Canonical Bigravity

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The Lagrangian is as follows

$$\mathcal{L} = \mathcal{L}^{(f)} + \mathcal{L}^{(g)} - \sqrt{-g}\, U(f_{\mu\nu}, g_{\mu\nu}),$$

where

$$\mathcal{L}^{(f)} = \frac{1}{16\pi G^{(f)}} \sqrt{-f}\, f^{\mu\nu} R^{(f)}_{\mu\nu},$$

and

$$\mathcal{L}^{(g)} = \frac{1}{16\pi G^{(g)}} \sqrt{-g}\, g^{\mu\nu} R^{(g)}_{\mu\nu} + \mathcal{L}^{(g)}_{M}(\phi^A, g_{\mu\nu}),$$

and

$$U = \frac{m^2}{2\kappa} F(g^{\mu\alpha} f_{\alpha\nu}).$$

Bigravity and bimetric theory: history

- Bimetric theory: *N. Rosen (1940)*
- f-g gravity: *Isham, Salam, Strathdee (1971)*
- Strong gravitation: *Zumino (1971)*
- Relativistic Theory of Gravitation (RTG) *Logunov, Mestvirishvili (1984)*
- Bigravity: *Damour, Kogan (2002)*
- dRGT potential *de Rham, Gabadadze, Tolley (2011)*
- Bimetric gravity (= bigravity): *Hassan, R. Rosen (2011)*
- Multi-metric theories: *Hinterbichler, R. Rosen (2012)*

The difference between gauge and coordinate transformations

$$\delta A_\mu = \partial_\mu \phi$$

$$\delta q^i = \dot{q}^i \delta t$$

In the General Relativity the role of A_μ is played by space-time metric $g_{\mu\nu}$ and the coordinate transformations include dynamics

$$x^\mu \to x'^\mu(x^\alpha) = x^\mu + \xi^\mu(x^\alpha), \tag{1}$$
$$\delta g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} = \tag{2}$$
$$= \dot{g}_{\mu\nu}\delta t + g_{\mu\nu,k}\xi^k + g_{\mu\alpha}\xi^\alpha_{,\nu} + g_{\alpha\nu}\xi^\alpha_{,\mu} \tag{3}$$

The flexible time

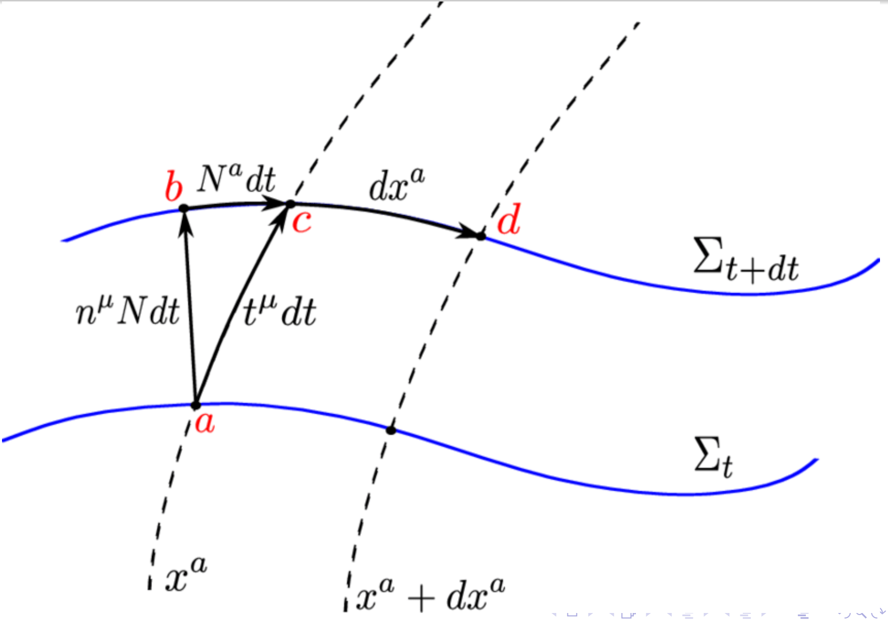
In the classical mechanics one may obtain invariance under general time transformations

$$t \to \tau = \tau(t), \qquad t = t(\tau)$$

by adding a new variable $q(t) \equiv t(\tau)$ to the old variables $q_i(t)$, then the corresponding velocity $N = dt/d\tau$ appears in the Hamiltonian as a Lagrange multiplyer standing at the constraint equation

$$H_{new} = NR, \qquad R = p + H_{old} \approx 0$$

Embedding of space into space-time



Lapse and shift

Field theory in 4-dimensional space-time requires 4 Lagrange multiplyers and 4 constraints. These multiplyers arise in the decomposition of 4-vector by the base formed of a unit normal to the spatial hypersurface and three tangentional vectors to this hypersurface

