Bianchi I cosmological solutions in teleparallel gravity

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$$e_A(x^\mu),$$

$$g_{\mu\nu} = \eta_{AB} e^A{}_\mu e^B{}_\nu,$$

$$T^\lambda{}_{\mu\nu} \equiv \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu} = e_A{}^\lambda (\partial_\mu e^A{}_\nu - \partial_\nu e^A{}_\mu),$$

where $e_A{}^\mu$ denotes inverse tetrad, which satisfies $e_A{}^\mu e^A{}_\nu = \delta^\mu_\nu$ and $e_A{}^\mu e^B{}_\mu = \delta^B_A$.

$$K^{\mu\nu}{}_\lambda \equiv -\frac{1}{2}\left(T^{\mu\nu}{}_\lambda - T^{\nu\mu}{}_\lambda - T_\lambda{}^{\mu\nu}\right),$$

$$S_\lambda{}^{\mu\nu} \equiv (K^{\mu\nu}{}_\lambda + \delta^\mu_\lambda T^{\alpha\nu}{}_\alpha - \delta^\nu_\lambda T^{\alpha\mu}{}_\alpha),$$

$$T \equiv \frac{1}{2} S_{\lambda}{}^{\mu\nu} T^{\lambda}{}_{\mu\nu} = \frac{1}{4} T^{\lambda\mu\nu} T_{\lambda\mu\nu} + \frac{1}{2} T^{\lambda\mu\nu} T_{\nu\mu\lambda} - T_{\lambda\mu}{}^{\lambda} T^{\nu\mu}{}_{\nu}.$$

$$S = \frac{1}{2\kappa^2} \int d^4x \, e \, f(T),$$

where $e = det(e^A_\mu) = \sqrt{-g}$ and κ^2 is the gravitational constant.

$$e^{-1} \partial_\mu (e S_A{}^{\mu\nu}) f' - e_A{}^{\lambda} T^{\rho}{}_{\mu\lambda} S_{\rho}{}^{\nu\mu} f' + S_A{}^{\mu\nu} \partial_\mu (T) f'' - \frac{1}{4} e_A{}^{\nu} f = \kappa^2 e_A{}^{\rho} T^{m}_{\rho}{}^{\nu},$$

$$f' (\overset{\circ}{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \overset{\circ}{R}) + \frac{1}{2} g_{\mu\nu} [f(T) - f' T] + f'' S_{\nu\mu\lambda} \nabla^{\lambda} T = \kappa^2 T^{m}{}_{\mu\nu},$$

$$e_\mu^A = \mathrm{diag}\big(1, \mathrm{a}(t), \mathrm{b}(t), \mathrm{c}(t)\big),$$

$$g_{\mu\nu} = \mathrm{diag}\big(1, -\mathrm{a}(t)^2, -\mathrm{b}(t)^2, -\mathrm{c}(t)^2\big)$$

Now introducing three Hubble parameters $H_\mathrm{a} \equiv \frac{\dot{\mathrm{a}}}{\mathrm{a}}$, $H_\mathrm{b} \equiv \frac{\dot{\mathrm{b}}}{\mathrm{b}}$ and $H_\mathrm{c} \equiv \frac{\dot{\mathrm{c}}}{\mathrm{c}}$, we find for torsion scalar

$$T = -2\big(H_\mathrm{a}H_\mathrm{b} + H_\mathrm{a}H_\mathrm{c} + H_\mathrm{b}H_\mathrm{c}\big).$$

$$\frac{1}{2}f - Tf' = \kappa^2\rho,$$

where we denote $f' \equiv df/dT$ and introduce isotropic fluid in the right-hand side $p = w\rho$,

$$\dot{\rho} + \left(1 + w\right)\left(H_{\rm a} + H_{\rm b} + H_{\rm c}\right)\rho = 0,$$

$$\left(H_{\rm b} + H_{\rm c}\right)\dot{T} f'' + \frac{1}{2} f + \tag{1}$$
$$f'\left(\dot{H}_{\rm b} + H_{\rm b}^2 + \dot{H}_{\rm c} + H_{\rm c}^2 + 2H_{\rm b}H_{\rm c} + H_{\rm a}H_{\rm b} + H_{\rm a}H_{\rm c}\right) = -\kappa^2 w\rho,$$

