FROM $4d$ SUPERCONFORMAL INDICES TO 6j-SYMBOLS FOR THE LORENTZ GROUP

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THE LORENTZ GROUP AND $\mathrm{SL}(2,\mathbb{C})$

Let $x_\mu \in \mathbb{R}^{3,1}$, general Lorentz rotations $x'_\mu = \Lambda_\mu^\nu x_\nu$ preserve

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = \det \chi, \qquad \chi = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix},$$

where $\chi^\dagger = \chi$. Proper Lorentz rotations from $SO(3,1)^+$ are equivalent to the $SL(2,\mathbb{C})$-transformations

$$\chi' = g\chi g^\dagger, \quad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1, \quad \det \chi' = \det \chi.$$

$s\ell(2,\mathbb{C})$ algebra generators $S_j, \bar{S}_j$, $j = 0, \pm$, $[S_j, \bar{S}_j] = 0$,

$$[S_+, S_-] = 2S_0, \qquad [S_0, S_\pm] = \pm S_\pm.$$

Explicit realization in the space of complex functions $\Phi(z,\bar{z})$:

$$S_- = -\partial_z, \qquad S_0 = z\partial_z - s, \qquad S_+ = z^2\partial_z - 2sz, \quad s \in \mathbb{C}.$$

$\bar{S}_j$ are obtained by $z \to \bar{z} = z^*$ and $s \to s' \neq s^*$.

Casimir operator: $K = S_+S_- + S_0(S_0 - 1) = s(s+1)$.

Equivalent representations

$$s \to -1 - s, \quad \text{or} \quad a \to -a \quad \text{for} \quad a = 2s + 1.$$

General principal series representation

$$[\mathrm{T}_a(g)\,\Phi]\,(z,\bar{z}) = [\beta z+\delta]^{a-1}\,\Phi\left(\frac{\alpha z+\gamma}{\beta z+\delta},\frac{\bar{\alpha}\bar{z}+\bar{\gamma}}{\bar{\beta}\bar{z}+\bar{\delta}}\right),$$

where

$$[z]^a := z^a\bar{z}^{a'} = |z|^{2a'}z^{a-a'}$$

Restrictions on the spin variables a, a':

1) single-valuedness $\Rightarrow a - a' = m \in \mathbb{Z}$

2) unitarity with respect to the scalar product

$$\int d^2z\,\overline{f_1(z,\bar{z})}f_2(z,\bar{z}) \qquad \Rightarrow$$

$$a = \frac{m+i\nu}{2}, \quad a' = \frac{-m+i\nu}{2} = -a^*, \quad \nu \in \mathbb{R}.$$

Then,

$$[z]^a := z^a\bar{z}^{a'} = |z|^{2a'}z^{a-a'} = z^m|z|^{i\nu-m}.$$

Infinite-dimensional unitary principal series representation.

Tensor product of two representations

$$\mathrm{T}_{a_1} \otimes \mathrm{T}_{a_2} \xrightarrow{\mathrm{P}(a_1,a_2|a_3)} \mathrm{T}_{a_3}$$

The projection operator

$$\Phi(z_1, z_2) \xrightarrow{\mathrm{P}(a_1,a_2|a_3)} [\mathrm{P}(a_1, a_2|a_3)\, \Phi]\,(z_3)$$
$$= \int \mathrm{d}^2\, z_1 \mathrm{d}^2\, z_2\, W \begin{pmatrix} a_1, a_2, a_3 \\ z_1, z_2, z_3 \end{pmatrix} \Phi(z_1, z_2),$$

where $d^2z = dxdy$ (for $z = x + iy$) Naimark, 1957

$$W \begin{pmatrix} a_1, a_2, a_3 \\ z_1, z_2, z_3 \end{pmatrix} = \frac{1}{[z_2 - z_1]^{\frac{1+a_1+a_2+a_3}{2}} [z_3 - z_1]^{\frac{1+a_1-a_2-a_3}{2}} [z_2 - z_3]^{\frac{1-a_1+a_2-a_3}{2}}},$$

with $m_1 + m_2 + m_3 \in 2\mathbb{Z}$. This means that

$$[\beta z_3 + \delta]^{a_3-1} [\mathrm{P}(a_1, a_2|a_3)\, \Phi] \left(\frac{\alpha z_3 + \gamma}{\beta z_3 + \delta}\right) = \int \mathrm{d}^2\, z_1 \mathrm{d}^2\, z_2$$
$$\times\, W \begin{pmatrix} a_1, a_2, a_3 \\ z_1, z_2, z_3 \end{pmatrix} [\beta z_1 + \delta]^{a_1-1}\, [\beta z_2 + \delta]^{a_2-1}\, \Phi\left(\frac{\alpha z_1 + \gamma}{\beta z_1 + \delta}, \frac{\alpha z_2 + \gamma}{\beta z_2 + \delta}\right).$$

