Canonical quantization of a massive scalar field in the Schwarzschild spacetime

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Real massive scalar field

Real massive scalar field in the Schwarzschild spacetime:

$$\sqrt{-g}g^{00}\ddot{\phi} + \partial_i \left(\sqrt{-g}g^{ij}\partial_j\phi\right) + m^2\sqrt{-g}\phi = 0.$$

T_{00} component of the energy-momentum tensor is

$$T_{00} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}g_{00}g^{ij}\partial_i\phi\partial_j\phi + \frac{m^2}{2}g_{00}\phi^2.$$

The conservation law looks like

$$\nabla_\mu T^\mu_\nu = \frac{1}{\sqrt{-g}}\frac{\partial\left(\sqrt{-g}\,T^\mu_\nu\right)}{\partial x^\mu} - \frac{1}{2}\frac{\partial g_{\mu\rho}}{\partial x^\nu}T^{\mu\rho} = 0.$$

For $\nu = 0$

$$\frac{\partial}{\partial x^0}\int\sqrt{-g}\,T^0_0 d^3x = 0.$$

The Hamiltonian of the system is

$$H = \int\sqrt{-g}\,g^{00}\,T_{00}d^3x = \int\sqrt{-g}\,g^{00}\left(\dot{\phi}^2 - \ddot{\phi}\,\phi\right)d^3x.$$

Outside the black hole

$$\phi_{lm}(r,\theta,\varphi,E) = Y_{lm}(\theta,\varphi) f_l(r,E),$$

where $f_l(r,E)$ satisfies the equation

$$E^2 \frac{r}{r-r_0} f_l(r,E) - m^2 f_l(r,E) + \frac{1}{r^2}\frac{d}{dr}\left(r(r-r_0)\frac{df_l(r,E)}{dr}\right) - \frac{l(l+1)}{r^2} f_l(r,E) = 0.$$

Here r_0 is the Schwarzschild radius. The orthogonality condition is

$$\int_{r_0}^{\infty} \frac{r^3}{r-r_0} f_l(r,E) f_l(r,E') dr = 0$$

for $E \neq E'$.

$$x = r/r_0, \qquad M = mr_0, \qquad \epsilon = Er_0.$$

Introducing the tortoise coordinate

$$z = x + \ln(x-1)$$

and defining $f_l(z,\epsilon) = \frac{u_l(z,\epsilon)}{x(z)}$, one gets

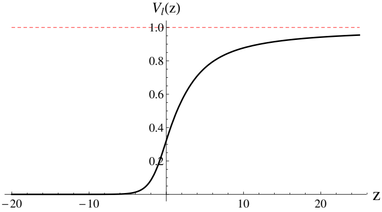
$$-\frac{d^2 u_l(z,\epsilon)}{dz^2} + V_l(z)u_l(z,\epsilon) = \epsilon^2 u_l(z,\epsilon),$$

with

$$V_l(z) = \frac{x(z)-1}{x(z)}\left(M^2 + \frac{l(l+1)}{x^2(z)} + \frac{1}{x^3(z)}\right).$$

The orthogonality condition is

$$\int\limits_1^\infty \frac{x^3}{x-1} f_l^*(x,\epsilon) f_l(x,\epsilon) dx = \int\limits_{-\infty}^\infty u_l^*(z,\epsilon) u_l(z,\epsilon) dz.$$



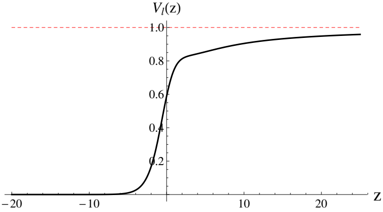
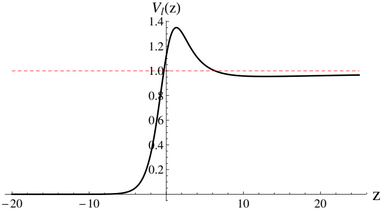


Figure 1: $V_l(z)$ for $M = 1$: $l = 0$ (left plot) and $l = 1$ (right plot).