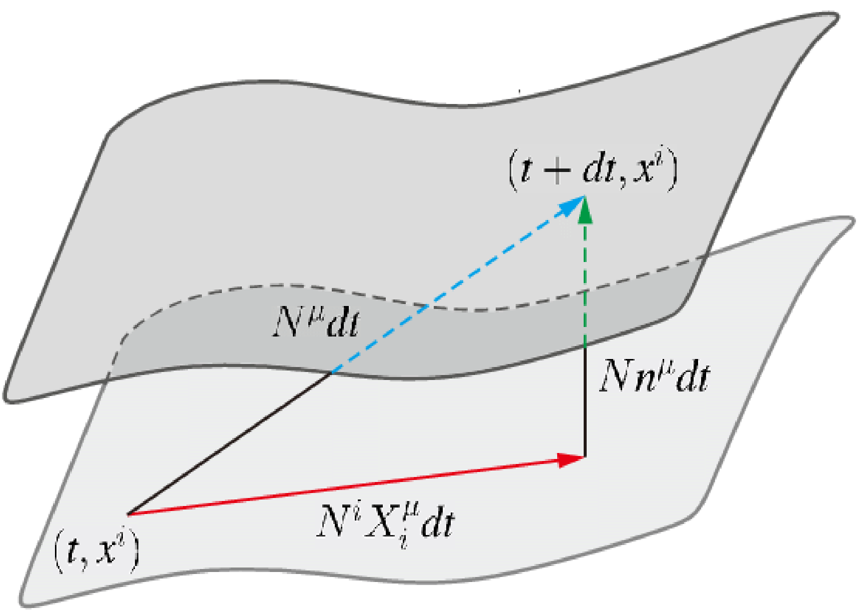
$$N^\alpha(\tau, x^i) \equiv \dot{e}^\alpha(\tau, x^i) = Nn^\alpha + N^i e_i^\alpha.$$

The Hamiltonian of General Relativity or any other generally covariant theory should have the following form

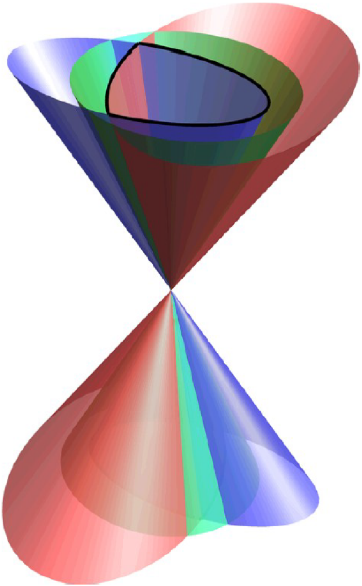
$$\mathrm{H} = \int \left(N\mathcal{R} + N^i \mathcal{R}_i \right) d^3x,$$

where first class constraints are derived by varying of N, N^i.

Embedding of space in space-time (2)



Bigravity: the two lightcones (Fig. by Kocic)



Kuchař's approach

- Working with two coordinate frames: X^α and (τ, x^i).
- Embedding functions $e^\alpha(\tau, x^i)$, and one-to-one map:

$$X^\alpha = e^\alpha(\tau, x^i),$$

three tangential vectors $e_i^\alpha = \frac{\partial e^\alpha}{\partial x^i}$.

- Two unit normal vectors: n^α and $\bar{n}^\alpha$:

$$g_{\mu\nu}\bar{n}^\mu \bar{n}^\nu = -1, \qquad g_{\mu\nu}\bar{n}^\mu e_i^\nu = 0,$$
$$f_{\mu\nu} n^\mu n^\nu = -1, \qquad f_{\mu\nu} n^\mu e_i^\nu = 0.$$

- Two metrics in their local bases $(\bar{n}^\alpha, e_i^\alpha)$ and (n^α, e_i^α):

$$g_{\mu\nu} = -\bar{n}_\mu \bar{n}_\nu + \gamma_{ij}\bar{e}^i_\mu \bar{e}^j_\nu, \qquad f_{\mu\nu} = -n_\mu n_\nu + \eta_{ij} e^i_\mu e^j_\nu.$$

A Tale of Two Metrics and Two Bases

With two space-time metrics we have two different unit normals and two bases

$$(n^\alpha, e_i^\alpha), \qquad (\bar{n}^\alpha, e_i^\alpha),$$

therefore we get two lapse functions and two shift vectors

$$N, N^i, \qquad \bar{N}, \bar{N}^i.$$

It is suitable to replace $\bar{N}$ and $\bar{N}^i$ by new variables u, u^i

$$\boxed{u = \frac{\bar{N}}{N}, \qquad u^i = \frac{\bar{N}^i - N^i}{N}}$$

These variable provide the coefficients of transformation between the two bases

$$\bar{n}_\mu = u n_\mu, \qquad \bar{e}^i_\mu = e^i_\mu - u^i n_\mu, \qquad \bar{n}^\mu = \frac{1}{u} n^\mu - \frac{u^i}{u} e^\mu_i .$$

