One-parametric extensions of the Starobinsky inflationary model

Sergey Yu. Vernov

Skobeltsyn Institute of Nuclear Physics,
Lomonosov Moscow State University

based on

V.R. Ivanov, S.V. Ketov, E.O. Pozdeeva, S.Yu. Vernov,
JCAP **2203** (2022) 058 [arXiv:2111.09058]

*International Conference on Quantum Field Theory,
High-Energy Physics, and Cosmology,
DUBNA, JINR, 19.07.2022*

The Starobinsky R^2 inflationary model

Starobinsky model of inflation, whose action is given by

$$S_{\rm Star.}[g^J_{\mu\nu}] = \frac{M_{Pl}^2}{2}\int d^4x\sqrt{-g_J}\left(R_J + \frac{1}{6m^2}R_J^2\right) \ , \qquad (1)$$

the only parameter is the mass m.
A.A. Starobinsky, *Phys. Lett. B* **91** (1980) 99,
A.A. Starobinsky, *Phys. Lett. B* **117** (1982) 175.
The action (1) is dual to the quintessence (or scalar-tensor gravity) action

$$S_{\rm quint.}[g_{\mu\nu},\phi] = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V_{\rm Star.}(\phi)\right] \quad (2)$$

in terms of the canonical scalar ϕ and another metric $g_{\mu\nu}$ in the Einstein frame, related to $g^J_{\mu\nu}$ (in the Jordan frame) by a Weyl transformation. The induced scalar potential is given by

$$V_{\rm Star.}(\phi) = \frac{3}{4}M_{Pl}^2 m^2\left[1 - \exp\left(-\sqrt{\frac{2}{3}}\frac{\phi}{M_{Pl}}\right)\right]^2 \ . \qquad (3)$$

The main cosmological parameters of inflation are given by the scalar tilt n_s and the tensor-to-scalar ratio r, whose values are constrained by the combined Planck, WMAP and BICEP/Keck observations of CMB as

$$n_s = 0.9649 \pm 0.0042 \quad (68\%\mathrm{CL}) \qquad \text{and} \qquad r < 0.036 \quad (95\%\mathrm{CL}) \ .$$

The Starobinsky model is known as the excellent model of large-field slow-roll cosmological inflation with very good agreement to the observation data.

ϕ_i/M_{Pl}	5.2262	5.4971
n_s	0.961	0.969
r	0.0043	0.0027
N_e	49.258	62.335

The values of the inflationary parameters are sensitive to the duration of inflation and the initial value of the inflaton field, ϕ_i.

The $F(R)$ gravity action can be rewritten as

$$S_J[g^J_{\mu\nu},\sigma] = \int d^4x\sqrt{-g^J}\,[F_{,\sigma}(R_J-\sigma)+F] \ , \tag{4}$$

where the new scalar field σ has been introduced, and $F_{,\sigma}(\sigma) = \frac{dF(\sigma)}{d\sigma}$. After the Weyl transformation of the metric

$$g_{\mu\nu} = \frac{2F_{,\sigma}(\sigma)}{M^2_{Pl}} g^J_{\mu\nu} \tag{5}$$

one gets the following action in the Einstein frame:

$$S_E[g_{\mu\nu},\sigma] = \int d^4x\sqrt{-g}\left[\frac{M^2_{Pl}}{2}R - \frac{h(\sigma)}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - V\right], \tag{6}$$

where we have introduced the functions

$$h(\sigma) = \frac{3M^2_{Pl}}{2F^2_{,\sigma}}F^2_{,\sigma\sigma} \quad \text{and} \quad V(\sigma) = M^4_{Pl}\frac{F_{,\sigma}\sigma - F}{4F^2_{,\sigma}} \ . \tag{7}$$

Introducing the canonical scalar field ϕ instead of σ as

$$\phi = \sqrt{\frac{3}{2}} M_{Pl} \ln\left[\frac{2}{M_{Pl}^2} F_{,\sigma}\right] \tag{8}$$

allows one to rewrite the action S_E to the standard (quintessence or scalar-tensor) form:

$$S_E[g_{\mu\nu},\phi] = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{9}$$

The inverse transformation reads as follows:

$$R_J = \left[\frac{\sqrt{6}}{M_{Pl}}V_{,\phi} + \frac{4V}{M_{Pl}^2}\right]\exp\left(\sqrt{\frac{2}{3}}\frac{\phi}{M_{Pl}}\right) \quad , \tag{10}$$

$$F = \frac{M_{Pl}^2}{2}\left[\frac{\sqrt{6}}{M_{Pl}}V_{,\phi} + \frac{2V}{M_{Pl}^2}\right]\exp\left(2\sqrt{\frac{2}{3}}\frac{\phi}{M_{Pl}}\right) \quad , \tag{11}$$

where $V_{,\phi} = \frac{dV}{d\phi}$, defining the function $F(R_J)$ in the parametric form with the parameter ϕ.