$$\left(H_{\rm a} + H_{\rm c}\right)\dot{T} f'' + \frac{1}{2} f + \tag{2}$$
$$f'\left(\dot{H}_{\rm a} + H_{\rm a}^2 + \dot{H}_{\rm c} + H_{\rm c}^2 + 2H_{\rm a}H_{\rm c} + H_{\rm a}H_{\rm b} + H_{\rm b}H_{\rm c}\right) = -\kappa^2 w\rho,$$

$$\left(H_{\rm a} + H_{\rm b}\right)\dot{T} f'' + \frac{1}{2} f + \tag{3}$$
$$f'\left(\dot{H}_{\rm a} + H_{\rm a}^2 + \dot{H}_{\rm b} + H_{\rm b}^2 + 2H_{\rm a}H_{\rm b} + H_{\rm a}H_{\rm c} + H_{\rm b}H_{\rm c}\right) = -\kappa^2 w\rho.$$

$$\left(T,\ \ H \equiv H_{\rm a} + H_{\rm b} + H_{\rm c}\right),$$

summing equations (1) + (2) + (3):

$$2\dot{T}Hf'' + \frac{3}{2}f + 2f'(\dot{H} + H^2) = -3w\kappa^2\rho,$$

summing with the coefficients $(-2H_a)\cdot$ (1) $+(-2H_b)\cdot$ (2) $+(-2H_c)\cdot$ (3):

$$2\dot{T}Tf'' - Hf + \dot{T}f' + 2f'TH = 2w\kappa^2 H\rho,$$

$$\dot{\rho} + \left(1 + w\right)H\rho = 0.$$

For the expanding Universe we expect decreases of all dynamical variables with time, so let us try to find solution in the form

$$T = \mathrm{t}_0 t^{-m}, \;\; H = h_0 t^{-n}, \;\; \rho = \rho_0 t^{-k}$$

with some positive m, n, k. Substituting it into

$$\dot{\rho} + \left(1 + w\right) H\rho = 0,$$

we find $n = 1$, which is quite natural. Now we can see that for any function f that can be expanded in the Taylor series $f = \sum_{i=1}^{\infty} f_i T^i$ will keep only the terms with the lowest i because all other will decrease more rapidly. It is quite natural to suppose that the lowest term is T because $f = T$ is equivalent to GR, and we would like to have it as a limit. It means that instead of previous system for this kind of solution we have an approximate system

$$\frac{3}{2}T + 2\left(\dot{H} + H^2\right) = \quad -3\kappa^2 w\rho = -3w(\frac{1}{2}T - T),$$
$$HT + \dot{T} = \quad 2\kappa^2 wH\rho = 2wH(\frac{1}{2}T - T),$$

where we used constraint equation. Substituting there our solution, we find $m = k = 2$. Let us introduce the parameter a: $T = -aH^2$. Now for the parameters a, h_0 we have

$$-\frac{3}{2}ah_0^2 - 2h_0 + 2h_0^2 = \quad -\frac{3}{2}wh_0^2 a,$$
$$-h_0 ah_0^2 + 2ah_0^2 = \quad wh_0 ah_0^2,$$

which has the unique solution $h_0(1 + w) = 2$, $a = 2/3$ corresponding to the expanding ($h_0 > 0$, $\forall w$) isotropic solution.

$$
\begin{aligned}
+\frac{3}{2}(1+w)f - 3wT_0f' + 2f'H_0^2 &= 0,\\
-(1+w)H_0f + 2(1+w)H_0T_0f' &= 0.
\end{aligned}
$$

In the most general case there are three (groups) stationary points.

P1. The first one is $H_0 = 0,\ T_0 = 0$. This point exists for any shape of the function with $f(0) = 0$, which is quite natural, any eos w and corresponds to the Minkowski solution.

P2. The second point is $H_0 = 0,\ T_0 \neq 0$.
$T_0:\ 2wT_0f' = (1+w)f$. Note that this point exists only if the last equation admits solution with $T_0 < 0$ and can present a group of points for the polynomial function f.

