Evaluating integrals for four photon amplitudes by functional reduction method

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July 19, Dubna, Russia

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One-loop n-point integrals

$$I_n^{(d)}(\{m_j^2\};\{s_{ik}\}) = \frac{1}{i\pi^{d/2}} \int \frac{d^d k_1}{D_1 \ldots D_n}$$

where

$$D_j = (k_1 - p_j)^2 - m_j^2 + i\eta, \qquad s_{ij} = (p_i - p_j)^2,$$

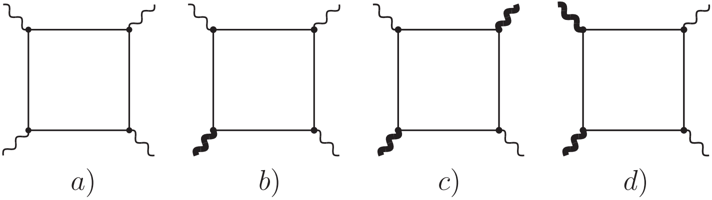
depending on $n(n+1)/2$ arbitrary kinematic variables and masses can be written as a linear combination of integrals depending on n variables:

$$I_n^{(d)}(m_1^2, ..., m_n^2; \{s_{kj}\}) = \sum_{r=1}^{n!} Q_r I_n^{(d)}(\{R_i\}_r; \{s_{kj} = R_j - R_k | j > k\})$$

where $\{R_i\}_r = \{\Delta(i)/\lambda(i)\}_r$ are sets of ratios of modified Cayley over Gram determinants and Q_r are products of ratios of polynomials in masses and kinematic variables.

Functional reduction of box integrals

As an example we condider functional reduction of box type integrals:



These integrals are encountered in calculating radiative corrections to

- light by light scattering (ATLAS, CMS)
- photon splitting (VEPP-4M, Novosibirsk, 2002)
- Delbrück scattering
- $\gamma\gamma \to \gamma Z$
- reduction of 5-, 6-point diagrams

R. Karplus and M. Neuman *Phys. Rev.*, 83:776–784, 1951; *Phys. Rev.*, 80:380–385, 1950.
B. De Tollis, Nuovo Cim., 32:757, 1964;
V. Constantini, B. De Tollis, and G. Pistoni. *Nuovo Cimento*, 2A:733–787, 1971.

Functional reduction formula of the box integral with arbitrary kinematics

$$I_4^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2; s_{12}, s_{23}, s_{34}, s_{14}, s_{24}, s_{13}) = \sum_{k=1}^{24} Q_k B_4^{(d)}(r_{\{....\}}, r_{\{...\}}, r_{\{..\}}, r_{\{.\}})$$

where $r_{\{...\}} = \Delta_{\{...\}}/\lambda_{\{...\}}$ and

$$B_4^{(d)}(R_1, R_2, R_3, R_4)$$
$$= I_4^{(d)}(R_1, R_2, R_3, R_4;\ R_2 - R_1, R_3 - R_2, R_4 - R_3, R_4 - R_1, R_4 - R_2, R_3 - R_1)$$

$$= \Gamma\left(4 - \frac{d}{2}\right) R_1^{\frac{d}{2}-4} \int_0^1 \int_0^1 \int_0^1 x_1^2 x_2\ h_4^{\frac{d}{2}-4} dx_1 dx_2 dx_3.$$

and

$$h_4 = 1 - z_1 x_1^2 - z_2 x_1^2 x_2^2 - z_3 x_1^2 x_2^2 x_3^2$$

with

$$z_1 = 1 - \frac{R_2}{R_1}, \quad z_2 = \frac{R_2 - R_3}{R_1}, \quad z_3 = \frac{R_3 - R_4}{R_1},$$

Master integral for light by light scattering

$$s_{12} = s_{23} = s_{34} = s_{14} = 0, \qquad m_j^2 = m^2, \quad j = 1, ..., 4.$$