Being motivated by the potential (3), we find useful to introduce the non-canonical dimensionless field

$$y \equiv \exp\left(-\sqrt{\frac{2}{3}}\frac{\phi}{M_{Pl}}\right) = \frac{M_{Pl}^2}{2F_{,\sigma}} > 0 \qquad (12)$$

because it is (physically) *small* during slow-roll inflation. Defining $\tilde{V}(y) = V(\phi)$ and using

$$\frac{dV}{d\phi} = -\sqrt{\frac{2}{3}}\frac{y}{M_{Pl}}\frac{d\tilde{V}}{dy} \ ,$$

we simplify Eqs. (10) and (11) as follows:

$$R_J = \frac{2}{M_{Pl}^2}\left(2\frac{\tilde{V}}{y} - \tilde{V}_{,y}\right), \qquad (13)$$

$$F = \frac{\tilde{V}}{y^2} - \frac{\tilde{V}_{,y}}{y}\ , \qquad (14)$$

respectively.

In the Starobinsky model, we have

$$\tilde{V}_{\rm Star.}(y) = V_0(1-y)^2\ , \quad \text{where} \quad V_0 = \frac{3}{4}m^2 M_{Pl}^2\ . \qquad (15)$$

In the spatially flat FLRW universe with the metric

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right) ,$$

the action (9) leads to the standard system of evolution equations:

$$6M_{Pl}^2 H^2 = \dot{\phi}^2 + 2V, \tag{16}$$

$$2M_{Pl}^2 \dot{H} = -\dot{\phi}^2, \tag{17}$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0, \tag{18}$$

where $H = \dot{a}/a$ is the Hubble parameter,
$a(t)$ is the scale factor,
and the dots denote the derivatives with respect to the cosmic time t.

In the inflationary model building, the e-foldings number

$$N_e = \ln\left(\frac{a_{\rm end}}{a}\right) \,, \qquad (19)$$

where $a_{\rm end}$ is the value of a at the end of inflation, is considered instead of the time variable. Using the relation $d/dt = -H\,d/dN_e$, one can rewrite Eq. (16) as follows:

$$Q = \frac{2V}{6M_{Pl}^2 - \chi^2} \,, \qquad (20)$$

where $Q \equiv H^2$ and $\chi = \phi' = -\dot{\phi}/H$, and the primes denote the derivatives with respect to N_e.
Equations (17) and (18) yield the dynamical system of equations:

$$Q' = \frac{1}{M_{Pl}^2} Q\chi^2 \,, \quad \phi' = \chi \,, \quad \chi' = 3\chi - \frac{1}{2M_{Pl}^2}\chi^3 - \frac{1}{Q}\frac{dV}{d\phi} \,. \qquad (21)$$

We rewrite the last equation as

$$\chi' = 3\chi - \frac{1}{2M_{Pl}^2}\chi^3 - \frac{6M_{Pl}^2 - \chi^2}{2V}\frac{dV}{d\phi} \,. \qquad (22)$$

In the Einstein frame, the slow-roll parameters are

$$\epsilon = \frac{M_{Pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 = \frac{y^2}{3}\left(\frac{\tilde{V}_{,y}}{\tilde{V}}\right)^2 ,$$

$$\eta = M_{Pl}^2\left(\frac{V_{,\phi\phi}}{V}\right) = \frac{2y}{3\tilde{V}}\left(\tilde{V}_{,y} + y\tilde{V}_{,yy}\right) .$$

The scalar spectral index n_s and the tensor-to-scalar ratio r in terms of the slow-roll parameters are given by

$$n_s = 1 - 6\epsilon + 2\eta, \qquad r = 16\epsilon . \tag{23}$$

In the slow-roll approximation, the function $\phi(N_e)$ can be found as a solution of

$$\chi \equiv \phi' \simeq \frac{M_{Pl}^2}{V} V_{,\phi} \tag{24}$$

when demanding that $\epsilon = 1$ corresponds to the end of inflation with $a = a_{\rm end}$.