Biorthogonality relation:

$$\int \mathrm{d}^2\, z_1 \mathrm{d}^2\, z_2\, W\begin{pmatrix} -a_1, -a_2, -\tilde{a}_3 \\ z_1, z_2, z_3' \end{pmatrix} W\begin{pmatrix} a_1, a_2, a_3 \\ z_1, z_2, z_3 \end{pmatrix}$$

$$= \rho^{-1}(a_3)\delta_R(a_3 - \tilde{a}_3)\, \delta^2(z_3 - \tilde{z}_3) + B(a_1, a_2, a_3)\, \frac{\delta_R(a_3 + \tilde{a}_3)}{[z_3 - \tilde{z}_3]^{1-a_3}},$$

$$\delta_R(a - \tilde{a}) = \delta_{m\,\tilde{m}}\delta\left(\sigma - \tilde{\sigma}\right), \quad a = \frac{m}{2} + i\sigma, \quad \tilde{a} = \frac{\tilde{m}}{2} + i\tilde{\sigma},$$

$$\rho(a_3) = -\frac{a_3 a_3'}{4\pi^4}, \qquad B(a_1, a_2, a_3) = 4\pi^3 \frac{\mathbf{a}\left(\frac{1-a_1+a_2-a_3}{2}, 1 + a_3\right)}{\mathbf{a}\left(\frac{1-a_1+a_2+a_3}{2}\right)}.$$

Here $\mathbf{a}(\alpha) = \mathbf{\Gamma}(\alpha|\alpha')^{-1}$, the complex gamma function

$$\mathbf{\Gamma}(x, n) = \mathbf{\Gamma}(\alpha|\alpha') := \frac{\Gamma(\alpha)}{\Gamma(1 - \alpha')} = \frac{\Gamma(\frac{n+\mathrm{i}x}{2})}{\Gamma(1 + \frac{n-\mathrm{i}x}{2})}, \quad x \in \mathbb{C},\ n \in \mathbb{Z}.$$

Completeness relation — Naimark, 1957

$$\sum_{m\in 2\mathbb{Z}+\epsilon} \int_{\mathbb{R}} \mathrm{d}\sigma \int_{\mathbb{C}} \mathrm{d}^2 z\, \frac{\rho(a)}{2}\, W\begin{pmatrix} -a_1, -a_2, -a \\ z_3, z_4, z \end{pmatrix} W\begin{pmatrix} a_1, a_2, a \\ z_1, z_2, z \end{pmatrix}$$

$$= \delta^2(z_1 - z_3)\, \delta^2(z_2 - z_4), \quad m_1 + m_2 \in 2\mathbb{Z} + \epsilon, \quad \epsilon = 0, 1.$$

A triple tensor product decomposition

$$\mathrm{T}_{a_1} \otimes \mathrm{T}_{a_2} \otimes \mathrm{T}_{a_3} \xrightarrow{\mathrm{P}(a_1,a_2|\tilde{c})} \mathrm{T}_{\tilde{c}} \otimes \mathrm{T}_{a_3} \xrightarrow{\mathrm{P}(\tilde{c},a_3|\ell)} \mathrm{T}_{\ell},$$

realized as

$$\Phi(z_1, z_2, z_3) \xrightarrow{\mathrm{P}(\tilde{c},a_3|\ell)\mathrm{P}(a_1,a_2|\tilde{c})} \left[\mathrm{P}(\tilde{c}, a_3|\ell)\mathrm{P}(a_1, a_2|\tilde{c})\,\Phi\right](z)$$

$$= \int \mathrm{d}^2 z_0 \mathrm{d}^2 z_3 \int \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2 W \begin{pmatrix} \tilde{c}, a_3, \ell \\ z_0, z_3, z \end{pmatrix} W \begin{pmatrix} a_1, a_2, \tilde{c} \\ z_1, z_2, z_0 \end{pmatrix} \Phi(z_1, z_2, z_3).$$