On the interaction potential of the two metrics

The potential should be a scalar density constructed algebraically of the two space-time metric tensors

$$\sqrt{(-g)}U(g_{\mu\nu}, f_{\mu\nu})$$

or

$$\sqrt{(-f)}U(g_{\mu\nu}, f_{\mu\nu}).$$

Then function U should depend on the invariants of matrix $\mathsf{Y} = g^{-1}f = g^{\mu\alpha}f_{\alpha\nu}$, or on the invariants of some matrix function of Y. For example, Logunov et all in the Relativistic Theory of Gravitation have taken the following expression

$$\sqrt{-g}U = \sqrt{-g}\left(\frac{1}{2}\mathrm{Tr}\mathsf{Y} - 1\right) - \sqrt{-f}$$

On the orginal sin of a general potential

But, as Boulware, Deser had demonstrated (1972), in general the Hamiltonian of massive gravity (where the second metric was non-dymanical) did not have first class constraints. For the corresponding bigravity theory (where the second metric was dynamical) the Hamiltonian had only 4 first class constraints. In both cases the number of degrees of freedom was too large (one d.o.f. is superfluous)

$$6[\gamma_{ij}] + 6[\eta_{ij}] - 4[\mathcal{H}, \mathcal{H}_i] = 8 > 7 = 2[\text{massless}] + 5[\text{massive}].$$

This result follows from the non-linear dependence of the Hamiltonian on $\bar{N}$, $\bar{N}^i$, or on u, u^i

$$\boxed{u = \frac{\bar{N}}{N}, \qquad u^i = \frac{\bar{N}^i - N^i}{N}}$$

The role of potential

The diffeomorphism invariance requires

$$U(f_{\mu\nu}, g_{\mu\nu}) = U(\text{invariants of Y}),$$

where Y has the following components in space-time basis (n^μ, e_i^μ) constructed by means of metric $f_{\mu\nu}$:

$$\mathrm{Y} = g^{-1} f = u^{-2} \begin{pmatrix} -[n^\mu n_\nu] & u^i [n^\mu e_{\nu i}] \\ u^j [e_j^\mu n_\nu] & \left(-u^i u^j + u^2 \gamma^{ij}\right) [e_i^\mu e_{\nu j}] \end{pmatrix},$$

The problem is the Boulware-Deser ghost arising due to nonlinearity of $\sqrt{-g}U$ in auxiliary variable u.
But is it possible to find a potential linear in u?

de Rham, Gabadadze, Tolley and their potential

$$U = \sum_{n=0}^{4} \beta_n e_n(X), \qquad \mathrm{X} = \sqrt{\mathrm{Y}}, \qquad \mathrm{Y} = ||g^{\mu\alpha} f_{\alpha\nu}||$$

On the sinless potential

The simplest choice of matrix $\mathsf{Y} = ||g^{\mu\alpha} f_{\alpha\nu}||$ gives non-linear dependence on u of its invariants multiplyed on the invariant volume $dV_g = \sqrt{-g} = Nu\sqrt{\gamma}$ or on the volume $dV_f = \sqrt{-f} = N\sqrt{\eta}$

$$\mathsf{Y} = g^{-1} f = u^{-2} \left(\begin{array}{cc} -[n^\mu n_\nu] & u^i [n^\mu e_{\nu i}] \\ u^j [e^\mu_j n_\nu] & \left(-u^i u^j + u^2 \gamma^{ij}\right) [e^\mu_i e_{\nu j}] \end{array} \right),$$

but the matrix $\mathsf{X} = \sqrt{\mathsf{Y}}$ will get invariants linear in u^{-1}, therefore after multiplying them onto $dV_g = \sqrt{-g} = Nu\sqrt{\gamma}$ we get expressions linear in u:

$$\boxed{U_{\mathrm{dRGT}} = \sum_{i=0}^{i=4} \beta_i e_i(\mathsf{X})}.$$

Symmetric polinomials of matrix $X^{\mu}_{\nu} = \sqrt{\|g^{-1}f\|}^{\mu}_{\nu}$ written through traces

$$
\begin{aligned}
e_0 &= 1, \\
e_1 &= \mathrm{Tr} X, \\
e_2 &= \frac{1}{2}\left((\mathrm{Tr} X)^2 - \mathrm{Tr} X^2\right), \\
e_3 &= \frac{1}{6}\left((\mathrm{Tr} X)^3 - 3\mathrm{Tr} X \mathrm{Tr} X^2 + 2\mathrm{Tr} X^3\right), \\
e_4 &= \det X .
\end{aligned}
$$

Decomposition of matrices $\mathrm{Y} = g^{-1}f$ and $\mathrm{X} = \sqrt{\mathrm{Y}}$

$$\boxed{u = \frac{\bar{N}}{N}, \qquad u^i = \frac{\bar{N}^i - N^i}{N}}$$

$$\mathrm{Y} = g^{-1}f = u^{-2}\left(\begin{array}{cc} -[n^\mu n_\nu] & u^i[n^\mu e_{\nu i}] \\ u^j[e^\mu_j n_\nu] & \left(-u^i u^j + u^2\gamma^{ij}\right)[e^\mu_i e_{\nu j}] \end{array}\right),$$