Equation (24) is equivalent to

$$y' = \frac{2y^2\tilde{V}_{,y}}{3\tilde{V}} . \tag{25}$$

The $(R + R^2 + R^3)$ gravity models of inflation

To the best of our knowledge, adding the higher-order terms in R was first proposed in
J.D. Barrow and S. Cotsakis, *Phys. Lett. B* **214** (1988) 515.
A generic $(R + R^2 + R^3)$ gravity action is given by

$$S_{3-\text{gen.}} = \frac{M_{Pl}^2}{2}\int d^4x\sqrt{-g_J}\left[(1+\delta_1)R_J + \frac{(1+\delta_2)}{6m^2}R_J^2 + \frac{\delta_3}{36m^4}R_J^3\right] ,$$

where we have introduced the three dimensionless parameters δ_i.
The corresponding inflaton scalar potential (7) is given by

$$V(\sigma) = \frac{16V_0\tilde{\sigma}^2\left[3(1+\delta_2)+\delta_3\tilde{\sigma}\right]}{3\left[12(1+\delta_1)+4(1+\delta_2)\tilde{\sigma}+\delta_3\tilde{\sigma}^2\right]^2} ,$$

where the dimensionless variable $\tilde{\sigma} = \sigma/m^2$ has been introduced.
$V(0) = 0$, $V(\tilde{\sigma}) > 0$ at $\tilde{\sigma} > 0$, and V tends to zero at $\tilde{\sigma} \to +\infty$, while the potential has a maximum at some positive value of $\tilde{\sigma}$. The equation $V' = 0$ has only one positive root given by $\tilde{\sigma}_{\text{max.}} = 6\sqrt{\frac{1+\delta_1}{\delta_3}}$.

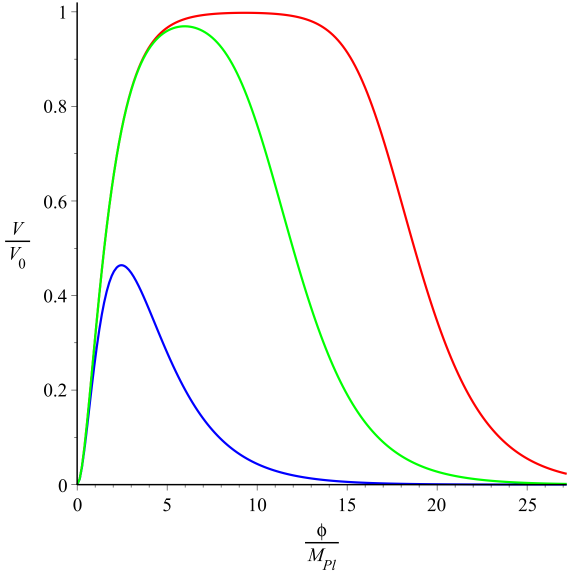

Figure: The normalized potential $V(\phi)/V_0$ with $\delta_1 = \delta_2 = 0$ for $\delta_3 = 0.000001$ (red), $\delta_3 = 0.000247$ (blue), and $\delta_3 = 1/3$ (green).

To study the impact of the R^3-term on inflation in more detail, let us consider the simplest non-trivial case with $\delta_1 = \delta_2 = 0$, in which Eq. (12) implies

$$\frac{1}{y} = 1 + \frac{1}{3}\tilde{\sigma} + \frac{\delta_3}{12}\tilde{\sigma}^2 \,. \tag{26}$$

Equation (26) is a quadratic equation on $\tilde{\sigma}$ as a function of y. The only positive root of this equation is given by

$$\tilde{\sigma} = \frac{2}{\delta_3}\left[\sqrt{1+3\delta_3\left(y^{-1}-1\right)}-1\right] = \frac{2}{\delta_3}\left[\sqrt{1+3\delta_3\left(\mathrm{e}^{\sqrt{\frac{2}{3}}\phi/M_{Pl}}-1\right)}-1\right] .$$

Using Eqs. (7) and (12), we find the scalar potential in terms of y or the inflaton field ϕ as follows:

$$\tilde{V}(y) = \frac{4V_0}{27\delta_3^2 y}\left[y + 2\sqrt{y(y+3\delta_3(1-y))}\right]\left(y - \sqrt{y(y+3\delta_3(1-y))}\right)^2 .$$

It is worth noticing that $\tilde{V}_{\mathrm{Star.}}(y)$ is reproduced in the limit $\delta_3 \to 0$.