For this kinematics functional reduction method gives

$$I_4^{(d)}(m^2, m^2, m^2, m^2; 0, 0, 0, 0, s_{24}, s_{13}) = \frac{\widetilde{s}_{13}}{(\widetilde{s}_{24} + \widetilde{s}_{13})} L^{(d)}(\widetilde{s}_{13}, M_2^2, m^2) + \frac{\widetilde{s}_{24}}{(\widetilde{s}_{24} + \widetilde{s}_{13})} L^{(d)}(\widetilde{s}_{24}, M_2^2, m^2),$$

where

$$L^{(d)}(\widetilde{s}_{ij}, M_2^2, m^2) = B_4^{(d)}\left(m^2 - M_2^2, m^2, m^2 - \widetilde{s}_{ij}, m^2\right)$$
$$= I_4^{(d)}\left(m^2 - M_2^2, m^2, m^2 - \widetilde{s}_{ij}, m^2; M_2^2, -\widetilde{s}_{ij}, \widetilde{s}_{ij}, M_2^2, 0, M_2^2 - \widetilde{s}_{ij}\right),$$

$$M_2^2 = \frac{\widetilde{s}_{24}\widetilde{s}_{13}}{(\widetilde{s}_{24} + \widetilde{s}_{13})}, \qquad \widetilde{s}_{ij} = \frac{s_{ij}}{4}.$$

Integral for light by light scattering

Analytic result for the integral $L^{(d)}(\widetilde{s}_{ij}, M_2^2, m^2)$ can be obtained by solving dimensional recurrence relation

$$(d-3)L^{(d+2)}(\widetilde{s}_{ij}, M_2^2, m^2) = -2(m^2 - M_2^2)L^{(d)}(\widetilde{s}_{ij}, M_2^2, m^2) - I_3^{(d)}(m^2, m^2, m^2; 4\widetilde{s}_{ij}, 0, 0),$$

Solution of this recurrence relation for arbitrary $d = 4 - 2\varepsilon$ reads

$$L^{(d)}(\widetilde{s}_{ij}, M_2^2, m^2) = -\frac{\pi^{\frac{1}{2}}(m^2)^{\frac{d}{2}-3}}{8\widetilde{s}_{ij}\Gamma\left(\frac{d-3}{2}\right)} \int_0^1 \frac{dz\; z^{2-\frac{d}{2}}(1-z)^{\frac{d-5}{2}}}{1 - z\frac{M_2^2}{m^2}} \ln\left(1 - z\frac{\widetilde{s}_{ij}}{m^2}\right).$$

Using this expression we get

$$I_4^{(d)}(m^2, m^2, m^2, m^2; 0, 0, 0, 0, s_{24}, s_{13}) = -\frac{\pi^{\frac{1}{2}}(m^2)^{\frac{d}{2}-3}}{2(s_{24}+s_{13})\Gamma\left(\frac{d-3}{2}\right)} \times \int_0^1 \frac{dz\; z^{2-\frac{d}{2}}(1-z)^{\frac{d-5}{2}}}{1 - z\frac{M_2^2}{m^2}} \left[\ln\left(1 - z\frac{s_{13}}{4m^2}\right) + \ln\left(1 - z\frac{s_{24}}{4m^2}\right)\right].$$

At $d = 4$ from the one-fold integral representation we get:

$$L^{(4)}(\widetilde{s}_{ij}, M_2^2, m^2) = \frac{-1}{8M_2^2\,\widetilde{s}_{ij}\beta_2}\left[\ln^2\frac{\beta_{ij}+1}{\beta_2-\beta_{ij}} - \frac{1}{2}\ln^2\frac{\beta_2-\beta_{ij}}{\beta_2+\beta_{ij}} - \ln\frac{\beta_{ij}-1}{\beta_{ij}+1}\ln\frac{\beta_2-\beta_{ij}}{\beta_2+\beta_{ij}}\right.$$