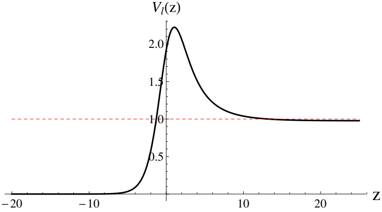


Figure 2: $V_l(z)$ for $M = 1$: $l = 2$ (left plot) and $l = 3$ (right plot).

- J. Barranco, A. Bernal, J. C. Degollado, A. Diez-Tejedor, M. Megevand, M. Alcubierre, D. Nunez and O. Sarbach, "Are black holes a serious threat to scalar field dark matter models?", Phys. Rev. D **84** (2011) 083008 [arXiv:1108.0931 [gr-qc]].

For $z \to -\infty$

$$\frac{d^2 u_l(z,\epsilon)}{dz^2} + \epsilon^2 u_l(z,\epsilon) = 0,$$

leading to

$$u_l(z,\epsilon) \sim \cos(\epsilon z + \gamma(\epsilon, l)).$$

For $z \to \infty$

$$\frac{d^2 u_l(z,\epsilon)}{dz^2} + (\epsilon^2 - M^2) u_l(z,\epsilon) = 0.$$

For $\epsilon^2 < M^2$ solutions take the form

$$u_l(z,\epsilon) \sim e^{-\sqrt{M^2-\epsilon^2}\, z}.$$

Solutions for $\epsilon^2 < M^2$ at $z \to -\infty$, normalized to δ-function, have the form

$$u_l(z,\epsilon) = \sqrt{\frac{2}{\pi}} \cos(\epsilon z + \gamma(\epsilon, l)).$$

For $\epsilon^2 > M^2$ it is convenient to take the scattering states, which, in the initial Schwarzschild coordinates at $r \to \infty$, have the form

$$\phi(\vec{k},\vec{x}) \sim e^{i\vec{k}\vec{x}} + A(\vec{n})\frac{e^{ikr}}{r},$$

where $A(\vec{n})$ is the scattering amplitude, $k = |\vec{k}|$ and $r = |\vec{x}|$. Explicitly:

$$\phi(\vec{k},\vec{x}) = \frac{1}{4\pi k}\sum_{l=0}^{\infty}(2l+1)e^{i\left(\frac{\pi l}{2}+\delta_l(k)\right)}P_l\left(\frac{\vec{k}\vec{x}}{kr}\right)f_l(k,r),$$

where $P_l(..)$ are the Legendre polynomials, $\delta_l(k)$ are phase shifts.

The completeness relation is

$$\sum_{lm}\int_0^M \phi^*_{lm}(\vec{x},E)\phi_{lm}(\vec{y},E)dE+\int \phi^*(\vec{k},\vec{x})\phi(\vec{k},\vec{y})\, d^3k = \frac{\delta^{(3)}(\vec{x}-\vec{y})}{\sqrt{-g}g^{00}}.$$

The orthogonality conditions are

$$\int \sqrt{-g}g^{00}\phi^*_{lm}(\vec{x},E)\phi_{l'm'}(\vec{x},E')\, d^3x = \delta_{ll'}\delta_{mm'}\delta(E-E'),$$

$$\int \sqrt{-g}g^{00}\phi^*(\vec{k},\vec{x})\phi(\vec{k}',\vec{x})\, d^3x = \delta^{(3)}(\vec{k}-\vec{k}'),$$

$$\int \sqrt{-g}g^{00}\phi^*_{lm}(\vec{x},E)\phi(\vec{k},\vec{x})\, d^3x = 0.$$

Canonical quantization

The canonical coordinate is

$$\phi(t,\vec{x}).$$

The canonically conjugate momentum is

$$\pi(t,\vec{x}) = \sqrt{-g(\vec{x})}g^{00}(\vec{x})\dot{\phi}(t,\vec{x}).$$

Canonical commutation relations are

$$[\phi(t,\vec{x}),\pi(t,\vec{y})] = \sqrt{-g(\vec{y})}g^{00}(\vec{y})[\phi(t,\vec{x}),\dot{\phi}(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}),$$
$$[\phi(t,\vec{x}),\phi(t,\vec{y})] = 0,$$
$$[\pi(t,\vec{x}),\pi(t,\vec{y})] = 0 \quad \to \quad [\dot{\phi}(t,\vec{x}),\dot{\phi}(t,\vec{y})] = 0.$$