P3. And the third one is $H_0 \neq 0,\ T_0 \neq 0$. As we can see from second equation, in this case we have $T_0 :\ 2T_0 f' = f, \forall w$ and therefore by using the first equation, we find $2H_0^2 = -3T_0$ that corresponds to the isotropic de Sitter solution. We can see that this solution actually corresponds to two de Sitter points one for an expanding and the other for a contracting Universe, and it exists only for functions that admit solutions with $T_0 < 0$.

$$\dot{\rho} + \big(1 + w\big) H\rho = 0,$$

$H > 0$ – expanding Universe; $H < 0$ – contracting Universe

$$2\dot{T}Hf'' + \frac{3}{2}f + 2f'(\dot{H} + H^2) = -3w\kappa^2\rho,$$

$$2\dot{T}Tf'' - Hf + \dot{T}f' + 2f'TH = 2w\kappa^2 H\rho,$$

$$\frac{1}{2}f - Tf' = \kappa^2\rho,$$

$$f = T + f_2 T^2,$$

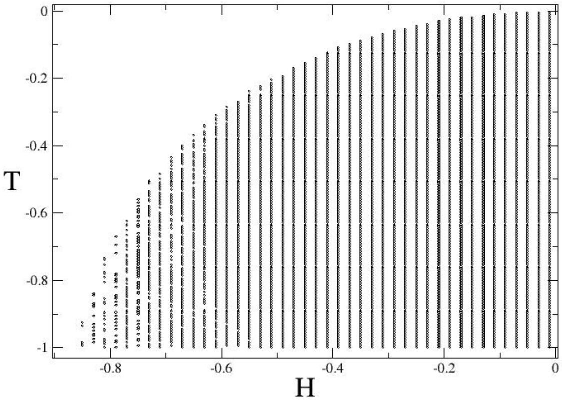

Рис.: Bounce solutions for $f_2 = 0.1,\ N = 2,\ w = 0,\ \kappa^2 = 1.$

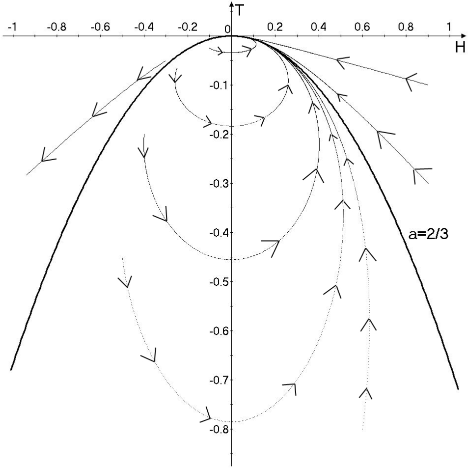

Рис.: Future isotropic attractor for $f_2 = 0.1$, $N = 2$, $w = 0$, $\kappa^2 = 1$, $H_0 = -10^{-2}$, $T_0 = -10^{-4}$.

The only point **P1** exists.

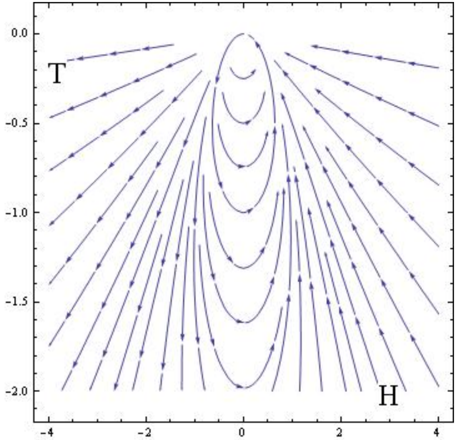

Рис.: Phase portrait for $f_2 = -0.2$, $N = 2$, $w = 0$, $\kappa^2 = 1$.

The points **P1** and **P2** exist.

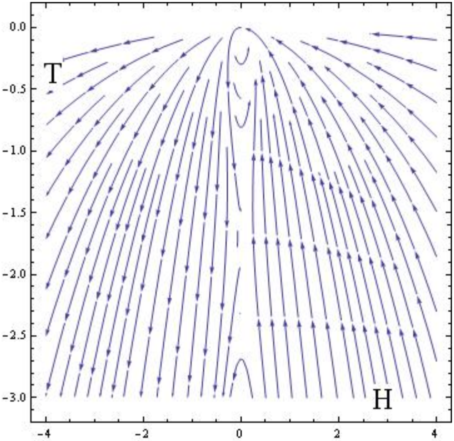

Рис.: Phase portrait for $f_2 = -0.2$, $N = 2$, $w = \frac{2}{3}$, $\kappa^2 = 1$.

The points **P1** and **P3** exist.