The condition $\phi_i < \phi_{\rm max.}$ yields the additional restriction on the possible initial values of ϕ, being represented by the blue curve on the left-hand-side of Fig. 2.

The upper bound on the parameter δ_3 can be estimated by assuming the observable value of n_s to be calculated at the maximum of the potential. Then we find

$$n_s(\phi_{\rm max.}) = 1 - \frac{8\sqrt{\delta_3}\left(1+4\sqrt{\delta_3}+4\delta_3\right)}{3\left(3\sqrt{\delta_3}+1\right)\left(2\sqrt{\delta_3}+1\right)^2} \quad . \tag{27}$$

Since observations require $n_s > 0.960$, we get $\delta_3 < 0.0002467$. The dependence of n_s upon δ_3 is given on the right-hand-side of Fig. 2. Therefore, the domain of allowed values of δ_3 and ϕ is highly restricted.

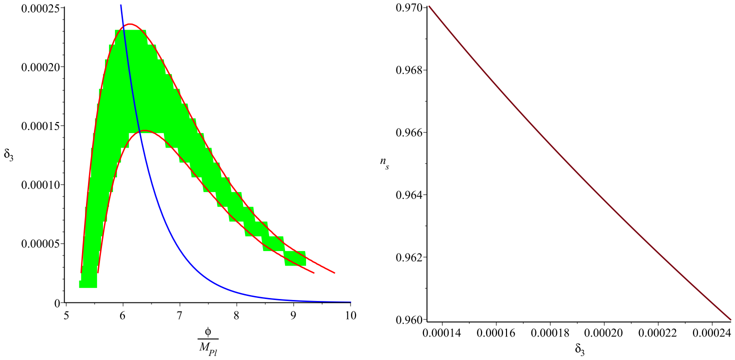

Figure: The allowed range of δ_3 and ϕ from the observational constraints (Planck): $0.961 < n_s < 0.969$ (left), and the dependence of n_s upon δ_3 (right), under the assumption that inflation started at the maximum of the potential.

Deforming the scalar potential in the Starobinsky model with analytic F-functions

The field y is *small* during slow-roll inflation. The inflaton potential (3) as a function of y is

$$V(\phi) = \tilde{V}(y) = V_0 \left[1 - 2y + y^2\right] \ , \tag{28}$$

where only the first two terms are essential for the CMB observables. The inflaton potential (3) can therefore be modified as

$$\tilde{V}(y) = V_0 \left[1 - 2y + y^2 \omega(y)\right] \ , \tag{29}$$

with arbitrary analytic function $\omega(y)$ *without* changing the CMB observables predicted by the Starobinsky model, at least for those values of ω that are not very large. The Starobinsky model appears at $\omega = 1$. The stability conditions should be satisfied with the potential (29):

$$y > 0, \qquad F_{,\sigma\sigma}(y) = \frac{M_{Pl}^2}{3m^2\left(2 + 2y^3\frac{d\omega}{dy} + y^4\frac{d^2\omega}{dy^2}\right)} > 0 \ . \tag{30}$$

Equation (13) reads

$$\tilde{\sigma} \equiv \frac{R_J}{m^2} = 3\left(\frac{1}{y} - 1 - \frac{1}{2}y^2\frac{d\omega}{dy}\right) \quad , \tag{31}$$

and Eq. (14) is given by

$$F = V_0\left(\frac{1}{y^2} - \omega - y\frac{d\omega}{dy}\right) \ . \tag{32}$$

As a check, in the Starobinsky case, $\omega = 1$ and $V = V_0(1-y)^2$, and Eq. (13) gives

$$y = \left(1 + \frac{R_J}{3m^2}\right)^{-1} \ . \tag{33}$$

Substituting it into Eq. (14), we get

$$F_{Star.}(R_J) = \frac{M_{Pl}^2}{2}\left(R_J + \frac{R_J^2}{6m^2}\right) \ , \tag{34}$$

as it should be. Moreover, when ω is an arbitrary constant, we find

$$F(R_J) = F_{Star.}(R_J) - \Lambda \ , \tag{35}$$

where $\Lambda = V_0(1-\omega)$ is a cosmological constant.

