \lambda_{123} & \lambda_{23} & \lambda_3 \end{Bmatrix} = \tag{9}$$

$$= \begin{Bmatrix} \lambda_2 & \lambda_1 & \lambda_{12} \\ \overline{\lambda_{123}} & \overline{\lambda_3} & \overline{\lambda_{23}} \end{Bmatrix}, \tag{10}$$

where $\overline{\lambda}$ denotes conjugate representation: $\lambda \otimes \overline{\lambda} \ni \varnothing$

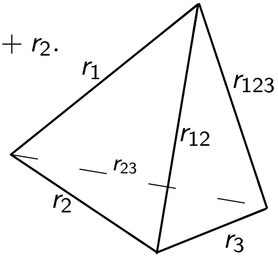
Tetrahedron II

Representations of $U_q(sl_2)$ enumerated by $[r] = \square\square\square\square\square\square\square$.

Tensor product rule is given by :

$$[r_1] \otimes [r_2] = [r_{12}], \quad r_{12} = |r_1 - r_2|, \ldots, r_1 + r_2.$$

$$\left\{ \begin{array}{ccc} [r_1] & [r_2] & [r_{12}] \\ [r_3] & [r_{123}] & [r_{23}] \end{array} \right\} =$$



Tetrahedral symmetry group S_4 contains $4! = 24$ elements.

Additional **Regge symmetries** [*T.Regge,1959*]:

$$\left\{ \begin{array}{ccc} [r_1] & [r_2] & [r_{12}] \\ [r_3] & [r_4] & [r_{23}] \end{array} \right\} = \left\{ \begin{array}{ccc} [p-r_1] & [p-r_2] & [r_{12}] \\ [p-r_3] & [p-r_4] & [r_{23}] \end{array} \right\}, \qquad (11)$$

where $p = \frac{1}{2}(r_1 + r_2 + r_3 + r_4)$. In total, for $N = 2$ we have 144 symmetries, full symmetry group $S_4 \times S_3$.

Symmetries of 6j-symbols: Racah identity

The Racah back-coupling rule is a general property of 6-j symbols:

$$q^{C_2(\lambda_1)+C_2(\lambda_3)+C_2(\lambda_{12})+C_2(\lambda_{23})-C_2(\lambda_2)-C_2(\lambda_{123})} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_{123} & \lambda_{23} \end{Bmatrix} = $$
$$= \sum_{\nu} \pm D_\nu q^{C_2(\nu)} \begin{Bmatrix} \lambda_{23} & \nu & \lambda_{12} \\ \lambda_3 & \lambda_{123} & \lambda_1 \end{Bmatrix} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \overline{\lambda_{23}} & \nu & \overline{\lambda_3} \end{Bmatrix} \tag{12}$$

Symmetries of 6j-symbols: pentagon I

There are 5 possibilities to decompose the tensor product of $T_{\lambda_1} \otimes T_{\lambda_2} \otimes T_{\lambda_3} \otimes T_{\lambda_{123}}$ into irreducible representations:

$$((T_{\lambda_1} \otimes T_{\lambda_2}) \otimes T_{\lambda_3}) \otimes T_{\lambda_{123}} \tag{13}$$

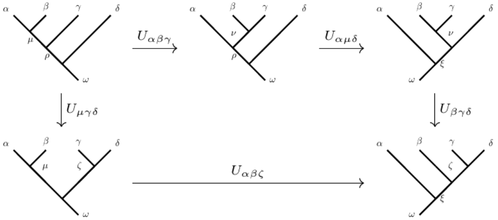
$$(T_{\lambda_1} \otimes (T_{\lambda_2} \otimes T_{\lambda_3})) \otimes T_{\lambda_{123}} \tag{14}$$

$$(T_{\lambda_1} \otimes T_{\lambda_2}) \otimes (T_{\lambda_3} \otimes T_{\lambda_{123}}) \tag{15}$$

$$T_{\lambda_1} \otimes ((T_{\lambda_2} \otimes T_{\lambda_3}) \otimes T_{\lambda_{123}}) \tag{16}$$

$$T_{\lambda_1} \otimes (T_{\lambda_2} \otimes (T_{\lambda_3} \otimes T_{\lambda_{123}})) \tag{17}$$

We can go from (13) to (17) by (13) → (14) → (16) → (17) and also by the chain (13) → (15) → (17) using 6j-symbols at each step.