Another option

$$\mathrm{T}_{a_1} \otimes \mathrm{T}_{a_2} \otimes \mathrm{T}_{a_3} \xrightarrow{\mathrm{P}(a_2,a_3|c)} \mathrm{T}_{a_1} \otimes \mathrm{T}_{c} \xrightarrow{\mathrm{P}(a_1,c|\ell)} \mathrm{T}_{\ell},$$

realized as

$$\Phi(z_1, z_2, z_3) \xrightarrow{\mathrm{P}(a_1,c|\ell)\mathrm{P}(a_2,a_3|c)} \left[\mathrm{P}(a_1, c|\ell)\mathrm{P}(a_2, a_3|c)\,\Phi\right](z)$$

$$= \int \mathrm{d}^2 z_0 \mathrm{d}^2 z_1 \int \mathrm{d}^2 z_2\, \mathrm{d}^2 z_3\, W \begin{pmatrix} a_1, c, \ell \\ z_1, z_0, z \end{pmatrix} W \begin{pmatrix} a_2, a_3, c \\ z_2, z_3, z_0 \end{pmatrix} \Phi(z_1, z_2, z_3).$$

Definition of the Racah coefficients, or $6j$-symbols

$$\mathrm{P}(a_1, c|\ell)\mathrm{P}(a_2, a_3|c) = \int \mathrm{D}_R\tilde{c}\, \frac{\rho(\tilde{c})}{2}\, \mathrm{R}_\ell(c, \tilde{c})\, \mathrm{P}(\tilde{c}, a_3|\ell)\mathrm{P}(a_1, a_2|\tilde{c}),$$

where $\tilde{c} = m/2 + i\sigma$ and

$$\int \mathrm{D}_R\tilde{c} = \sum_{m\in 2\mathbb{Z}\,\mathrm{or}\,2\mathbb{Z}+1} \int_{\mathbb{R}} \mathrm{d}\sigma$$

depending on whether $m_1 + m_2$ is even or odd. Explicitly,

$$\begin{aligned}
&\int \mathrm{d}^2 z_0\, W\begin{pmatrix} a_2, a_3, c \\ z_2, z_3, z_0 \end{pmatrix} W\begin{pmatrix} a_1, c, \ell \\ z_1, z_0, z \end{pmatrix} \\
&= \int \mathrm{D}_R\tilde{c}\, \frac{\rho(\tilde{c})}{2}\, \mathrm{R}_\ell(c, \tilde{c}) \int \mathrm{d}^2 z_0\, W\begin{pmatrix} a_1, a_2, \tilde{c} \\ z_1, z_2, z_0 \end{pmatrix} W\begin{pmatrix} \tilde{c}, a_3, \ell \\ z_0, z_3, z \end{pmatrix}.
\end{aligned}$$

Solution of this integral equation with the help of $2d$ Feynman diagrams technique Derkachov, V.S., 2017

$$\mathrm{R}_\ell(c,\tilde c) = \int \mathrm{d}^2 z\, \Phi_2(a_1,a_2,a_3|\ell,c,z)\overline{\Phi_1(a_1,a_2,a_3|\ell,\tilde c,z)},$$

$$\overline{\Phi_1} = \int \frac{\mathrm{d}^2 y}{[y-1]^{\frac{1-a_1+a_2+\tilde c}{2}}[-y]^{\frac{1+a_1-a_2+\tilde c}{2}}[z-y]^{\frac{1-a_3-\ell+\tilde c}{2}}},$$

$$\Phi_2 = \frac{1}{[z]^{\frac{1+a_2+a_3+c}{2}}}\int \frac{\mathrm{d}^2 z_0}{[z_0-1]^{\frac{1+a_1+\ell+c}{2}}[-z_0]^{\frac{1+a_2-a_3-c}{2}}[z_1-z]^{\frac{1-a_2+a_3-c}{2}}}.$$

Earlier derivation: Ismagilov (2006), but $\tilde c \to -\tilde c, \quad m_j \in 2\mathbb{Z}.$