The quantized scalar field can be represented as

$$\hat{\phi}(t,\vec{x}) = \sum_{lm} \int\limits_0^M \frac{dE}{\sqrt{2E}} \left(e^{-iEt}\phi_{lm}(\vec{x},E)a_{lm}(E) + e^{iEt}\phi^*_{lm}(\vec{x},E)a^\dagger_{lm}(E) \right)$$

$$+ \int \frac{d^3k}{\sqrt{2\sqrt{\vec{k}^2+M^2}}} \left(e^{-i\sqrt{\vec{k}^2+M^2}\,t}\phi(\vec{k},\vec{x})a(\vec{k}) + e^{i\sqrt{\vec{k}^2+M^2}\,t}\phi^*(\vec{k},\vec{x})a^\dagger(\vec{k}) \right)$$

with

$$[a_{lm}(E), a^\dagger_{l'm'}(E')] = \delta_{ll'}\delta_{mm'}\delta(E-E'),$$
$$[a(\vec{k}), a^\dagger(\vec{k}')] = \delta^{(3)}(\vec{k}-\vec{k}').$$

The resulting Hamiltonian is

$$\hat{H} = \sum_{lm} \int\limits_0^M dE\, E\, a^\dagger_{lm}(E)a_{lm}(E) + \int d^3k \sqrt{\vec{k}^2+M^2}\, a^\dagger(\vec{k})a(\vec{k}).$$

Kruskal–Szekeres coordinates

Outside the black hole

$$T + X = 2e^{\frac{t+z}{2}} > 0, \qquad T - X = -2e^{-\frac{t-z}{2}} < 0.$$

For $x \to 1$ ($z \to -\infty$)

$$ds^2 = e^z \left(dt^2 - dz^2\right) \to dT^2 - dX^2.$$

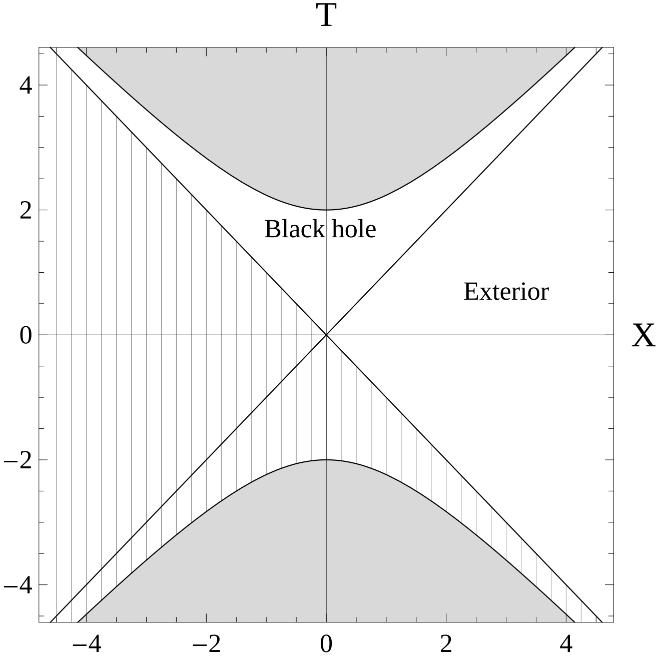
Inside the black hole

$$y = -x - \ln(1 - x), \qquad 0 \leq y < \infty,$$

$$T + X = 2e^{-\frac{t+y}{2}} > 0, \qquad T - X = 2e^{\frac{t-y}{2}} > 0.$$

For $x \to 1$ ($y \to \infty$)

$$ds^2 = e^{-y} \left(dt^2 - dz^2\right) \to dT^2 - dX^2.$$


T
X
Black hole
Exterior
4
2
0
−2
−4
−4
−2
0
2
4

For $z \to -\infty$

$$\frac{d^2 u_l(t,z)}{dt^2} - \frac{d^2 u_l(t,z)}{dz^2} = 0 \quad \to \quad u_l(t,z) = g(t \pm z).$$