Symmetries of 6j-symbols: pentagon II

By this way we get Biedenharn-Elliott identity, or pentagon identity:

$$\sum_{\lambda_{23}} D_{\lambda_{23}} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_{123} & \lambda_{23} \end{Bmatrix} \begin{Bmatrix} \lambda_1 & \lambda_{23} & \lambda_{123} \\ \lambda_4 & \lambda_{1234} & \lambda_{234} \end{Bmatrix} \begin{Bmatrix} \lambda_2 & \lambda_3 & \lambda_{23} \\ \lambda_4 & \lambda_{234} & \lambda_{34} \end{Bmatrix} = \\ = \begin{Bmatrix} \lambda_{12} & \lambda_3 & \lambda_{123} \\ \lambda_4 & \lambda_{1234} & \lambda_{34} \end{Bmatrix} \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_{34} & \lambda_{1234} & \lambda_{234} \end{Bmatrix}$$

Theorem (Butler'1981)

Non-primitive 6j-symbols can always be converted to primitive ones.

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \varnothing \\ \lambda_3 & \lambda_{123} & \lambda_{23} \end{Bmatrix}, \begin{Bmatrix} \lambda_1 & \lambda_2 & \square \\ \lambda_3 & \lambda_{123} & \lambda_{23} \end{Bmatrix} \Rightarrow \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda_{123} & \lambda_{23} \end{Bmatrix}$$

Known results

- *$U_q(sl_2)$, A.N.Kirillov and N.Y.Reshetikhin, 1989*

$$\left\{ \begin{matrix} r_1 & r_2 & i \\ r_3 & r_4 & j \end{matrix} \right\}^{11}_{11} = \sqrt{[2i+1][2j+1]}\,(-1)^{\sum_{m=1}^{4} r_m}\,\theta(r_1,r_2,i)$$

$$\times\theta(r_3,r_4,i)\,\theta(r_4,r_1,j)\theta(r_2,r_3,j)\sum_{k\geq 0}(-1)^k[k+1]!$$

$$\Big([k-r_1-r_2-i]![k-r_3-r_4-i]![k-r_1-r_4-j]![k-r_2-r_3-j]!$$

$$[r_1+r_2+r_3+r_4-k]![r_1+r_3+i+j-k]![r_2+r_4+i+j-k]! \Big)^{-1},$$

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}},\ \theta(a,b,c) = \sqrt{\frac{[a-b+c]![b-a+c]![a+b-c]!}{[a+b+c+1]!}}$$

- *$U_q(sl_N)$, S. Alisauskas, 1995,* $\left\{ \begin{matrix} [r_1] & \lambda_2 & \lambda_{12} \\ [r_3] & \lambda_{123} & \lambda_{23} \end{matrix} \right\}^{11}_{11}$

Solution of the pentagon identity in $U_q(sl_2)$

K.S.Rao, T.S.Santhanam, R.A.Gustafson, 1987

$\lambda_4 = [2], \lambda_1 = [r_1], \ \lambda_2 = [r_2], \ \lambda_3 = [r_3], \ \lambda_{12} = [x], \ \lambda_{23} = [y_i]$

$\lambda_{34} = \lambda_3 \otimes \lambda_4 = \{[\mathbf{r_3 - 2}], [\mathbf{r_3}], [\mathbf{r_3 + 2}]\} = [r_3]$

$\lambda_{123} = \lambda_{1234} = [R], \ \lambda_{234} = [y_0]$

$$\sum_{y_i} D_{y_i} \begin{Bmatrix} r_1 & r_2 & x \\ r_3 & R & y_i \end{Bmatrix} \begin{Bmatrix} r_1 & y_i & R \\ 2 & R & y_0 \end{Bmatrix} \begin{Bmatrix} r_2 & r_3 & y_i \\ 2 & y_0 & r_3 \end{Bmatrix} = \\ = \begin{Bmatrix} x & r_3 & R \\ 2 & R & r_3 \end{Bmatrix} \begin{Bmatrix} r_1 & r_2 & x \\ r_3 & R & y_0 \end{Bmatrix}$$