The Mellin-Barnes type representation ($s = \frac{1}{2}(n+i\nu)$)

$$\mathrm{R}_\ell(c,\tilde c) = (-1)^{\tilde c - \tilde c'}\,\frac{\pi^2}{4}\frac{\mathbf{a}\left(\frac{1-a_3-\ell+\tilde c}{2},\frac{1+a_1+c+\ell}{2}\right)}{\mathbf{a}\left(\frac{1+a_1-a_2+\tilde c}{2},\frac{1+a_2-a_3+c}{2}\right)}\sum_{n\in\mathbb{Z}}\int_L d\nu$$

$$\times\frac{\mathbf{a}\left(\frac{1+a_1-a_2+\tilde c}{2}+s,\frac{1-a_1-a_2+\tilde c}{2}+s,\frac{1+a_3+\ell+\tilde c}{2}+s,\frac{1-a_3+\ell+\tilde c}{2}+s\right)}{\mathbf{a}\left(s,\tilde c+s,\frac{\tilde c+\ell-a_2-c}{2}+s,\frac{c+\tilde c+\ell-a_2}{2}+s\right)}.$$

Change of notation $s = N/2 + \mathrm{i}u/2$ and

$$a_1 = N_1/2 + \mathrm{i}\sigma_1, \qquad a_3 = N_3/2 + \mathrm{i}\sigma_3, \qquad c = M_1/2 + \mathrm{i}\rho_1,$$
$$a_2 = N_2/2 + \mathrm{i}\sigma_2, \qquad \ell = N_4/2 + \mathrm{i}\sigma_4, \qquad \tilde{c} = M_2/2 + \mathrm{i}\rho_2.$$

Then the above $6j$-symbols take the form

$$\left\{ \begin{smallmatrix} \sigma_1,N_1 & \sigma_2,N_2 \\ \sigma_3,N_3 & \sigma_4,N_4 \end{smallmatrix} \middle| \begin{smallmatrix} \rho_1,M_1 \\ \rho_2,M_2 \end{smallmatrix} \right\} = \frac{\pi^2}{4} \frac{\mathbf{\Gamma}\left(\sigma_1 - \sigma_2 + \rho_2 - \mathrm{i}, A_1\right) \mathbf{\Gamma}\left(\sigma_2 - \sigma_3 + \rho_1 - \mathrm{i}, A_2\right)}{\mathbf{\Gamma}\left(-\sigma_3 - \sigma_4 + \rho_2 - \mathrm{i}, A_3\right) \mathbf{\Gamma}\left(\sigma_1 + \sigma_4 + \rho_1 - \mathrm{i}, A_4\right)}$$

$$\times (-1)^{M_2 - N_2 + N_4} \sum_{N \in \mathbb{Z}} \int_{u \in L} \prod_{j=1}^{4} \mathbf{\Gamma}(R_j - u, S_j - N) \mathbf{\Gamma}(U_j + u, T_j + N)\, du,$$

$$R_1 = -\sigma_1 + \sigma_2 - \rho_2 - \mathrm{i},\ U_1 = -\rho_1 - \sigma_2 + \sigma_4 + \rho_2,\ S_1 = (-N_1 + N_2 - M_2)/2,$$
$$R_2 = \sigma_1 + \sigma_2 - \rho_2 - \mathrm{i}, \quad U_2 = \rho_1 - \sigma_2 + \sigma_4 + \rho_2, \quad S_2 = (N_1 + N_2 - M_2)/2,$$
$$R_3 = -\sigma_3 - \sigma_4 - \rho_2 - \mathrm{i},\ U_3 = 0, \qquad S_3 = -(N_3 + N_4 + M_2)/2,$$
$$R_4 = \sigma_3 - \sigma_4 - \rho_2 - \mathrm{i}, \quad U_4 = 2\rho_2, \qquad S_4 = (N_3 - N_4 - M_2)/2,$$
$$T_1 = (-M_1 - N_2 + N_4 + M_2)/2,\ T_2 = (M_1 - N_2 + N_4 + M_2)/2,\ T_3 = 0,\ T_4 = M_2,$$
$$A_1 = \tfrac{M_2 + N_1 - N_2}{2},\ A_2 = \tfrac{M_1 + N_2 - N_3}{2},\ A_3 = \tfrac{M_2 - N_3 - N_4}{2},\ A_4 = \tfrac{M_1 + N_1 + N_4}{2},$$

with $A_1 + A_2 = A_3 + A_4$ and the balancing conditions

$$\sum_{a=1}^{4} (R_a + U_a) = -4\mathrm{i} \quad \text{and} \quad \sum_{a=1}^{4} (S_a + T_a) = 0.$$