If Hassan-Rosen transform of variables

$$\boxed{u^i = v^i + uD^i{}_j v^j}, \qquad \varepsilon^{-1} = \sqrt{1 - \eta_{ij}v^i v^j}.$$

is applied then

$$\mathrm{X} = \sqrt{\mathrm{Y}} = \varepsilon u^{-1}\left(\begin{array}{cc} -[n^\mu n_\nu] & v^i[n^\mu e_{\nu i}] \\ v^j[e^\mu_j n_\nu] & \left(-v^i v^j + \varepsilon^{-2}uD^{ij}\right)[e^\mu_i e_{\nu j}] \end{array}\right).$$

The main problem

How can we calculate the matrix square root?

There are three ways:

1. To apply implicit functions
2. To use tetrads
3. To go to mini-superspace

History of canonical approach to bigravity

2011: S. F. Hassan, Rachel A. Rosen, 1106.3344, 1109.3515, 1111.2070, S. F. Hassan, Rachel A. Rosen, Angnis Schmidt-May, 1109.3230, J. Kluson, 1109.3052.

2012: K. Hinterbichler, R. A. Rosen, 1203.5783, D. Comelli, M. Crisostomi, F. Nesti, L. Pilo, 1204.1027, J. Kluson, 1211.6267, V.O. Soloviev, M.V. Tchichikina, 1211.6530, S. Alexandrov, K. Krasnov, and S. Speziale, 1212.3614.

2013: J. Kluson, 1301.3296, D. Comelli, F. Nesti, L. Pilo, 1302.4447, V.O. Soloviev, M.V. Tchichikina, 1302.5096, J. Kluson, 1303.1652, D. Comelli, F. Nesti and L. Pilo, 1305.0236, J. Kluson, 1307.1974, S. Alexandrov, 1308.6586.

2014: C. de Rham, L. Heisenberg and R.H. Ribeiro, 1408.1678, S.F. Hassan, Mikica Kocic, Angnis Schmidt-May, 1409.1909, C. de Rham, L. Heisenberg and R.H. Ribeiro, 1409.3834. V.O. Soloviev, 1410.0048.

2015: V.O. Soloviev, 1505.00840.

2018: S.F. Hassan and A. Lundkvist, 1802.07267, M. Kocic, 1803.09752.

2020: V.O. Soloviev, 2006.16230.

Fawad Hassan, Anders Lundkvist, Mikica Kocic



Definition of implicit matrix $D^i{}_j$

There are two conditions:

1. Symmetry

$$\boxed{D^{ij} = D^{ji}},$$

2. Square-root-like relation to γ_{ij}

$$\boxed{\gamma^{ij} = D^i{}_k v^k D^j{}_m v^m + \varepsilon^{-2} D^{ik} D_k{}^j}.$$

Therefore

$$D^i{}_j = D^i{}_j(v^m, \gamma_{mn}, \eta_{mn}),$$

indices of $D^i{}_j$ are moved up and down by η_{ij}, η^{ij}.

On the fight with this problem (2011, 2018)

Hassan and Rosen invented a transformation of variables

$$u^i = v^i + uD^i{}_j v^j,$$

and supposed that the matrix square root could be expressed as follows

$$\mathrm{X} = \begin{pmatrix} (-\frac{\varepsilon}{u})[n^\mu n_\nu] & \frac{\varepsilon v^j}{u}[n^\mu e_{\nu j}] \\ \frac{\varepsilon v^i}{u}[e^\mu_i n_\nu] & \left(-\frac{\varepsilon v^i v^j}{u} + \frac{1}{\varepsilon} D^{ij}\right)[e^\mu_i e_{\nu j}] \end{pmatrix},$$

where $\varepsilon = 1/\sqrt{1 - \eta_{ij} v^i v^j}$. Then the neccesary conditions are

$$D^{ij} = D^{ji}, \qquad \gamma^{ij} = D^i_k v^k D^j_m v^m + \varepsilon^{-2} D^{ik} D_k{}^j.$$

The main point is the introduction of a new implicit function of the two spatial metrics $D^i{}_j$. The detailed calculations were provided 7 years later, in 2018.