Let us consider the case of

$$\omega(y) = \omega_0 + \omega_1 y \ , \tag{36}$$

where $\omega_0 \leqslant 1$ and $\omega_1 > 0$ are constants.
The constant ω_1 should be positive for the potential V bounded from below. The inequality $\omega_0 \leqslant 1$ is needed for positivity of a cosmological constant, see Eq. (35).
Equation (31) leads to the depressed cubic equation

$$y^3 + \frac{2}{\omega_1}\left(1 + \frac{R_J}{3m^2}\right) y - \frac{2}{\omega_1} \equiv y^3 + py + q = 0 \tag{37}$$

with the negative discriminant

$$\Delta = -\left(4p^3 + 27q^2\right) = -\frac{32}{\omega_1^3}\left(1 + \frac{R_J}{3m^2}\right)^3 - \frac{108}{\omega_1^2} < 0 \ ,$$

so that it has only one real root.

The explicit $F(R)$ function in this case is as follows:

$$\frac{F}{V_0} = \frac{1}{y^2} - \omega_0 - 2\omega_1 y$$

$$= \omega_1^{\frac{2}{3}} \left[\left(1 + \sqrt{1 + \frac{8}{27\omega_1} \left(1 + \frac{R_J}{3m^2} \right)^3} \right)^{\frac{1}{3}} + \left(1 - \sqrt{1 + \frac{8}{27\omega_1} \left(1 + \frac{R_J}{3m^2} \right)^3} \right)^{\frac{1}{3}} \right]^{-2}$$

$$- 2\omega_1^{\frac{2}{3}} \left[\left(1 + \sqrt{1 + \frac{8}{27\omega_1} \left(1 + \frac{R_J}{3m^2} \right)^3} \right)^{\frac{1}{3}} + \left(1 - \sqrt{1 + \frac{8}{27\omega_1} \left(1 + \frac{R_J}{3m^2} \right)^3} \right)^{\frac{1}{3}} \right]$$

$$- \omega_0 \ ,$$

where

$$\omega_0 = \omega_1^{2/3} \left[\left(1 + \sqrt{1 + \frac{8}{27\omega_1}} \right)^{1/3} + \left(1 - \sqrt{1 + \frac{8}{27\omega_1}} \right)^{1/3} \right]^{-2}$$

$$- 2\omega_1^{2/3} \left[\left(1 + \sqrt{1 + \frac{8}{27\omega_1}} \right)^{1/3} + \left(1 - \sqrt{1 + \frac{8}{27\omega_1}} \right)^{1/3} \right] .$$

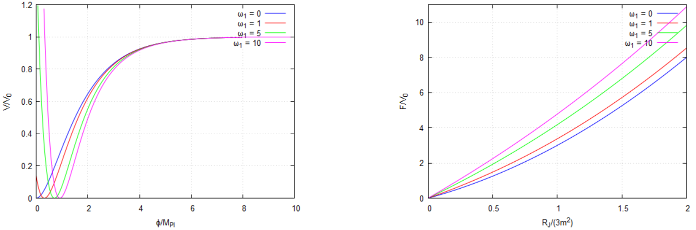

Figure: The scalar potential of the canonical inflaton field ϕ (left) and the related $F(R)$-function (right) in the case of the deformation of the Starobinsky model for some values of the parameter ω_1: 0, 1, 5, and 10.

From Eq. (30) we get a simple formula for the second derivative $F_{,\sigma\sigma}$ as

$$F_{,\sigma\sigma}(y) = \frac{M_{Pl}^2}{6m^2\left(1+\omega_1 y^3\right)} \quad , \tag{38}$$

so that the considering $F(R)$ gravity model satisfies the stability conditions at $\omega_1 \geqslant 0$.

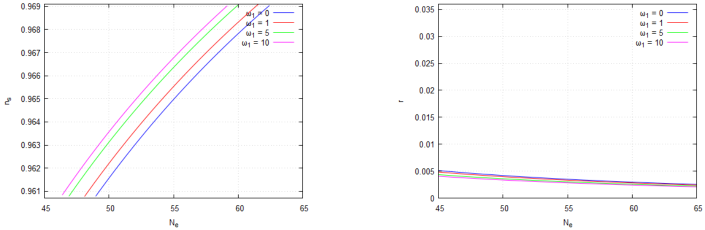

Figure: The index n_s (left) of scalar perturbations and the tensor-to-scalar ratio r (right) as the functions of e-folds N_e in the case I of the deformation of the Starobinsky model for some values of the parameter ω_1: 0, 1, 5, and 10.