These $6j$-symbols appear as a limit of an **elliptic analogue** of the ${}_2F_1$ Euler-Gauss hypergeometric function V.S., 2000, 2003

$$V(t_1,\ldots,t_8;p,q)=\frac{(p;p)_\infty(q;q)_\infty}{4\pi i}\int_{\mathbb{T}}\prod_{k=\pm1}\frac{\prod_{j=1}^{8}\Gamma(t_jz^k;p,q)}{\Gamma(z^{2k};p,q)}\frac{dz}{z},$$

$|t_j|,|p|,|q|<1$ and $\prod_{j=1}^{8}t_j=p^2q^2$, $(z;p)_\infty=\prod_{j=0}^{\infty}(1-zp^j)$.

The elliptic gamma function

$$\Gamma(z;p,q)=\prod_{j,k=0}^{\infty}\frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^jq^k},\qquad |p|,|q|<1,$$

with the key property

$$\Gamma(qz;p,q)=\theta(z;p)\Gamma(z;p,q),\qquad \theta(z;p)=(z;p)_\infty(pz^{-1};p)_\infty.$$

$W(E_7)$-group transformation law:

$$V(\underline{t};p,q)=\prod_{1\le j<k\le 4}\Gamma(t_jt_k,t_{j+4}t_{k+4};p,q)\,V(\underline{s};p,q),$$

where $|t_j|,|s_j|<1$ and

$$\begin{cases} s_j=\varepsilon t_j, & j=1,2,3,4\\ s_j=\varepsilon^{-1}t_j, & j=5,6,7,8\end{cases};\quad \varepsilon=\sqrt{\frac{pq}{t_1t_2t_3t_4}}=\sqrt{\frac{t_5t_6t_7t_8}{pq}}.$$

Superconformal index

Flat $4d$ $\mathcal{N}=1$ SUSY gauge field theory: $SU(2,2|1)\times G\times F$

Space-time (supercoformal) symmetry $SU(2,2|1)$:
$J_i, \overline{J}_i$ = $SU(2)$ subgroups generators, or Lorentz rotations,
$P_\mu, Q_\alpha, \overline{Q}_{\dot\alpha}$ = supertranslations,
$K_\mu, S_\alpha, \overline{S}_{\dot\alpha}$ = special superconformal transformations,
H = dilations and $R = U(1)_R$-rotations.
Internal symmetries: gauge G and flavor F groups

For $Q\propto \overline{Q}_1$ and $Q^\dagger \propto \overline{S}_1$, one has $Q^2=(Q^\dagger)^2=0$ and

$$\{Q,Q^\dagger\}=2\mathcal{H},\qquad \mathcal{H}=H-2\overline{J}_3-3R/2$$

The superconformal index:
Romelsberger, 2005; Kinney, Minwalla, Maldacena, Raju, 2005

$$I(y;p,q)=\mathrm{Tr}_G\Big((-1)^{\mathcal{F}}p^{\mathcal{R}/2+J_3}q^{\mathcal{R}/2-J_3}\prod_k y_k^{F_k}\,e^{-\beta\mathcal{H}}\Big),$$

$\mathcal{F}$ = the fermion number, $\mathcal{R}=H-R/2$, $p,q,y_k,e^{-\beta}$ are group parameters (fugacities) for maximal Cartans commuting with Q (F_k - generators of F).

Flat $4d$ space-time $\Rightarrow S^3 \times S^1 \Rightarrow$ SCI is preserved (the only SUSY index) $\Rightarrow$ no divergencies. H is the Hamiltonian.

Counting of BPS states $Q|\psi\rangle = Q^\dagger|\psi\rangle = \mathcal{H}|\psi\rangle = 0$ or cohomology space of Q, $Q^\dagger$ operators (hence, no β-dependence).