The alternative approach (2013)

The potential itself is treated as an implicit function

$$N\tilde{U}, \qquad \tilde{U} = \sqrt{\eta}U(u, u^i, \eta_{ij}, \gamma_{ij}),$$

the Hamiltonian is as follows

$$\mathrm{H} = \int \left(N\mathcal{R} + N^i\mathcal{R}_i\right) d^3x,$$

then the first class constrained are obtained by varying in N, N^i,

$$\mathcal{R} = \mathcal{H} + u\bar{\mathcal{H}} + u^i\bar{\mathcal{H}}_i + \tilde{U}, \qquad \mathcal{R}_i = \mathcal{H}_i + \bar{\mathcal{H}}_i,$$

and the second class constraints appear when the Hamiltonian is varied in variables u, u^i

$$\mathcal{S} = \bar{\mathcal{H}} + \frac{\partial \tilde{U}}{\partial u}, \qquad \mathcal{S}_i = \bar{\mathcal{H}}_i + \frac{\partial \tilde{U}}{\partial u^i}.$$

The constraints algebra

As we know that 4 constraints are to be first class, we obtain new equations after calculating their Poisson brackets. These equations are linear in the potential and its first partial derivatives

$$2\eta_{jk}\frac{\partial\tilde{U}}{\partial\eta_{ij}} + 2\gamma_{jk}\frac{\partial\tilde{U}}{\partial\gamma_{ij}} - u^i\frac{\partial\tilde{U}}{\partial u^k} = \delta^i_k\tilde{U},$$

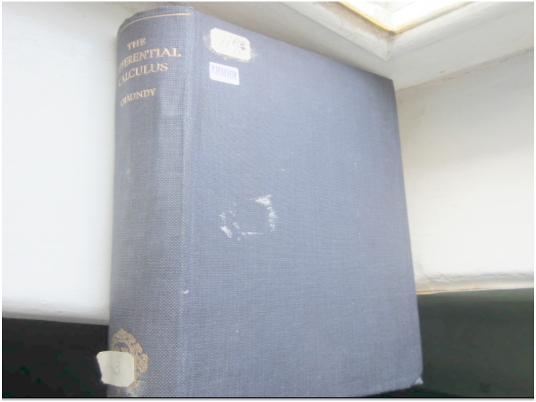
$$2u^j\gamma_{jk}\frac{\partial\tilde{U}}{\partial\gamma_{k\ell}} - u^\ell u\frac{\partial\tilde{U}}{\partial u} + \left(\eta^{k\ell} - u^2\gamma^{k\ell} - u^k u^\ell\right)\frac{\partial\tilde{U}}{\partial u^k} = 0.$$

In its turn, the Poisson brackets of the other 4 constraints lead to the homogeneous Monge-Ampere equation constructed of the second derivatives of the potential in variables u, u^i

$$\det\frac{\partial^2\tilde{U}}{\partial u^a\partial u^b}(x) = 0.$$

On the magic of mathematics

D. Fairlie, A.N. Leznov, General solutions of the Monge-Ampère equation in n-dimensional space, Journal of Geometry and Physics 16, 385 (1995)



Th. Chaundy, The Differential Calculus, Oxford, 1935

How the problem is solved?

Constraint $\mathcal{S}$ has weakly vanishing Poisson bracket with itself. To preserve constraint $\mathcal{S}$ in the Hamiltonian evolution we need to fulfil

$$\{\mathcal{S}, \mathrm{H}\} = 0,$$

and this condition gives the secondary constraint Ω. The constraints $\mathcal{S}$ and Ω do not commute, it means they are second class.

$$\{\mathcal{S}(x), \mathcal{S}(y)\} = -\bar{U}^i \mathcal{S}(x)\delta_{,i}(x,y) + \bar{U}^i \mathcal{S}(y)\delta_{,i}(y,x),$$
$$\{\mathcal{R}(x), \mathcal{S}(y)\} = (u^i - u\bar{U}^i)\mathcal{S}(x)\delta_{,i}(x,y) - \left(u(\bar{U}^i\mathcal{S})_{,i} + \Omega\right)\delta(x,y),$$
$$\{\mathcal{S}(x), \Omega(y)\} \neq 0.$$

These relations are sufficient to avoid the Boulware-Deser ghost.

Kurt Hinterbichler, Rachel Rosen

The tetrads

Vierbeins (tetrads) are square root of metric

$$g = E^T E, \qquad g_{\mu\nu} = E_{\mu A} E^A_\nu,$$

$$g^{-1} = E^{-1}(E^{-1})^T, \qquad g^{\mu\nu} = E^\mu_A E^{A\nu},$$

Therefore we can extract the square root of matrix Y

$$\mathrm{X} = \sqrt{g^{-1} f} = \sqrt{E^{-1}(E^{-1})^T F^T F} = E^{-1} F^T,$$

if only symmetry conditions are fulfilled

$$(FE^{-1})^T = FE^{-1}.$$

The null tetrad gauge

There is a diagonal Lorentz symmetry generated by

$$L^{+}_{AB} = \left(\begin{array}{cc} 0 & L^{+}_{0b} \\ L^{+}_{a0} & L^{+}_{ab} \end{array} \right) ,$$

We can sacrify L^{+}_{a0} to achieve the null tetrad gauge for $E_{A\mu}$

$$E_{0\mu} = \bar{n}_{\mu} .$$

Then parametrisation of a boost

$$\Lambda^{A}{}_{B} = \left(\begin{array}{cc} \varepsilon & \varepsilon v_b \\ \varepsilon v^a & \mathcal{P}^a_b \end{array} \right) , \quad \mathcal{P}^a_b = \delta^a{}_b + \frac{\varepsilon^2}{\varepsilon + 1} v^a v_b ,$$

allows to take the second tetrad $F_{A\mu}$ in the form

$$F^{A}_{\mu} = \Lambda^{A}{}_{B} \mathcal{F}^{B}_{\mu}$$

where $\mathcal{F}^{B}_{\mu}$ is a second tetrad given in the null gauge.