$y_i \in 2 \otimes y_0 = y_0 - 2 \oplus y_0 \oplus y_0 + 2$

$$\sum_{i=-1}^{1} c_i \begin{Bmatrix} r_1 & r_2 & x \\ r_3 & R & y_i \end{Bmatrix} = \begin{Bmatrix} x & r_3 & R \\ 2 & R & r_3 \end{Bmatrix} \begin{Bmatrix} r_1 & r_2 & x \\ r_3 & R & y_0 \end{Bmatrix}$$

q-Racah polynomial

$$\mathfrak{R}_n(\nu(x);\alpha,\beta,\gamma,\delta|q) = {}_4\Phi_3\left(\begin{matrix} -n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1 \\ \alpha+1, \beta+\delta+1, \gamma+1 \end{matrix}; q, q\right)$$

where $n = 0, 1, \dots, L$ is the degree of the polynomial in variable $\nu(x) := q^{-x} + q^{\gamma+\delta+x+1}$

Three-term recurrence relation:

$$[x]_q[x+\gamma+\delta+1]_q\, \mathfrak{R}_n(\nu(x)) = A_n\, \mathfrak{R}_{n+1}(\nu(x)) - (A_n+C_n)\, \mathfrak{R}_n(\nu(x)) + C_n\, \mathfrak{R}_{n-1}(\nu(x))$$

with coefficients specified for q-Racah polynomial:

$$A_n = \frac{[n+\alpha+1]_q[n+\alpha+\beta+1]_q[n+\beta+\delta+1]_q[n+\gamma+1]_q}{[2n+\alpha+\beta+1]_q[2n+\alpha+\beta+2]_q}$$

$$C_n = \frac{[n]_q[n+\alpha+\beta-\gamma]_q[n+\alpha-\delta]_q[n+\beta]_q}{[2n+\alpha+\beta]_q[2n+\alpha+\beta+1]_q}$$

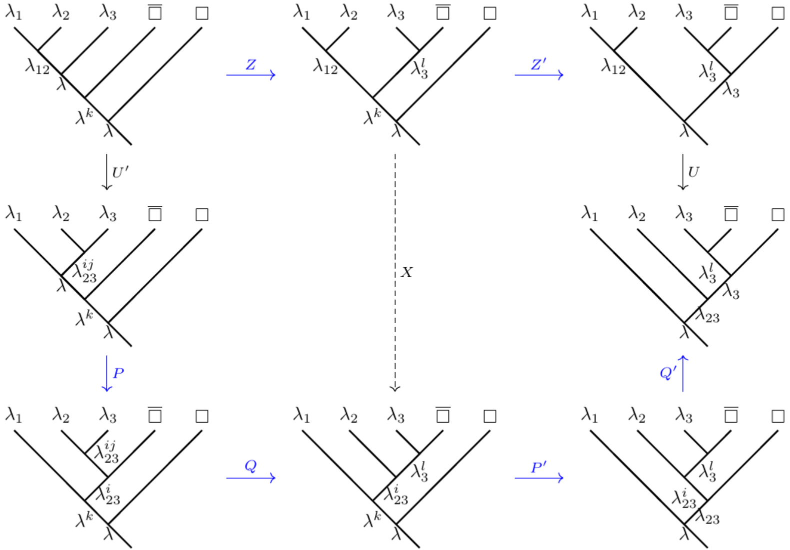
$$\sum_x \mu(x)\, \mathfrak{R}_n(\nu(x))\, \mathfrak{R}_m(\nu(x)) = h_n\delta_{nm} \qquad (18)$$

Troubles with $U_q(sl_N)$

$$\lambda_1 = [r_1],\ \lambda_2 = [r_2],\ \lambda_3 = [r_3],\ \lambda_{12} = [x],\ \lambda_{23} = [y_i]$$
$$\lambda_4 = \mathrm{adj} = [2, 1^{N-2}],\ \lambda_{123} = \lambda_{1234} = R,\ \lambda_{234} = [y_0]$$
$$\lambda_{34} = [r_3]$$
$$y_i = \mathrm{adj} \otimes y_0\ \cap\ [r_2] \otimes [r_3] = \{y_0 - 2, y_0, y_0 + 2\}$$

$$\sum_{i=-1}^{1} D_{y_i} \begin{Bmatrix} r_1 & r_2 & x \\ r_3 & R & y_i \end{Bmatrix} \begin{Bmatrix} r_1 & y_i & R \\ \mathrm{adj} & R & y \end{Bmatrix}^{21}_{13} \begin{Bmatrix} r_2 & r_3 & y_i \\ \mathrm{adj} & y & r_3 \end{Bmatrix}^{21}_{12} =$$
$$= \begin{Bmatrix} x & r_3 & R \\ \mathrm{adj} & R & r_3 \end{Bmatrix}^{21}_{13} \begin{Bmatrix} r_1 & r_2 & x \\ r_3 & R & y \end{Bmatrix}$$