The $(R + R^{3/2} + R^2)$ gravity model of inflation

Let

$$F(R_J) = \frac{M_{Pl}^2}{2}\left[R_J + \frac{1}{6m^2}R_J^2 + \frac{\delta}{m}R_J^{3/2}\right] , \qquad (39)$$

where we have introduced the dimensionless parameter δ.
The $R^{3/2}$ term appears in the (chiral) modified supergravity
S.V. Ketov and A.A. Starobinsky, Phys. Rev. D 83 (2011) 063512 [1011.0240].
S.V. Ketov and S. Tsujikawa, Phys. Rev. D 86 (2012) 023529 [1205.2918].
The $R^{3/2}$-term in $F(R)$ gravity arises in an approximate description of the Higgs field with a small cubic term in its scalar potential and a large non-minimal coupling to R
J.S. Martins, O.F. Piattella, I.L. Shapiro and A.A. Starobinsky, 2010.14639.

Given $\tilde{\sigma} > 0$, we find

$$F_{,R_J} = \frac{M_{Pl}^2}{6}\left(\sqrt{R_J} + \frac{9\delta}{4}\right)^2 - \frac{3}{16}\left(27\delta^2 - 16\right) > 0 \qquad (40)$$

when $\delta > -4\sqrt{3}/9$, and

$$F_{,R_J R_J} = \frac{M_{Pl}^2}{24m^2}\left(4 + \frac{9\delta}{\sqrt{R_J}}\right) > 0 \qquad (41)$$

only when $\delta > 0$.
Hence, the condition $\delta > 0$ is necessary to get a stable $F(R)$ gravity model for all $R_J > 0$.

The corresponding scalar potential (7) is given by

$$V = \frac{4V_0\tilde{\sigma}(3\delta\sqrt{\tilde{\sigma}} + \tilde{\sigma})}{(6 + 9\delta\sqrt{\tilde{\sigma}} + 2\tilde{\sigma})^2} \quad . \tag{42}$$

Equation (12) in this case is a quadratic equation on $\sqrt{\tilde{\sigma}}$, and its only real solution is

$$\tilde{\sigma} = \frac{3(1-y)}{y} + \frac{9\delta}{8y}\left[9\delta\, y - \sqrt{3y(27\delta^2 y - 16y + 16)}\right] \quad . \tag{43}$$

The potential can be rewritten as

$$\begin{aligned} \tilde{V} &= \frac{V_0}{2304y^2}\,(s + 3\delta\, y)\,(s - 9\delta\, y)^3 \\ &= \frac{243V_0\delta^4 y^2}{256}\left(3\sqrt{1 + \frac{16(1-y)}{27\delta^2 y}} + 1\right)\left(\sqrt{1 + \frac{16(1-y)}{27\delta^2 y}} - 1\right)^3 \quad , \end{aligned}$$

where we have introduced $s = \sqrt{3y(27\delta^2 y - 16y + 16)}$.

When $\delta = 4\sqrt{3}/9$, the $F_{,\sigma}$ function is a perfect square, and the potential simplifies as

$$\tilde{V}_{\rm special}(y) = \frac{V_0}{3}\,(3+\sqrt{y})\,(1-\sqrt{y})^3\;, \tag{44}$$

or

$$V_{\rm special}(\phi) = \frac{V_0}{3}\left({\rm e}^{\phi/(\sqrt{6}M_{Pl})}-1\right)^3\left(1+3{\rm e}^{\phi/(\sqrt{6}M_{Pl})}\right){\rm e}^{-2\sqrt{2/3}\,\phi/M_{Pl}}\;.$$

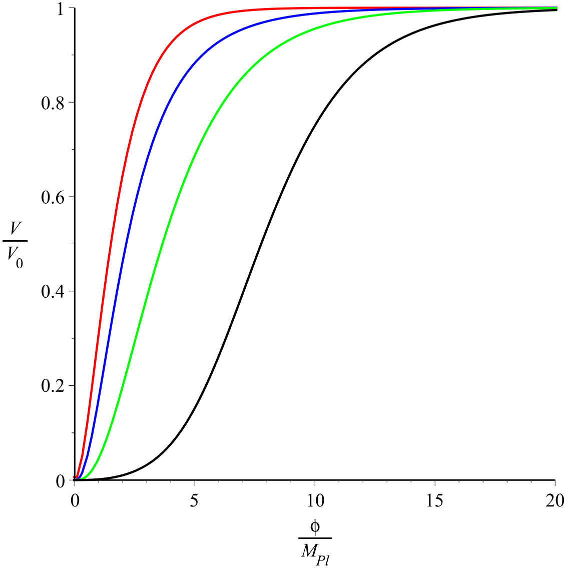

Figure: The potential $V(\phi)$ for $\delta = 0$ (red), $\delta = 1/5$ (blue), $\delta = 4\sqrt{3}/9$ (green), and $\delta = 5$ (black).