There are 21 pairs of canonically conjugate variables:

$$(e_{ai}, \pi^i_a), \ (\tilde{f}_{ai}, \Pi^i_a), (\tilde{v}_i, \Pi^i_0),$$

other variables are Lagrange multipliers:

$$N, N^i, u, u^i,$$

Degrees of freedom calculation

$$\mathrm{DOF} = \frac{1}{2}\left(n - 2n_{f.c.} - n_{s.c.}\right).$$

	BiGr (general)	Bi-Gr (dRGT)	Bi-Gr (tetrads)
(q,p) n	$\gamma_{ij}, \pi^{ij}, \eta_{ij}, \Pi^{ij}$ 24	$\gamma_{ij}, \pi^{ij}, \eta_{ij}, \Pi^{ij}$ 24	$e_{ia}, \pi^{ia}, \tilde{f}_{ia}, \Pi^{ia}, \tilde{v}_i, \Pi^i_0$ 42
1st class $n_{f.c.}$	$\mathcal{R}, \mathcal{R}_i$ 4	$\mathcal{R}, \mathcal{R}_i$ 4	$\mathcal{R}, \mathcal{R}_i, L^+_{ab}$ 7
2nd class $n_{s.c.}$	— 0	$\mathcal{S}, \Omega$ 2	$\mathcal{S}, \Omega, L^-_{ab}, G_{ab}, \mathcal{S}_i, L_{a0}$ 14
DoF	8	7	7

Lagrange multiplyers and constraints

GR (metric): N, N^i. GR (tetrads): N, N^i, λ_{ab}.
Bigravity (metrics): N, N^i, u, u^i.
Bigravity (tetrads): N, N^i, u, u^i, λ^+_{ab}, λ_a, λ^-_{ab}.

Lagrange multiplyer	primary constraint	constraint preservation condition	consequence	consequence 2
N	$\mathcal{R} \approx 0$			
N^i	$\mathcal{R}_i \approx 0$			
λ^+_{ab}	$L^+_{ab} \approx 0$			
λ^a	$L_{a0} = 0$	u^i		
λ^-_{ab}	$L^-_{ab} = 0$	$G_{ab} = 0$	$\{L^-_{ab}, G_{cd}\} \neq 0$	λ^-_{ab}
u	$\mathcal{S}' = 0$	$\Omega = 0$	$\{\mathcal{S}, \Omega\} \neq 0$	u
u^i	$\mathcal{S}_i = 0$	$\{\mathcal{S}_i, L_{a0}\} \neq 0$	λ^a	

Deciphering of the implicit functions

The Hassan-Rosen transformation becomes

$$u^i = v^i + uD^i{}_j v^j \equiv \left(\mathrm{f}^{ia} + u\mathrm{e}^{ia}\right) \mathrm{v}_a.$$

The coefficients in the second class constraints algebra is now

$$\bar{U}^i = || \frac{\partial^2 \tilde{U}}{\partial u^i \partial u^j} ||^{-1} \frac{\partial^2 \tilde{U}}{\partial u \partial u^j} = -\mathrm{e}^{ia}\mathrm{v}_a.$$

The potential becomes linear in variables u, u^i

$$\tilde{U} = W + u^i V_i + uV,$$

or

$$\tilde{U} = W' + uV',$$

where u^i is replaced by the r.h.s. of the first equation.

Deciphering of the potential

$$V' = e\Big(\beta_1 e_1(w) + \beta_2 e_2(w) + \beta_3 e_3(w)\Big) + \beta_0 e,$$
$$W' = \frac{e}{\varepsilon}\Big(\beta_1 e_0(z) + \beta_2 e_1(z) + \beta_3 e_2(z)\Big) + \beta_4 f.$$

where

$$z_{ab} = \mathcal{P}_{ac} x_{cb} \equiv \tilde{f}_{ia} \mathrm{e}^{ib}, \qquad \mathcal{P}_{ac} = \delta_{ac} + \frac{\varepsilon^2 \mathrm{v}_a \mathrm{v}_c}{\varepsilon + 1},$$
$$w_{ab} = \mathcal{P}^{-1}_{ac} x_{cb} \equiv \tilde{f}^{ia} \eta_{ij} \mathrm{e}^{jb}, \qquad \mathcal{P}^{-1}_{ac} = \delta_{ac} - \frac{\varepsilon \mathrm{v}_a \mathrm{v}_c}{\varepsilon + 1},$$
$$x_{cb} = \mathrm{f}_{ic} \mathrm{e}^{ib}, \qquad \tilde{f}_{ia} = \mathcal{P}_{ac} \mathrm{f}_{ic}, \qquad \tilde{f}^{ia} = \mathcal{P}^{-1}_{ac} \mathrm{f}^{ic},$$