"Physical" (not rigorous) computation yields the matrix integral:

$$I(y;p,q) \;=\; \int_G d\mu(z)\,\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{ind}\big(p^n,q^n,z^n,y^n\big)\right),$$

$d\mu(z) =$ the Haar measure, the single particle states index

$$\mathrm{ind}(p,q,z,y) = \frac{2pq-p-q}{(1-p)(1-q)}\chi_{adj_G}(z)$$
$$+\sum_j \frac{(pq)^{R_j/2}\chi_{r_F,j}(y)\chi_{r_G,j}(z) - (pq)^{1-R_j/2}\chi_{\bar{r}_F,j}(y)\chi_{\bar{r}_G,j}(z)}{(1-p)(1-q)},$$

$\chi_{R_F,j}(y)$ and $\chi_{R_G,j}(z) =$ characters of field representations, and R_j are the R-charges.

Romelsberger conjecture (2007): SCIs of Seiberg-like dual theories coincide.

The proof (Dolan, Osborn, 2008; Vartanov, V.S.,2008-2014) is based on the theory of elliptic hypergeometric integrals.

Particular electromagnetic Seiberg-type duality:

Electric theory: $SU(2)$, $F = SU(8)$, vector superfield + chiral superfield $(f; f)$, $I_E \equiv V(\underline{t}; p, q)$.

Magnetic theory: $G = SU(2)$, $F = SU(4)_l \times SU(4)_r \times U(1)$, vector superfield + chiral superfields $(1; T_a, 1)$ and $(1; 1, T_a)$.

Equality of superconformal indices $I_E = I_M$ coincides with the key $W(E_7)$ identity for the V-function.

Hyperbolic degeneration of the V-function

Parametrize

$$t_j = e^{-2\pi v g_j}, \quad z = e^{-2\pi v u}, \quad p = e^{-2\pi v \omega_1}, \quad q = e^{-2\pi v \omega_2}.$$

In the limit $v \to 0^+$, Ruijsenaars, 1997

$$\Gamma(e^{-2\pi v u}; e^{-2\pi v \omega_1}, e^{-2\pi v \omega_2}) \underset{v\to 0^+}{=} e^{-\pi \frac{2u-\omega_1-\omega_2}{12 v \omega_1 \omega_2}} \gamma^{(2)}(u;\omega_1,\omega_2).$$

Faddeev's (1994) modular dilogarithm, or hyperbolic gamma function

$$\gamma^{(2)}(u;\omega) = \gamma^{(2)}(u;\omega_1,\omega_2) := e^{-\frac{\pi \mathrm{i}}{2} B_{2,2}(u;\omega)} \gamma(u;\omega),$$

second order multiple Bernoulli polynomial

$$B_{2,2}(u;\omega) = \frac{1}{\omega_1\omega_2}\left((u - \frac{\omega_1+\omega_2}{2})^2 - \frac{\omega_1^2+\omega_2^2}{12}\right),$$

$$\gamma(u;\omega) := \frac{(\tilde{\mathbf{q}} e^{2\pi \mathrm{i}\frac{u}{\omega_1}}; \tilde{\mathbf{q}})_\infty}{(e^{2\pi \mathrm{i}\frac{u}{\omega_2}}; \mathbf{q})_\infty} = \exp\left(- \int_{\mathbb{R}+\mathrm{i}0} \frac{e^{ux}}{(1-e^{\omega_1 x})(1-e^{\omega_2 x})} \frac{dx}{x}\right),$$

$$\mathbf{q} = e^{2\pi \mathrm{i}\frac{\omega_1}{\omega_2}}, \qquad \tilde{\mathbf{q}} = e^{-2\pi \mathrm{i}\frac{\omega_2}{\omega_1}}.$$

Well defined for $\omega_1, \omega_2 > 0$ ($|\mathbf{q}| = 1$) and $0 < \mathrm{Re}(u) < \omega_1 + \omega_2$.

Then, Rains, 2006

$$V(e^{-2\pi v u_k}; e^{-2\pi v \omega_1}, e^{-2\pi v \omega_2}) \underset{v\to 0^+}{\propto} e^{\frac{\pi}{4v}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)} I_h(\underline{u}),$$

$$I_h(\underline{u}) = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\prod_{j=1}^{8}\gamma^{(2)}(u_j \pm z;\omega)}{\gamma^{(2)}(\pm 2z;\omega)} \frac{dz}{2\mathrm{i}\sqrt{\omega_1\omega_2}},$$

$$\mathrm{Re}(u_j) > 0, \qquad \sum_{j=1}^{8} u_j = 2(\omega_1+\omega_2).$$

Asymmetric parametrization

$$u_a = \nu_a + \mathrm{i}\xi, \quad u_{a+4} = \mu_a - \mathrm{i}\xi, \quad a = 1,2,3,4,$$

so that $\sum_{a=1}^{4}(\nu_a+\mu_a) = 2(\omega_1+\omega_2)$. Shift the integration variable $z \to z - \mathrm{i}\xi$ and take the limit $\xi \to -\infty$.