The inflationary parameters

The inflationary parameters are given by

$$n_s = 1 + \frac{8y\left(3s(3\delta(9\delta^2-16)s+720\delta^2-256)y^2 - s^3\delta(39\delta y+s)\right)}{(-9\,\delta\,y+s)^2\,(3\,\delta\,y+s)^2\,s} - \frac{8y\left[72\delta(4-9\delta^2)(27\delta^2-16)y^4 + [(768-1215\delta^4-432\delta^2)s - 144\delta(45\delta^2-16)]y^3\right]}{(-9\,\delta\,y+s)^2\,(3\,\delta\,y+s)^2\,s} \tag{45}$$

and

$$r = \frac{768\,y^2\left(-9\,\delta^2 y + s\delta + 8\,y\right)^2}{(-9\,\delta\,y+s)^2\,(3\,\delta\,y+s)^2} \quad . \tag{46}$$

The amplitude of scalar perturbations is given by

$$A_s = \frac{(-9\,\delta\,y+s)^5\,(3\,\delta\,y+s)^3\,m^2}{3538944y^4\pi^2\left(-9\,\delta^2 y + s\delta + 8\,y\right)^2} \quad . \tag{47}$$

The observed value of A_s determines the value of the parameter m.

The slow-roll evolution equation allows us to relate N_e with y at the end of inflation,

$$N_e = \left(\frac{9}{8} - \delta^{-2}\right) \ln\left[9\delta^3 \left(9\delta\, y - s\right) + 24\left(1 - 4y\right)\delta^2 + 8\left(\delta s + 4y\right)\right]$$
$$+ \left(\delta^{-2} - \frac{3}{8}\right) \ln y + \frac{s}{4\delta\, y} - N_0 \ ,$$

where the integration constant N_0 is fixed by the condition $N_e(y_{end}) = 0$. The analytic formula for $N_0(\delta)$ is obtained by substituting $N_e = 0$ and $y = y_{end}$.
The condition $\epsilon = 1$ gives

$$y_{end} = \frac{3(4 - 3\delta^2 + \sqrt{3}\delta^2) - \sqrt{9(4 - 3\delta^2 + \sqrt{3}\delta^2)^2 - 72(2 - 3\delta^2)}}{2(2 - 3\delta^2)(3 + 2\sqrt{3})} \ .$$

It is worth noticing that this solution has no singularity at $\delta = \sqrt{2/3}$, while $y_{end}(\delta)$ is a smooth monotonically decreasing function.

The slow-roll parameters ϵ and η remain finite in the limit $\delta \to +\infty$ at fixed y,

$$\epsilon_\infty(y) = \frac{(2y+1)^2}{3(1-y)^2} \quad , \quad \eta_\infty(y) = \frac{2(4y^2+y+1)}{3(1-y)^2} \ . \tag{48}$$

Since the value of y at the end of inflation is determined by the condition $\epsilon(y_{end}) = 1$, y_{end} also approaches a finite limit as $\delta \to +\infty$, which is given by a solution to the equation

$$\epsilon_\infty = \frac{(2y+1)^2}{3(1-y)^2} = 1. \tag{49}$$

This equation has only one positive solution $y_{end}|_{\delta\to+\infty} = 3\sqrt{3} - 5 \approx 0.196$.

When $\delta = 4\sqrt{3}/9$, the function $F_{,\sigma}$ simplifies as

$$F_{,\sigma} = \frac{M_{Pl}^2}{6}\left(\sqrt{\tilde{\sigma}} + \sqrt{3}\right)^2 .$$

Accordingly, the slow-roll parameters are also simplified as

$$\epsilon = \frac{4y^2\left(2\sqrt{y}+y\right)^2}{3\left(\sqrt{y}-y\right)^2\left(3\sqrt{y}+y\right)^2} \tag{50}$$

and

$$\eta = \frac{4y^2\left(2\sqrt{y}+2y-1\right)}{3\left(\sqrt{y}-y\right)^2\left(3\sqrt{y}+y\right)} . \tag{51}$$

In this special case we find

$$n_s = 1 - \frac{8y^2\left(7y+4y\sqrt{y}+y^2+3\sqrt{y}\right)}{3\left(\sqrt{y}-y\right)^2\left(3\sqrt{y}+y\right)^2} \tag{52}$$

and

$$r = \frac{64y^2\left(2\sqrt{y}+y\right)^2}{3\left(\sqrt{y}-y\right)^2\left(3\sqrt{y}+y\right)^2} . \tag{53}$$

The inflationary parameters in the special case are also given in Table 1.