For $y \to \infty$

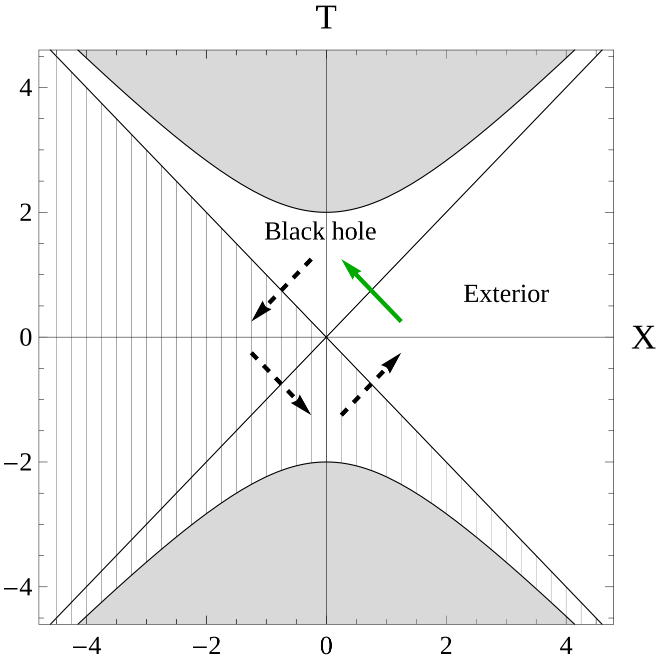
$$\frac{d^2 u_l(t,y)}{dt^2} - \frac{d^2 u_l(t,y)}{dy^2} = 0 \quad \to \quad u_l(t,y) = \tilde{g}(t \pm y).$$

Incoming wave packet for $T^2 - X^2 \to 0$ (going into BH):

$$g_i(t+z) = g_i\left(2\ln\left(\frac{T+X}{2}\right)\right) = g_i(-(t+y)).$$

Outgoing wave packet for $T^2 - X^2 \to 0$ (coming out from BH):

$$g_o(t-z) = g_o\left(2\ln\left(-\frac{T-X}{2}\right)\right)$$
$$\neq g_o(\pm(t-y)) = g_o\left(\pm 2\ln\left(\frac{T-X}{2}\right)\right).$$


T
X
Black hole
Exterior
4
2
0
−2
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−4
−2
0
2
4

- A half of the Kruskal–Szekeres plane cannot provide a description of all solutions of the initial differential equation. Perhaps, it does not provide a complete set of eigenfunctions. In the latter case, consistent quantization in the whole Schwarzschild spacetime is impossible.

- In any case, the approach of D.G. Boulware, "Quantum Field Theory in Schwarzschild and Rindler Spaces", Phys. Rev. D **11** (1975) 1404 (starting from construction of the wave packets that cross the horizon) cannot be realized.

- Complete Kruskal–Szekeres plane?

- G. 't Hooft, "Virtual Black Holes and Space-Time Structure", Found. Phys. **48** (2018) 1134,
  G. 't Hooft, "The quantum black hole as a theoretical lab, a pedagogical treatment of a new approach", [arXiv:1902.10469 [gr-qc]].

  Antipodal mapping between two parts of the Kruskal–Szekeres plane (similar to orbifolds in models with extra dimensions).

- G. 't Hooft, "Quantum clones inside black holes", [arXiv:2206.04608 [gr-qc]].

  Antipodal mapping leads to contradiction $\Rightarrow$ cloning of quantum states, which leads to physical (not geometrical) identification of two exterior regions (interior regions turn out to play no role at all). However, the problem of closed time-like curves emerges.

Conclusion

- Go further – only a single quadrant of the Kruskal–Szekeres plane (i.e., the exterior region of our black hole) is physically relevant.
- The exterior region of the Schwarzschild spacetime allows one to construct a consistent QFT.
- Time t is a global QFT time? No problems with constructing the -in and -out states. Similar to models with warped extra dimensions.
- Matter (represented by fields) cannot penetrate into the Schwarzschild black hole from the point of view of a distant observer (which is also represented by fields). Black hole is completely isolated?
- Matter can move from the horizon towards $r \to \infty$ (recall that there exist wave packets $g(t \pm z)$). However, it is not the Hawking radiation as it is usually regarded.
- Collapsar is a different story.

Thank you for your attention!