As a result, $I_h(\underline{u}) \to J_h(\underline{\mu},\underline{\nu})$,

$$J_h(\underline{\mu},\underline{\nu}) = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \prod_{a=1}^{4} \gamma^{(2)}(\mu_a - z;\omega)\gamma^{(2)}(\nu_a + z;\omega) \frac{dz}{\mathrm{i}\sqrt{\omega_1\omega_2}}.$$

A different degeneration: set $u_1 = 2(\omega_1+\omega_2) - \sum_{k=2}^8 u_k$ and take $u_8 \to \infty$. After renumerating $u_k \to u_{k-1} \Rightarrow I_h(\underline{u}) \to E_h(\underline{u})$,

$$E_h(\underline{u}) = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\prod_{a=1}^6 \gamma^{(2)}(u_a \pm z;\omega)}{\gamma^{(2)}(\pm 2z;\omega)} \frac{dz}{2\mathrm{i}\sqrt{\omega_1\omega_2}}.$$

Consequences of the V-function symmetry transformations:

$$J_h(\underline{\mu},\underline{\nu}) = \prod_{j,k=1}^{2} \gamma^{(2)}(\mu_j+\nu_k;\omega) \prod_{j,k=3}^{4} \gamma^{(2)}(\mu_j+\nu_k;\omega)$$

$$\times J_h(\mu_1+\eta,\mu_2+\eta,\mu_3-\eta,\mu_4-\eta,\nu_1+\eta,\nu_2+\eta,\nu_3-\eta,\nu_4-\eta),$$

where $\eta = \frac{1}{2}(\omega_1+\omega_2-\mu_1-\mu_2-\nu_1-\nu_2)$,

$$J_h(\underline{\mu},\underline{\nu}) = \prod_{a=1}^{3} \gamma^{(2)}(\mu_a+\nu_4;\omega)\gamma^{(2)}(\nu_a+\mu_4;\omega)$$

$$\times E_h(\mu_1+\eta,\mu_2+\eta,\mu_3+\eta,\nu_1-\eta,\nu_2-\eta,\nu_3-\eta),$$

where $2\eta = \omega_1+\omega_2-\nu_4-\sum_{a=1}^3 \mu_a$.

Limits to complex hypergeometric functions

Take

$$\gamma(u;\omega) = \frac{(e^{2\pi \mathrm{i}\frac{u}{\omega_1}} e^{-2\pi \mathrm{i}\frac{\omega_2}{\omega_1}}; e^{-2\pi \mathrm{i}\frac{\omega_2}{\omega_1}})_\infty}{(e^{2\pi \mathrm{i}\frac{u}{\omega_2}}; e^{2\pi \mathrm{i}\frac{\omega_1}{\omega_2}})_\infty}$$

and set

$$b := \sqrt{\frac{\omega_1}{\omega_2}} = \mathrm{i} + \delta, \quad \delta \to 0^+.$$

Special choice of the argument u:

$$u = \mathrm{i}\sqrt{\omega_1\omega_2}(n + x\delta), \quad n \in \mathbb{Z},\ x \in \mathbb{C}.$$

Then, uniformly on the compacta Sarkissian, V.S., 2020

$$\gamma^{(2)}(\mathrm{i}\sqrt{\omega_1\omega_2}(n + x\delta);\omega) \underset{\delta\to 0^+}{\approx} (4\pi\delta)^{\mathrm{i}x-1} e^{\frac{\pi \mathrm{i}}{2}n^2} \mathbf{\Gamma}(x,n).$$

$b \to \mathrm{i}$ degeneration of hyperbolic hypergeometric integrals.