$$\left.+\frac{\pi^2}{3} + 2\mathrm{Li}_2\left(\frac{\beta_{ij}-\beta_2}{\beta_{ij}+1}\right) - 2\mathrm{Li}_2\left(\frac{\beta_{ij}-1}{\beta_2+\beta_{ij}}\right) + \mathrm{Li}_2\left(\frac{\beta_{ij}^2-1}{\beta_{ij}^2-\beta_2^2}\right)\right].$$

where

$$\beta_{ij} \equiv \sqrt{1-\frac{m^2}{\widetilde{s}_{ij}}}, \qquad (ij = 13, 24) \qquad\qquad \beta_2 \equiv \sqrt{1-\frac{m^2}{M_2^2}}.$$

This expression for $L^{(4)}$ gives (in agreement with result by A.Davydychev, 1993)

$$I_4^{(4)}(m^2, m^2, m^2, m^2; 0, 0, 0, 0, s_{24}, s_{13})$$

$$= \frac{1}{8\widetilde{s}_{13}\widetilde{s}_{24}\beta_2}\left\{2\ln^2\left(\frac{\beta_2+\beta_{13}}{\beta_2+\beta_{24}}\right) + \ln\left(\frac{\beta_2-\beta_{13}}{\beta_2+\beta_{13}}\right)\ln\left(\frac{\beta_2-\beta_{24}}{\beta_2+\beta_{24}}\right) - \frac{\pi^2}{2}\right.$$

$$\left.+\sum_{i=13,24}\left[2\mathrm{Li}_2\left(\frac{\beta_i-1}{\beta_2+\beta_i}\right) - 2\mathrm{Li}_2\left(-\frac{\beta_2-\beta_i}{\beta_i+1}\right) - \ln^2\left(\frac{\beta_i+1}{\beta_2+\beta_i}\right)\right]\right\},$$

Integral for the light by light scattering

Solving differential equation for $L^{(d)}$ with respect to s_{ij} we obtained:

$$L^{(4)}(\widetilde{s}_{ij}, M_2^2, m^2) = \frac{1}{8\widetilde{s}_{ij} M_2^2 \beta_2}$$
$$\times \left[\mathrm{Li}_2(1 - y_{ij} y_2) + \mathrm{Li}_2\left(1 - \frac{y_2}{y_{ij}}\right) - 2\mathrm{Li}_2(1 - y_2) + \frac{1}{2} \ln^2 y_{ij} \right],$$

where

$$y_{ij} = \frac{\beta_{ij} - 1}{\beta_{ij} + 1}, \qquad y_2 = \frac{\beta_2 - 1}{\beta_2 + 1}. \qquad \beta_{ij} = \sqrt{1 - \frac{m^2}{\widetilde{s}_{ij}}}, \qquad \beta_2 = \sqrt{1 - \frac{m^2}{M_2^2}},$$

Substituting this expression into the functional relation we get

$$I_4^{(4)}(m^2, m^2, m^2, m^2; 0, 0, 0, 0, s_{24}, s_{13})$$
$$= \frac{2}{s_{13} s_{24} \beta_2} \left[\frac{1}{2} \ln^2 y_{13} + \frac{1}{2} \ln^2 y_{24} - 4\mathrm{Li}_2(1 - y_2) \right.$$
$$\left. + \mathrm{Li}_2(1 - y_{13} y_2) + \mathrm{Li}_2(1 - y_{24} y_2) + \mathrm{Li}_2\left(1 - \frac{y_2}{y_{13}}\right) + \mathrm{Li}_2\left(1 - \frac{y_2}{y_{24}}\right) \right].$$

This expression is invariant with respect to $y_{ij} \to 1/y_{ij}$ as well as $y_2 \to 1/y_2$.