Linear recursion with primitive 6j-symbols, I



Linear recursion with primitive 6j-symbols, II

Theorem Any 6j-symbol satisfies the linear system

$$\sum_{i,j,p,m} E_{kl}(\lambda_{23},i,j)^{pm}_{r_3r_4}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_{12}\\ \lambda_3 & \lambda & \lambda_{23}^{ij}\end{Bmatrix}^{r_1r_2}_{pm} = \sum_t x_{kl}(\lambda_{12})^{r_2}_t \begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_{12}\\ \lambda_3 & \lambda & \lambda_{23}\end{Bmatrix}^{r_1t}_{r_3r_4}$$

$$x_{kl}(\mu)^{r_2}_t = \sum_{\tilde t}\begin{Bmatrix}\lambda_{12} & \lambda_3^l & \lambda^k\\ \Box & \lambda & \lambda_3\end{Bmatrix}^{\tilde t 1}_{1t}\begin{Bmatrix}\lambda_{12} & \lambda_3 & \lambda\\ \overline{\Box} & \lambda^k & \lambda_3^l\end{Bmatrix}^{r_2 1}_{1\tilde t}$$

$$E(\lambda_{23},i,j)^{pm}_{r_3r_4} = \sum_{s,r_5}\begin{Bmatrix}\lambda_1 & \lambda_{23}^i & \lambda^k\\ \Box & \lambda & \lambda_{23}\end{Bmatrix}^{r_5 1}_{1r_4}\begin{Bmatrix}\lambda_2 & \lambda_3^l & \lambda_{23}^i\\ \Box & \lambda_{23} & \lambda_3\end{Bmatrix}^{s1}_{1r_3}\times$$

$$\times\begin{Bmatrix}\lambda_1 & \lambda_{23}^{ij} & \lambda\\ \overline{\Box} & \lambda^k & \lambda_{23}^i\end{Bmatrix}^{m1}_{1r_5}\begin{Bmatrix}\lambda_2 & \lambda_3 & \lambda_{23}^{ij}\\ \overline{\Box} & \lambda_{23}^i & \lambda_3^l\end{Bmatrix}^{p1}_{1s}$$

Multidimensional orthogonal polynomials

It is known [C.F. Dunkl, Y.Xu' 2014] that for orthogonal polynomials of d variables there are d three-term relations. The set of independent variables is $\{x_r\}_{r=1}^{d}$. The generalized three-term recurrence relation has the form:

$$\sum_{b=1}^{d_{n+1}} A^r_{n,ab}\, p_{n+1,b}(x) + \sum_{b=1}^{d_n} B^r_{n,ab}\, p_{n,b}(x) + \sum_{b=1}^{d_{n-1}} C^r_{n,ab}\, p_{n-1,b}(x) = x_r\, p_{n,a}(x).$$

We introduce a height function $n(\nu) = -\nu_1 + const$. It allows us to split all terms with $E_{kl}(\lambda_{23}, i, j)$ in the linear recursion into 3 groups:

$$\left(\sum_{i,j:\,\Delta n=1} + \sum_{i,j:\,\Delta n=0} + \sum_{i,j:\,\Delta n=-1} \right) E_{kl}(\lambda_{23}, i, j) \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda & \lambda_{23}^{ij} \end{Bmatrix} = $$

$$= x_{kl}(\lambda_{12}) \begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda & \lambda_{23} \end{Bmatrix}$$

$$\Delta n := n(\lambda_{23}^{ij}) - n(\lambda_{23})$$

Results

Theorem

If the coefficients in a recurrence relation on multiplicity-free 6j-symbol are from 2-dimensional multiplicity-free Racah matrices, then 6j-symbol satisfy q-Racah recurrence relation.

Example 1. $\begin{Bmatrix} [r_1] & [r_2] & [r_{12}-x,x] \\ [r_3] & R & [r_{23}-y,y] \end{Bmatrix}_{11}^{11}$ for $U_q(sl_N)$

Example 2. $\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_{12} \\ \lambda_3 & \lambda & \lambda_{23} \end{Bmatrix}_{11}^{11}$ for $U_q(sl_N)$ with $\lambda = \lambda_1 + \lambda_2 + \lambda_3 - s \cdot \mathbf{e_1} + s \cdot \mathbf{e_2}$.

Thank you for your attention!