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All-loop Contribution To Effective Potential

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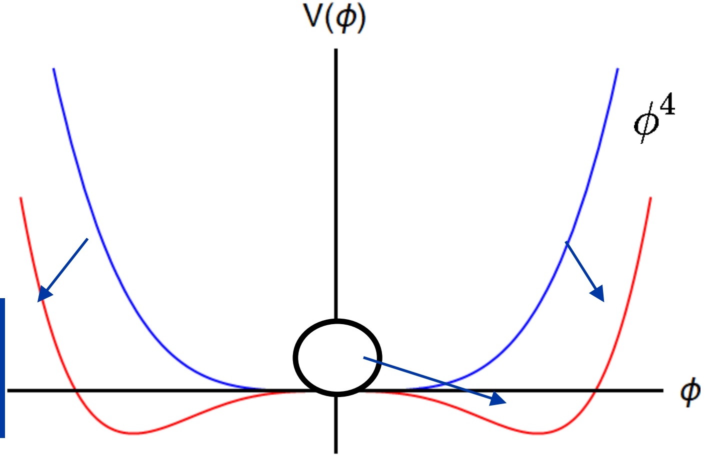
Content

- Effective potentials: Weinberg and Coleman mechanism
- Bogoliubov-Parasyuk theorem: R-operation in non-renormalizable theories
- Application to sextic potential
- Recurrence relations for all leading contributions to an arbitrary potential
- Numerical and analytic solutions:

Weinberg-Coleman results

In 1973 E. Weinberg and S. Coleman investigated the mechanism of appearance of an additional minimum in the effective potential after the addition of a one-loop quantum correction.

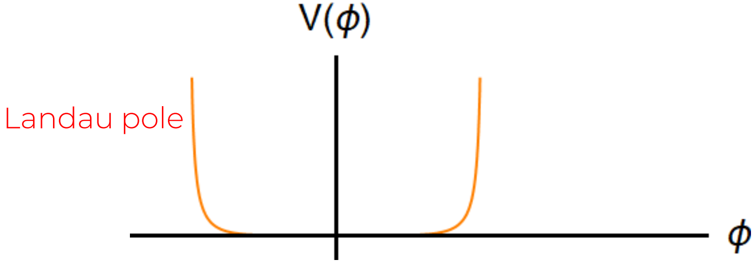


$$V_{classical}(\phi) \sim \frac{g}{4!}\phi^4$$

$$V_{eff}(\phi) \sim \frac{g}{4!}\phi^4 \left(1 + \frac{3}{2}g\log(\phi^2/\mu^2)\right)$$

But accounting for **all the corrections** in the effective potential (**RG**) leads to the restoration of the original minimum :

$$V_{all-loop}(\phi) \sim \frac{g\phi^4}{1 - \frac{3}{2}g\log(\phi^2/\mu^2)}$$



Non-renormalizable potentials was not considered.
Leading log φ2 terms do not depend on arbitrariness.

The effective potential

Path integral:

$$Z(J) = \int \mathcal{D}\phi \, \exp\left(i \int d^4x \, \mathcal{L}(\phi, d\phi) + J\phi \right)$$

Legendre transformation

$$W(J) = -i \log Z(J)$$

$$\varphi = \frac{\delta W}{\delta J}$$

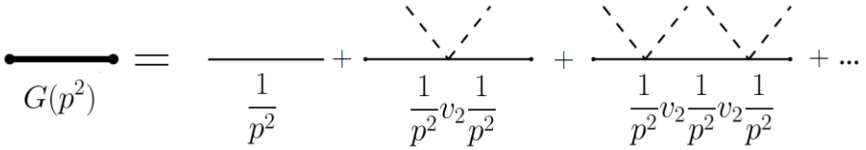
$$\Gamma[\Phi] \equiv W[j] - \int d^4y j(y)\Phi(y)$$

Definition

$$\Gamma(\varphi) = - \int d^4x \, V_{eff}(\varphi)$$

Shift

$$S(\varphi) \to S(\varphi + \phi)$$



$$G(p^2) = \frac{i}{p^2 + v_2}$$

$$v_2 = \frac{\partial^2 V}{\partial \phi^2}$$

Bogoliubov-Parasyuk theorem

If we consider a **divergent graph** $\mathbf{G}$ of any local field theory, then after **subtraction** all the divergent subgraphs, the remaining divergence will also be local