Consider box with one leg off shell

$$s_{23} = s_{34} = s_{14} = 0, \qquad m_i^2 = m^2.$$

Using formula of the functional reduction for $I_4^{(d)}$ we get for arbitrary d

$$I_4^{(d)}(m^2, m^2, m^2, m^2, s_{12}, 0, 0, 0, s_{24}, s_{13})$$

$$= \frac{s_{12}}{s_0} L^{(d)}(\widetilde{s}_{12}, M_3^2, m^2) - \frac{s_{13}}{s_0} L^{(d)}(\widetilde{s}_{13}, M_3^2, m^2) - \frac{s_{24}}{s_0} L^{(d)}(\widetilde{s}_{24}, M_3^2, m^2).$$

where

$$M_3^2 = -\frac{s_{24}s_{13}}{4s_0}, \qquad s_0 = s_{12} - s_{13} - s_{24}.$$

Thus, integral depending on 4 variables is a sum of integrals depending on 3 variables.
At $d = 4$

$$I_4^{(4)}(m^2, m^2, m^2, m^2, s_{12}, 0, 0, 0, s_{24}, s_{13})$$

$$= -\frac{[2\mathrm{Li}_2(1 - y_3) + F(y_{12}, y_3) + F(y_{13}, y_3) + F(y_{24}, y_3)]}{\beta_3 \widetilde{s}_{24} \widetilde{s}_{13}},$$

where

$$F(y_{ij}, y_k) = \mathrm{Li}_2\left(1 - y_{ij}y_k\right) + \mathrm{Li}_2\left(1 - \frac{y_k}{y_{ij}}\right) + \frac{1}{2}\ln^2 y_{ij}.$$

Two legs off shell

In a similar manner one can consider the box integral with two leggs off shell. There are two different kinematical situations. The easiest case

$$s_{23} = s_{14} = 0, \qquad m_k^2 = m^2, \quad k = 1, ..., 4.$$

From final functional reduction formula for this kinematics we get

$$\begin{aligned} I_4^{(d)}(m^2, m^2, m^2, m^2; s_{12}, 0, s_{34}, 0, s_{24}, s_{13}) \\ = \frac{s_{12}}{\delta_4} L^{(d)}(\widetilde{s}_{12}, M_4^2, m^2) - \frac{s_{13}}{\delta_4} L^{(d)}(\widetilde{s}_{13}, M_4^2, m^2) \\ - \frac{s_{24}}{\delta_4} L^{(d)}(\widetilde{s}_{24}, M_4^2, m^2) + \frac{s_{34}}{\delta_4} L^{(d)}(S_{34}, M_4^2, m^2), \end{aligned}$$

where

$$M_4^2 = \frac{s_{12}s_{34} - s_{13}s_{24}}{4\delta_4}, \qquad \delta_4 = s_{12} - s_{13} - s_{24} + s_{34}.$$

Two legs off shell

At $d=4$ substituting $L^{(4)}$ we get

$$I_4^{(d)}(m^2,m^2,m^2,m^2;s_{12},0,s_{34},0,s_{24},s_{13})$$
$$= \frac{F(y_{13},y_4)+F(y_{24},y_4)-F(y_{34},y_4)-F(y_{12},y_4)}{\beta_4\ (\widetilde{s}_{13}\widetilde{s}_{24}-\widetilde{s}_{12}S_{34})}.$$