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$$J_h(\underline{\mu}, \underline{\nu}) \underset{\delta\to 0}{\propto} \frac{1}{(4\pi\delta)^6} \mathcal{J}_{cr}(\underline{s}, \underline{n}; \underline{t}, \underline{m}),$$

$$\mathcal{J}_{cr}(\underline{s}, \underline{n}; \underline{t}, \underline{m}) = \frac{1}{4\pi} \sum_{N\in\mathbb{Z}+\varepsilon} \int_{-\infty}^{\infty} \prod_{a=1}^{4} \mathbf{\Gamma}(s_a - y, n_a - N) \mathbf{\Gamma}(t_a + y, m_a + N) dy,$$

where $n_a, m_a \in \mathbb{Z} + \varepsilon$ with the balancing condition

$$\sum_{a=1}^{4} (n_a + m_a) = 0, \qquad \sum_{a=1}^{4} (s_a + t_a) = -4\mathrm{i},$$

$\equiv$ the function entering $6j$-symbols for the Lorentz group.

Additional integral representation and symmetry transformations:

$$E_h(\underline{u}) \underset{\delta\to 0^+}{\to} (4\pi\delta)^{2\mathrm{i}\sum_{k=1}^{6} p_k - 9} \mathcal{E}_{cr}(\underline{p}, \underline{l}),$$

where $y, p_k \in \mathbb{C}$ and $N, l_k \in \mathbb{Z} + \varepsilon$, $\varepsilon = 0, \frac{1}{2}$, and

$$\mathcal{E}_{cr}(\underline{p}, \underline{l}) = \frac{1}{8\pi} \sum_{N\in\mathbb{Z}+\varepsilon} \int_{-\infty}^{\infty} (y^2 + N^2) \prod_{k=1}^{6} \mathbf{\Gamma}(p_k \pm y, l_k \pm N) dy.$$

Symmetry transformation I:

$$\mathcal{J}_{cr}(\underline{s},\underline{n};\underline{t},\underline{m}) = e^{\pi \mathrm{i} A} \prod_{j,k=1}^{2} \boldsymbol{\Gamma}(s_j+t_k, n_j+m_k) \prod_{j,k=3}^{4} \boldsymbol{\Gamma}(s_j+t_k, n_j+m_k)$$
$$\times\, \mathcal{J}_{cr}(s_1+Y, n_1+K, s_2+Y, n_2+K, s_3-Y, n_3-K, s_4-Y, n_4-K;$$
$$t_1+Y, m_1+K, t_2+Y, m_2+K, t_3-Y, m_3-K, t_4-Y, m_4-K),$$

$$K = -\frac{n_1+n_2+m_1+m_2}{2}, \qquad Y = -\frac{s_1+s_2+t_1+t_2+2\mathrm{i}}{2},$$
$$A = (n_1+n_2)(m_1+m_2)+(n_3+n_4)(m_3+m_4)+2(\varepsilon+\lambda)\Big(1+\sum_{a=1}^{4} m_a\Big).$$

Symmetry transformation II:

$$\mathcal{J}_{cr}(\underline{s}, \underline{n}; \underline{t}, \underline{m}) = e^{\pi \mathrm{i} A} \prod_{a=1}^{3} \mathbf{\Gamma}(s_a + t_4, n_a + m_4)\mathbf{\Gamma}(t_a + s_4, m_a + n_4)$$
$$\times\, \mathcal{E}_{cr}(\underline{s} + Z, \underline{n} + L, \underline{t} - Z, \underline{m} - L),$$

where

$$\mathcal{E}_{cr}(\underline{s} + Z, \underline{n} + L, \underline{t} - Z, \underline{m} - L) = \sum_{N \in \mathbb{Z}+\lambda} \int_{y \in \mathbb{R}} (y^2 + N^2)$$

$$\times \prod_{a=1}^{3} \mathbf{\Gamma}(s_a + Z \pm y, n_a + L \pm N)\mathbf{\Gamma}(t_a - Z \pm y, m_a - L \pm N) dy,$$

$$L = -\frac{1}{2}(m_4 + \sum_{a=1}^{3} n_a), \quad Z = -\frac{1}{2}(t_4 + 2\mathrm{i} + \sum_{a=1}^{3} s_a),$$

$$A = 2L^2 - \Big(\sum_{a=1}^{4} n_a\Big) - 2n_4 m_4 - \lambda + 2\varepsilon\Big(1 + \sum_{a=1}^{4} m_a\Big).$$

Discrete parameters $\varepsilon, \lambda = 0, \frac{1}{2}$: if L is integer $\Rightarrow \varepsilon = \lambda$, otherwise $\varepsilon \neq \lambda$.