R-operation:

$$\mathcal{R}G = \prod_{\gamma}(1 - K_{\gamma})G,$$

Incomplete R'-operation:

$$\mathcal{R}G = (1 - \mathcal{K}_{\gamma})\mathcal{R}'G$$

The remained leading divergence after applying the incomplete R-operation $\mathbf{R'}\ \mathbf{G(n)}$ looks like

$$\frac{A_n(\mu^2)^{n\epsilon}}{\epsilon^n} + \frac{A_{n-1}(\mu^2)^{(n-1)\epsilon}}{\epsilon^n} + \ldots + \frac{A_1(\mu^2)^{\epsilon}}{\epsilon^n},$$

$$\boxed{\frac{A_k(\mu^2)^{k\epsilon}}{\epsilon^n}}$$

Final result must not include these term:

$$(\log\mu)^k/\epsilon^m$$

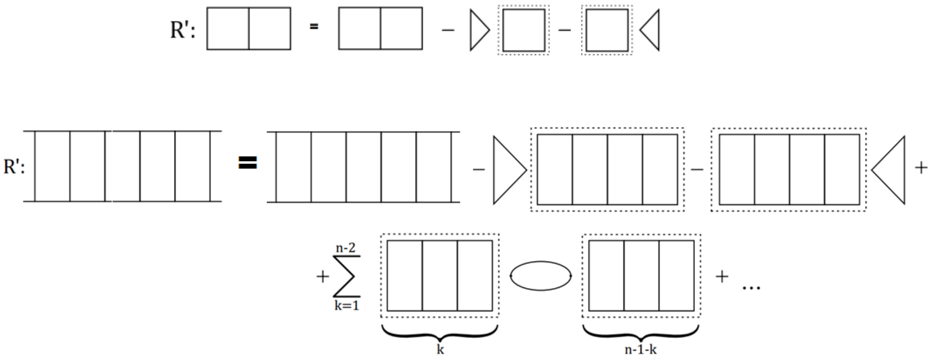
Such a restriction leads to recurrence relation:

$$\boxed{A_n = (-1)^{n+1}\frac{A_1}{n}.}$$

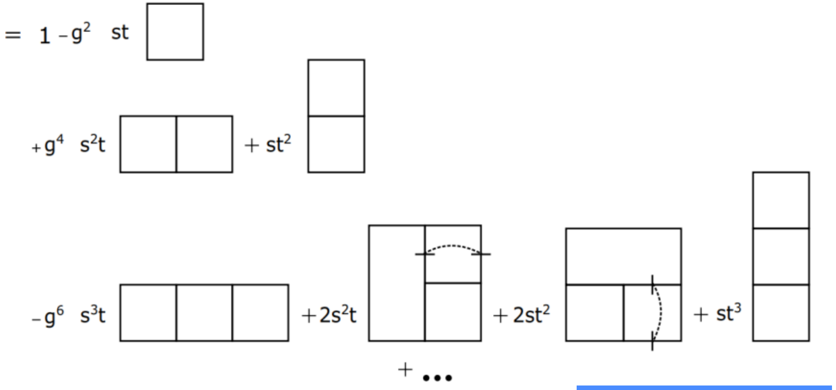
Example: SYM-theory

Kazakov, Tolkachev et al.'15

R'-operation for ladder-type diagrams:



$$\frac{A_4}{A_4^{(0)}} = 1 + \sum_L M_4^{(L)}(s,t) =$$



Recurrence relation for D=8 SYM:

$$nA_n = -\frac{2}{4!}A_{n-1} + \frac{2}{5!}\sum_{k=1}^{n-2} A_k A_{n-1-k},$$

$$\Sigma' = -\frac{1}{3!} + \frac{2}{4!}\Sigma - \frac{2}{5!}\Sigma^2.$$

All-loop recursive equation:

$$\frac{d}{dz}\Sigma(s,t,z) = -\frac{1}{12} + 2s^2\int_0^1 dx \int_0^x dy\; y(1-x)\;(\Sigma(s,t',z) + \Sigma(t',s,z))|_{t'=tx+yu}$$

$$-s^4\int_0^1 dx\; x^2(1-x)^2\sum_{p=0}^{\infty}\frac{1}{p!(p+2)!}\left(\frac{d^p}{dt'^p}(\Sigma(s,t',z)+\Sigma(t',s,z))|_{t'=-sx}\right)^2 (tsx(1-x))^p.$$

Sextic potential

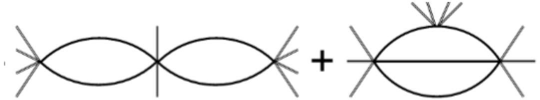
$$V = -\frac{g}{6!}\phi^6$$

$$v_n = \frac{\partial^n}{\partial \varphi^n} V(\varphi)$$

One-loop diagram:

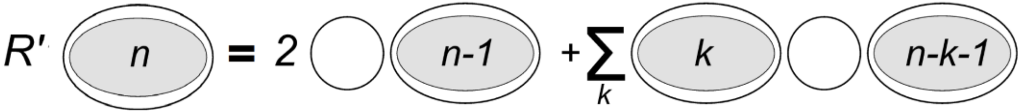
$$S_1 = v_2 \frac{1}{2}\frac{1}{\epsilon}$$

Two-loop diagram:



$$S_2 = -v_2 v_4 \frac{1}{4}\frac{1}{\epsilon} - v_3^2 v_2 \frac{1}{2}\frac{1}{2\epsilon}$$

Performing R'-operation we get:



$$D_2 = \frac{\partial^2}{\partial \varphi^2} \qquad S_2 = -\frac{1}{4} v_2 D_2 S_1$$

In the three-loop case, there are additional terms that give a non-linear contribution.

Thus, all loop corrections can be written as a recurrence equation:

$$nS_n = 2\frac{D_2}{4}\left(\frac{g\varphi^6}{6!}\right)\frac{D_2}{4}S_{n-1} + 2\sum_{k=3}^{n-2}\frac{D_2}{4}S_k\frac{D_2}{4}S_{n-k-1}$$

$$S_0 = V(\varphi)$$

Arbitrary power of interaction

Given the insensitivity of the equation to the form of the potential, the **recurrence equation** can be reduced to the following form

$$nS_n = -\sum_{k=0}^{n-1} \frac{D_2}{2} S_k \frac{D_2}{2} S_{n-k-1}$$

$$z = g/\epsilon$$

$$\Sigma(z) = \sum_{n=0}^{\infty} (-z)^n S_n$$

Differential equation:

$$\frac{d}{dz}\Sigma(z) = -\left(\frac{D_2}{2}\Sigma(z)\right)^2$$

Function for arbitrary power of interaction:

$$\Sigma(z) = \frac{\varphi^p}{p!} f\left(z\varphi^{p-4}\right)$$

Dimensionless variable

$$y = z\varphi^{p-4}$$

Equation:

$$-f'(y) = \frac{1}{4(p-2)!}\left(p(p-1)f(y) + (p-4)(3p-5)yf'(y) + (p-4)^2 y^2 f''(y)\right)^2$$

Initial conditions:

$$f(0) = 1, f'(0) = -\frac{1}{4}\frac{p(p-1)}{(p-2)!}$$

Analytical and numerical solutions

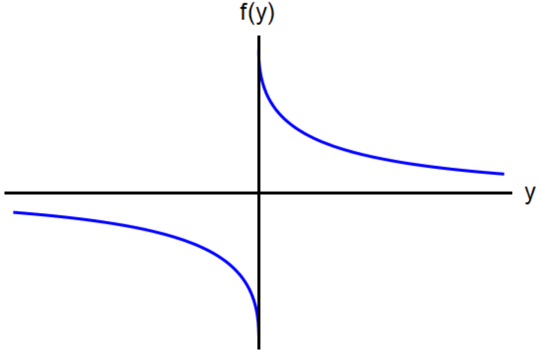
Equation for the quartic interaction:

$$f'(y) = -\frac{3}{2} f(y)^2$$

Coincide!

$$V(\phi) = \frac{g\phi^4}{1 - \frac{3}{2} g \log\left(\frac{\phi^2}{\mu^2}\right)}$$

The analytical solution maybe obtained for any series with **homogeneous diagram topology**

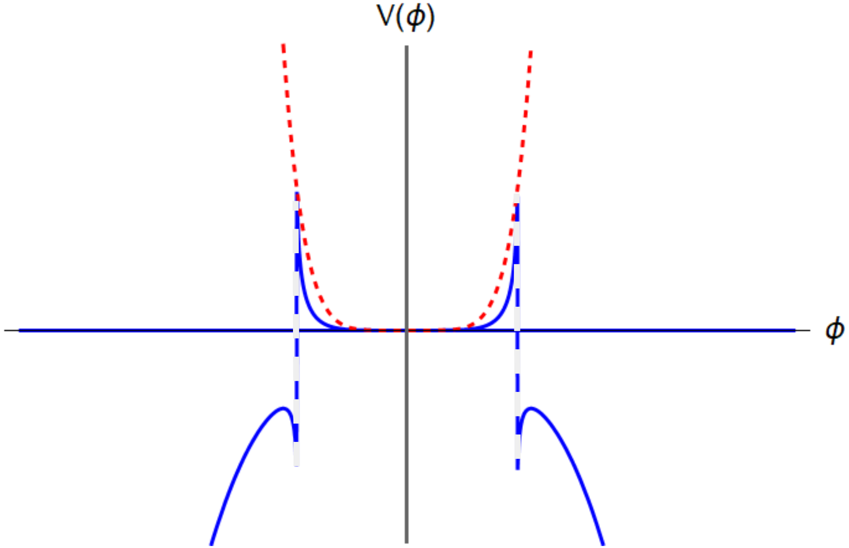


General form of solutions of the differential equation

Equation for the sextic interaction:

$$\frac{1}{24}\left(y^2 f''(y) + \frac{13}{2} y f'(z) - \frac{15}{2} f(z)\right)^2 = -f'(y)$$

$$f(0) = 1, f'(0) = -\frac{5}{16} \qquad z \to -\log(v_2/\mu^2)$$

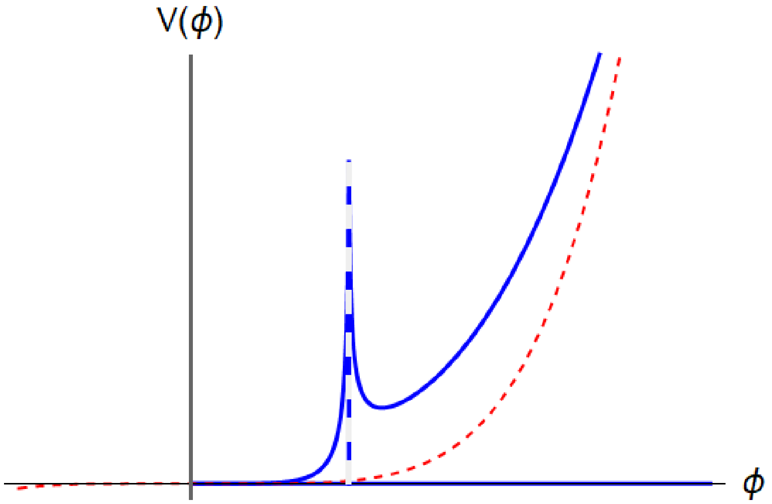


Analytical and numerical solutions

Equation for the 5-th order interaction:

$$-f'(y) = -\frac{1}{24}\left(y^2 f''(y) - 20 f(y)\right)^2$$

$$f(0) = 1,\ f'(0) = -5/6$$

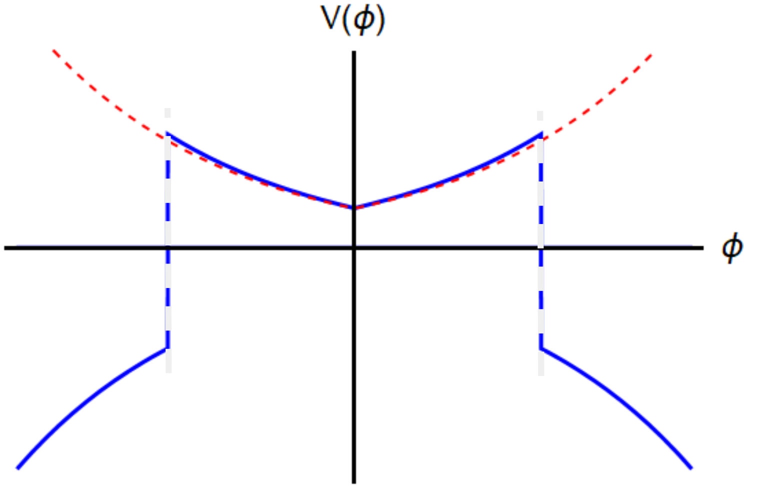


Equation for the exponential interaction:

$$V = g\exp\left(|\phi|/\mu\right) \quad \Sigma(z) = \exp\left(\frac{|\phi|}{m}\right) f(z/m^4)$$

$$f'(y) = -\frac{1}{4}\left(y^2 f''(y) + 3y f'(y) + f(y)\right)^2$$

$$f(0) = 1,\ f'(0) = -\frac{1}{4}$$



Conclusions

- In this work we found **recurrence relations** for leading divergences for scalar theories with arbitrary type of interactions.
- In separate cases we managed to obtain differential equations which <u>reproduce</u> the results in the literature and <u>generalize</u> them.
- We have obtained numerical solutions in the general case of a power potential.
- For even and odd power potentials, we obtained a solution in which symmetry is restored.. The solution contains a discontinuity so that the vacuum of the theory is **metastable**.

Further development

- It would be interesting to get a differential equation in the case of an arbitrary potential. One could, for example, consider potentials of the cosine type
- In principle, there is no obstacle in future to consider scalar electrodynamics with a non-renormalizable potential or a model with many scalar fields and so on…

Thanks for attention