Two legs off shell

The most complicated case corresponds to the kinematics

$$s_{34} = s_{14} = 0, \quad m_k^2 = m^2, \quad k = 1, ..., 4.$$

From final formula of functional reduction for this kinematics we get

$$\begin{aligned}
&2d_1 d_2 I_4^{(d)}(m^2, m^2, m^2, m^2; s_{12}, s_{23}, 0, 0, s_{24}, s_{13}) \\
&= d_1 s_{13} s_{24} L^{(d)}(\widetilde{s}_{13}, S_1, m^2) - d_1 s_{23} s_{24} L^{(d)}(\widetilde{s}_{23}, S_1, m^2) \\
&- d_1 s_{12} s_{24} L^{(d)}(\widetilde{s}_{12}, S_1, m^2) + 2d_1 s_{24}^2 L^{(d)}(\widetilde{s}_{24}, S_1, m^2) \\
&+ n_1 s_{12}(s_{12} - s_{13} - s_{23}) B_4^{(d)}(m_0^2, \widetilde{m}_0^2, m^2 - \widetilde{s}_{12}, m^2) \\
&- n_1 s_{13}(s_{12} - s_{13} + s_{23}) B_4^{(d)}(m_0^2, \widetilde{m}_0^2, m^2 - \widetilde{s}_{13}, m^2) \\
&- n_1 s_{23}(s_{12} + s_{13} - s_{23}) B_4^{(d)}(m_0^2, \widetilde{m}_0^2, m^2 - \widetilde{s}_{23}, m^2),
\end{aligned}$$

where

$$B_4^{(d)}(m_0^2, \widetilde{m}_0^2, m^2 - \widetilde{s}_{ij}, m^2) = I_4^{(d)}(m_0^2, \widetilde{m}_0^2, m^2 - \widetilde{s}_{ij}, m^2; -S_2, S_2 - \widetilde{s}_{ij}, \widetilde{s}_{ij}, S_1, S_2, S_1 - \widetilde{s}_{ij}).$$

$$m_0^2 = m^2 - S_1, \quad \widetilde{m}_0^2 = m^2 - S_2, \quad S_1 = \frac{s_{13} s_{24}^2}{4d_2}, \quad S_2 = -\frac{s_{12} s_{13} s_{23}}{d_1},$$

$$n_1 = 2s_{12}s_{23} - s_{12}s_{24} + s_{13}s_{24} - s_{23}s_{24}, \quad d_2 = s_{12}s_{23} - s_{12}s_{24} + s_{13}s_{24} - s_{23}s_{24} + s_{24}^2,$$

$$d_1 = s_{12}^2 + s_{13}^2 + s_{23}^2 - 2s_{12}s_{13} - 2s_{12}s_{23} - 2s_{13}s_{23},$$

Two leggs off shell

In order to evaluate the integrals $B_4^{(d)}(m_0^2, \widetilde{m}_0^2, m^2 - \widetilde{s}_{13}, m^2)$ we use Feynman parameter representation

$$B_4^{(d)}(r_{1234}, r_{234}, r_{34}, r_4) = \int_0^1 \int_0^1 \int_0^1 \frac{\Gamma\left(4 - \frac{d}{2}\right) x_1^2 x_2 \; dx_1 dx_2 dx_3}{[a - bx_1^2 - cx_1^2 x_2^2 - ex_1^2 x_2^2 x_3^2]^{4-\frac{d}{2}}}.$$

where

$$a = r_{1234}, \qquad b = r_{1234} - r_{234}, \qquad c = r_{234} - r_{34}, \qquad e = r_{34} - r_4.$$

At $d = 4$, changing variables $x_2 = \sqrt{u}$, $x_3 = z_3/x_1$, integrating over u, changing order of integration over x_1 and z_3, we get

$$B_4^{(4)}(r_{1234}, r_{234}, r_{34}, r_4) = \frac{1}{8} \int_0^1 \frac{dt}{\sqrt{1-t} \; (ac + be - bet)}$$

$$\times \left\{ \ln \left[1 + \frac{(b+c+e)t}{r_4} \right] - \ln \left[1 + \frac{et}{r_4} \right] - \ln \left[1 + \frac{bt}{a-b} \right] \right\}.$$