The advantages of the tetrad approach

- The potential (and the Hamiltonian) is linear in lapses and shifts N, $\bar{N}$, N^i, $\bar{N}^i$
- All the non-dynamical functions are Lagrange multipliers
- The tetrad symmetry conditions are derived as the secondary constraints
- The crucial Hassan-Rosen transformation is not guessed, but is derived
- Neither implicit functions, nor Dirac brackets are used
- The coefficients of the constraint algebra are explicit functions

Bigravity in the simplest mini-superspace

The spatial geometry is assumed flat. There is only time dependence.

$$
\begin{aligned}
f_{\mu\nu} &= (-N^2(t), \omega^2(t)\delta_{ij}), \\
g_{\mu\nu} &= (-\bar{N}^2(t), \xi^2(t)\delta_{ij}), \\
\rho_f &= \rho_f(t), \qquad \rho_g = \rho_g(t), \\
p_f &= p_f(t), \qquad p_g = p_g(t),
\end{aligned}
$$

Then let us introduce new variables

$$
u = \frac{\bar{N}}{N}, \qquad r = \frac{\omega}{\xi}.
$$

It is easy to calculate the square root matrix $g^{-1}f$:

$$
\begin{aligned}
Y^{\mu}_{\nu} &= \left(g^{-1}f\right)^{\mu}_{\nu} = g^{\mu\alpha}f_{\alpha\nu} = \mathrm{diag}\left(u^{-2}, r^2\delta_{ij}\right), \\
X &= \sqrt{Y} = \mathrm{diag}\left(+\sqrt{u^{-2}}, +\sqrt{r^2}\delta_{ij}\right) \equiv \mathrm{diag}\left(u^{-1}, r\delta_{ij}\right).
\end{aligned}
$$

The eigenvalues λ_i and the symmetric polynomials e_i are

$$
\begin{aligned}
\lambda_1 &= u^{-1}, \qquad \lambda_2 = \lambda_3 = \lambda_4 = r, \\
e_0 &= 1, \\
e_1 &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = u^{-1} + 3r, \\
e_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = 3ru^{-1} + 3r^2, \\
e_3 &= \lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_4 = 3r^2u^{-1} + r^3, \\
e_4 &= \lambda_1\lambda_2\lambda_3\lambda_4 = r^3u^{-1}.
\end{aligned}
$$

The potential in mini-superspace

Take the notation

$$B_i(r) = \beta_i + 3\beta_{i+1} r + 3\beta_{i+2} r^2 + \beta_{i+3} r^3,$$

then

$$U = \frac{2m^2}{\kappa} N u \xi^3 \left(B_0(r) + \frac{1}{u} B_1(r) \right) = \frac{2m^2}{\kappa} N \left(uV + W \right),$$

where

$$\begin{aligned} V &= \frac{1}{N} \frac{\partial U}{\partial u} = \xi^3 B_0(r), \\ W &= \frac{1}{N} \left(U - u \frac{\partial U}{\partial u} \right) = \xi^3 B_1(r) \equiv \omega^3 \frac{B_1(r)}{r^3}. \end{aligned}$$

Bigravity cosmology

Constraints

$$H_g^2 = \frac{\kappa_g}{6}\rho_g + \frac{\Lambda_g}{3}, \qquad \Lambda_g(r) = m^2 B_0(r)\left[\frac{\kappa_g}{\kappa}\right],$$
$$H_f^2 = \left[\frac{\kappa_f}{6}\rho_f\right] + \frac{\Lambda_f}{3}, \qquad \Lambda_f(r) = m^2 \frac{\kappa_f}{\kappa_g}\frac{B_1(r)}{r^3}\left[\frac{\kappa_g}{\kappa}\right],$$

Conservation laws

$$\dot\rho_g = -3NuH_g(\rho_g + p_g),$$
$$\dot\rho_f = -3NH_f(\rho_f + p_f),$$

Dynamical equations

$$\dot H_g = -\frac{Nu\kappa_g}{4}(\rho_g + p_g) + \frac{N}{6}m^2(1 - ur)B_0{}'(r)\left[\frac{\kappa_g}{\kappa}\right],$$
$$\dot H_f = \left[-\frac{N\kappa_f}{4}(\rho_f + p_f)\right] - \frac{N}{6}m^2\frac{\kappa_f}{\kappa_g}(1 - ur)\frac{B_0{}'(r)}{r^3}\left[\frac{\kappa_g}{\kappa}\right].$$