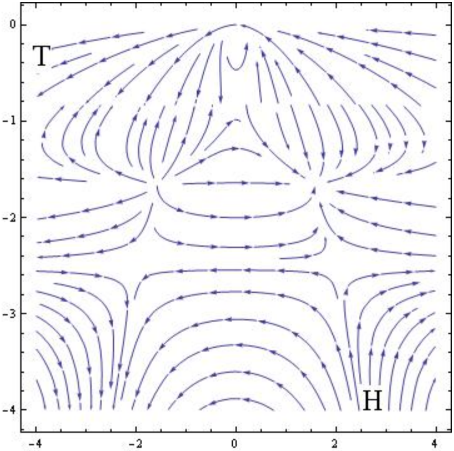

Рис.: Phase portrait for $f_2 = 0.2$, $N = 2$, $w = \frac{2}{3}$, $\kappa^2 = 1$.

All three types of points exist.

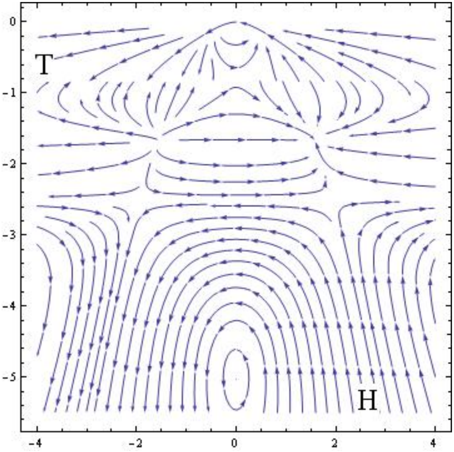

Рис.: Phase portrait for $f_2 = 0.2$, $N = 2$, $w = 0$, $\kappa^2 = 1$.

- Maria A. Skugoreva, Alexey V. Toporensky, Anisotropic cosmological dynamics in f(T) gravity in the presence of a perfect fluid, Eur. Phys. J. C 79 (2019) no. 10, 813; arXiv:1907.12538 [gr-qc]
- We confirm mains results of previous authors and find a number of new ones such as
- the existence of an essential number of bounce solutions
- and existence of point P2, which was not found in that research.

$$g_{\mu\nu} = \eta_{AB} e^A{}_\mu e^B{}_\nu,$$

$$e'^A{}_\mu = \Lambda^A{}_B e^B{}_\mu, \quad \omega'^A{}_{B\mu} = \Lambda^A{}_C \omega^C{}_{F\mu} \Lambda_B{}^F + \Lambda^A{}_C \partial_\mu \Lambda_B{}^C,$$

$$R^A{}_{B\mu\nu} = \partial_\mu \omega^A{}_{B\nu} - \partial_\nu \omega^A{}_{B\mu} + \omega^A{}_{C\mu} \omega^C{}_{B\nu} - \omega^A{}_{C\nu} \omega^C{}_{B\mu} = 0,$$

$$\Gamma^\lambda{}_{\mu\nu} = e_A{}^\lambda \left(\partial_\nu e^A{}_\mu + \omega^A{}_{B\nu} e^B{}_\mu \right),$$

$$\nabla_\mu e^A{}_\nu = \partial_\mu e^A{}_\nu + \omega^A{}_{B\mu} e^B{}_\nu - \Gamma^\lambda{}_{\nu\mu} e^A{}_\lambda = 0,$$

$$\omega^{AB}{}_{\mu} = -\omega^{BA}{}_{\mu},$$

$$T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu},$$

$$\overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_{\mu} g_{\sigma\nu} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu}),$$

$$K^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} - \overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} = \frac{1}{2}(T_{\mu}{}^{\rho}{}_{\nu} + T_{\nu}{}^{\rho}{}_{\mu} - T^{\rho}{}_{\mu\nu}),$$

$$f'(\overset{\circ}{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \overset{\circ}{R}) + \frac{1}{2} g_{\mu\nu}[f(T) - f'T] + f'' S_{\nu\mu\lambda} \nabla^{\lambda} T = \kappa^2 T^{m}{}_{\mu\nu},$$

connection field equation

$$\partial_\mu f_T \left[\partial_\nu (e e_{[A}{}^{\mu} e_{B]}{}^{\nu}) + 2 e e_C{}^{[\mu} e_{[A}^{\nu]} \omega^C{}_{B]\nu} \right] = 0,$$

- Martin Krssak, Emmanuel N. Saridakis, The covariant formulation of f(T) gravity, Class. Quantum Grav. 33 (2016) 115009; arXiv:1510.08432
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