Two leggs off shell

The integrals can be rewritten in terms of $L^{(4)}$

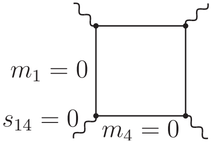
$$B_4(m^2-S_1,m^2-S_2,m^2-\widetilde{s}_{ij},m^2)$$
$$=\frac{m^2}{\delta_5}\left[S_1L^{(4)}\left(S_1,\frac{m^2(S_1-S_2)\widetilde{s}_{ij}}{\delta_5},m^2\right)-\widetilde{s}_{12}L^{(4)}\left(\widetilde{s}_{12},\frac{m^2(S_1-S_2)\widetilde{s}_{ij}}{\delta_5},m^2\right)\right.$$
$$\left.+\frac{m^2(S_1-S_2)}{S_2-m^2}L^{(4)}\left(\frac{m^2(S_1-S_2)}{m^2-S_2},\frac{m^2(S_1-S_2)\widetilde{s}_{ij}}{\delta_5},m^2\right)\right].$$

where

$$\delta_5=\widetilde{s}_{ij}(m^2-S_2)+S_2(S_1-m^2).$$

Thus the integral $I_4^{(4)}$ with two leggs off shell can be written in terms of $L^{(4)}$ functions.

Finite observables typically require the explicit cancellation of infrared divergences across different loop orders. In ref. Zhang J., Commun. Theor. Phys.,v.73 (2021) 105203 complicated IR divergent integral was considered



Substituting $m_1^2 = m_4^2 = s_{14} = 0$ into formula for functional reduction we get

$$I_4^{(d)}(0, m_2^2, m_3^2, 0; s_{12}, s_{23}, s_{34}, 0, s_{24}, s_{13}) = \alpha_1 B_4^{(d)}(r_1, 0, 0, 0) + \sum_{ij} \alpha_{ij} B_4^{(d)}(r_i, 0, r_j, 0)$$
$$+ \sum_{ijk} \alpha_{ijk} B_4^{(d)}(r_i, 0, r_j, r_k) + \sum_{ijk} \beta_{ijk} B_4^{(d)}(r_i, r_j, r_k, 0) + \sum_{ijkl} \alpha_{ijkl} B_4^{(d)}(r_i, r_j, r_k, r_l)$$

$I_4^{(d)}$ depends on 7 variables. After functional reduction IR divergences arise in integrals depending on 1, 2, 3 variables.

Three infrared divergent integrals:

$$B_4^{(d)}(r_i,0,0,0) = \frac{-\pi^{\frac{3}{2}} r_i^{\frac{d}{2}-3}}{8 r_j \Gamma\left(\frac{d-3}{2}\right) \sin\frac{\pi d}{2}},$$

$$B_4^{(d)}(r_i,0,r_j,0) = \frac{-\pi^{\frac{3}{2}} r_i^{\frac{d}{2}-3}}{8 r_j \Gamma\left(\frac{d-3}{2}\right) \sin\frac{\pi d}{2}} \left[\ln\left(1-\frac{r_j}{r_i}\right) + \frac{2}{d-4}\frac{r_j^{\frac{d}{2}-2}}{r_i^{\frac{d}{2}-2}}\, {}_2F_1\left(1,\frac{d-4}{2};\frac{d-2}{2};\frac{r_j}{r_i}\right)\right]$$

$$B_4^{(d)}(r_i,0,r_j,r_k) = \frac{\sqrt{\pi}\,\Gamma\left(4-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-2\right)}{\Gamma\left(\frac{d-3}{2}\right)\sqrt{r_j(r_i-r_j)}} r_i^{\frac{d}{2}-3} \operatorname{arctanh}\left(\sqrt{\frac{r_k-r_j}{r_j}}\right) + \sum_j \gamma_j L^{(4)}(\ldots) + O(\varepsilon).$$

Conclusions

- Functional reduction transforms integrals of interest to simpler integrals.
- At $d = 4$ all box integrals are expressible in terms of combinations of dilogarithms encapsulated in the $L^{(4)}$ functions.
- Application of functional reduction to the infrared divergent integrals split them into a simpler IR divergent integrals plus more complicated IR convergent part.