Algebra of the constraints

$$
\begin{aligned}
\dot{\mathcal{S}} &= \{\mathcal{S}, \mathrm{H}\} = N\{\mathcal{S}, \mathcal{R}'\} \equiv N\Omega = 0, \\
\Omega &\equiv \{\mathcal{S}, \mathcal{R}'\} = \frac{4m^2}{\kappa}\xi^2 \left(\omega H_f - \xi H_g\right) B_0{}'(r) = 0, \\
\Omega &= \Omega_1 \Omega_2,
\end{aligned}
$$

The secondary constraint is factorized, so there are two branches of cosmological solutions

$$
\begin{aligned}
\Omega_1 &= 0, \quad \leftrightarrow \quad H_g = rH_f, & (4) \\
\Omega_2 &= 0, \quad \leftrightarrow \quad \beta_1 + 2\beta_2 r + \beta_3 r^2 = 0 = B_0{}'(r). & (5)
\end{aligned}
$$

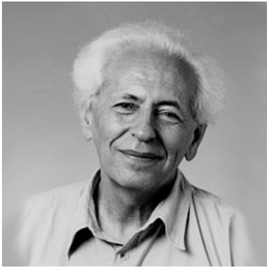
The first branch

$$
\begin{aligned}
H_f &= r^{-1} H_g, \\
\dot{r} &= N r(1 - ur) H_g, \qquad u = -\{\Omega_1, \mathcal{R}'\}/\{\Omega, \mathcal{S}\}, \\
H_g^2 &= \frac{8\pi G}{3}\rho + m^2 r \left(\left[\frac{\beta_0}{3r} + \right] \beta_1 + \beta_2 r + \frac{\beta_3}{3} r^2 \right), \\
H_g^2 &= \frac{m^2}{r} \left[\frac{\kappa_f}{\kappa_g} \right] \left(\frac{\beta_1}{3} + \beta_2 r + \beta_3 r^2 \left[+ \frac{\beta_4}{3} r^3 \right] \right), \\
\rho &= \frac{m^2}{8\pi G} \left(\left[\frac{\kappa_f}{\kappa_g} \right] \frac{B_1(r)}{r} - B_0(r) \right), \\
\rho + p &= \frac{m^2}{8\pi G} (1 - ur) \left(\left[\frac{\kappa_f}{\kappa_g} \right] \frac{D_1(r) - 2B_1(r)}{r^2} - D_1(r) \right), \\
\dot{\rho} &= -3 N u H_g (\rho + p), \\
\dot{H}_g &= N \left[-4\pi G u(\rho + p) + \frac{m^2}{6} (1 - ur)(\beta_1 + 2\beta_2 r + \beta_3 r^2) \right].
\end{aligned}
$$

The second branch

$$
\begin{aligned}
D_1(r) &\equiv \beta_1 + 2\beta_2 r + \beta_3 r^2 = 0, \qquad \dot{r} = 0, \\
u &= \frac{H_f}{H_g}, \qquad \dot{H}_f = 0, \\
H_g^2 &= \frac{8\pi G}{3}\rho + m^2 r \left(\left[\frac{\beta_0}{3r} + \right] \beta_1 + \beta_2 r + \frac{\beta_3}{3} r^2 \right), \\
H_f^2 &= \frac{\kappa_f}{\kappa_g} \frac{m^2}{r^3} \left(\frac{\beta_1}{3} + \beta_2 r + \beta_3 r^2 \left[+ \frac{\beta_4}{3} r^3 \right] \right), \\
\rho &= \frac{m^2}{8\pi G} \left(\left[\frac{\kappa_f}{\kappa_g} \right] \frac{B_1(r)}{u^2 r^3} - B_0(r) \right), \\
\dot{\rho} &= -3NuH_g(\rho + p), \\
\dot{H}_g &= -4\pi G N u (\rho + p),
\end{aligned}
$$

Acknowledgements



“What do you think?” shouted Razumihin, louder than ever, “you think I am attacking them for talking nonsense? Not a bit! I like them to talk nonsense. That’s man’s one privilege over all creation. Through error you come to the truth! I am a man because I err! You never reach any truth without making fourteen mistakes and very likely a hundred and fourteen. And a fine thing, too, in its way; but we can’t even make mistakes on our own account! Talk nonsense, but talk your own nonsense, and I’ll kiss you for it. To go wrong in one’s own way is better than to go right in someone else’s. In the first case you are a man, in the second you’re no better than a bird.”

– Да вы что думаете? – кричал Разумихин, еще более возвышая голос, – вы думаете, я за то, что они врут? Вздор! Я люблю, когда врут! Враньё есть единственная человеческая привилегия перед всеми организмами. **Соврёшь - до правды дойдешь!** Потому я и человек, что вру. Ни до одной правды не добирались, не соврав наперед раз четырнадцать, а может, и сто четырнадцать, а это почетно в своем роде; ну, а мы и соврать-то своим умом не умеем! Ты мне ври, да ври по-своему, и я тебя тогда поцелую. Соврать по-своему - ведь это почти лучше, чем правда по одному по-чужому; в первом случае ты человек, а во втором ты только что птица!