Table: The values of y, N_e and r corresponding to $n_s = 0.961$ and $n_s = 0.969$, respectively, and the values of y_{end} for some values of the parameter δ.

δ	y_{end}	$y_{in, n_s=0.961}$	$y_{in, n_s=0.969}$	$N_{e,0.961}$	$N_{e,0.969}$	$r_{n_s=0.961}$	$r_{n_s=0.969}$
0	0.464	0.0140	0.0112	49.3	62.3	0.0043	0.0027
0.2	0.395	0.00682	0.00505	45.0	56.8	0.0096	0.0065
$\frac{4\sqrt{3}}{9}$	0.299	0.00146	0.000968	48.1	60.9	0.0152	0.0099
1	0.279	0.000939	0.000616	49.4	62.4	0.0157	0.0102
5	0.205	$4.32 \cdot 10^{-5}$	$2.75 \cdot 10^{-5}$	56.3	69.7	0.0168	0.0108
10	0.199	$1.08 \cdot 10^{-5}$	$6.91 \cdot 10^{-6}$	58.7	72.0	0.0168	0.0108
25	0.197	$1.74 \cdot 10^{-6}$	$1.11 \cdot 10^{-6}$	61.4	74.8	0.0169	0.0108
50	0.196	$4.34 \cdot 10^{-7}$	$2.77 \cdot 10^{-7}$	63.5	76.9	0.0169	0.0108
100	0.196	$1.09 \cdot 10^{-7}$	$6.92 \cdot 10^{-8}$	65.5	79.1	0.0169	0.0108

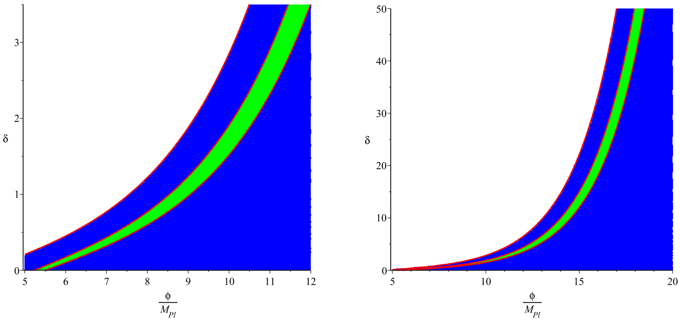

Figure: The inflaton field values against the values of the parameter δ. The green area corresponds to the observational restrictions on n_s and r. The blue area is defined by the restrictions on r only. When $\delta > 5$, the allowed domain is restricted by the lines $\delta\, y^2 = const$.

CONCLUSIONS

• We studied several extensions of the Starobinsky inflation model of the $(R + R^2)$ gravity in the context of $F(R)$ gravity and scalar-tensor gravity.

• By deforming the scalar potential of the Starobinsky model, and derived the corresponding F-function in the analytic form in a new model with a single parameter, and found the lower and upper bounds on the values of the parameters. The new model is very close to the original Starobinsky model of inflation, as regards their inflationary parameters. However, unlike the Starobinsky model, the inflaton (scalaron) acquires a non-vanishing vacuum expectation value in those models.

• The modification of the Starobinsky model by the $R^{3/2}$ term does not lead to significant constraints on its coefficient in slow-roll inflation, at least for $0 < \delta < 100$.

The $R^{3/2}$ term has a significant impact on the value of the tensor-to-scalar ratio r.

CONCLUSIONS

• We studied several extensions of the Starobinsky inflation model of the $(R + R^2)$ gravity in the context of $F(R)$ gravity and scalar-tensor gravity.

• By deforming the scalar potential of the Starobinsky model, and derived the corresponding F-function in the analytic form in a new model with a single parameter, and found the lower and upper bounds on the values of the parameters. The new model is very close to the original Starobinsky model of inflation, as regards their inflationary parameters. However, unlike the Starobinsky model, the inflaton (scalaron) acquires a non-vanishing vacuum expectation value in those models.

• The modification of the Starobinsky model by the $R^{3/2}$ term does not lead to significant constraints on its coefficient in slow-roll inflation, at least for $0 < \delta < 100$.

The $R^{3/2}$ term has a significant impact on the value of the tensor-to-scalar ratio r.

Thank for your attention