Classical and quantum chaos in nonlinear vector mechanics

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Classical and quantum chaos

- **Classical chaos** is closely related to the exponential sensitivity to initial conditions ("butterfly effect"):

$$\|\delta \mathbf{z}(t)\| \sim e^{\kappa_{cl} t} \|\delta \mathbf{z}(0)\|,$$

  where κ_{cl} is called the classical Lyapunov exponent
- **Quantum chaos** and quantum Lyapunov exponent are more subtle because there are no trajectories in quantum world
- Due to this reason, we need to find **alternative signatures of chaos** that are well defined in the quantum case and distinct chaotic and integrable systems in the semiclassical limit

OTOCs

- One of such signatures, which has recently grown popular, is the exponential growth of the **out-of-time-ordered correlation functions (OTOCs)**:

$$C(t) = \frac{1}{N^2} \sum_{i,j=1}^{N} \left\langle [\hat{q}_i(t), \hat{p}_j(0)]^\dagger [\hat{q}_i(t), \hat{p}_j(0)] \right\rangle$$

- In the semiclassical liimt, OTOCs capture the "butterfly effect":

$$C(t) \approx \frac{1}{N^2} \sum_{i,j=1}^{N} \{q_i(t), p_j(0)\}^2 = \frac{\hbar^2}{N^2} \sum_{i,j=1}^{N} \left| \frac{\partial q_i(t)}{\partial q_j(0)} \right|^2 \sim \hbar^2 \frac{\|\mathbf{z}(t)\|^2}{\|\mathbf{z}(0)\|^2} \sim \hbar^2 e^{2\kappa t}$$

- OTOCs allow us to define the **quantum Lyapunov exponent**:

$$\kappa_q \approx \frac{1}{2t} \log \left[\frac{1}{\hbar^2} \frac{1}{N^2} \sum_{i,j} C_{ij}(t) \right] \quad \text{as} \quad \frac{1}{\kappa_q} \ll t \ll \frac{1}{\kappa_q} \log \frac{1}{\hbar}$$

- Note that eventually OTOCs are saturated, which reflects the breakdown of the semsiclassical description (cf. the Ehrenfest time)

Correspondence

- Unfortunately, the correspondence between the classical and quantum chaos remains **relatively poorly studied**
- Therefore, it is useful to consider a tractable model, where this **correspondence can be checked directly**
- As an example of such a model, we propose the **vector mechanics** with a large number of degrees of freedom N and quartic interaction:

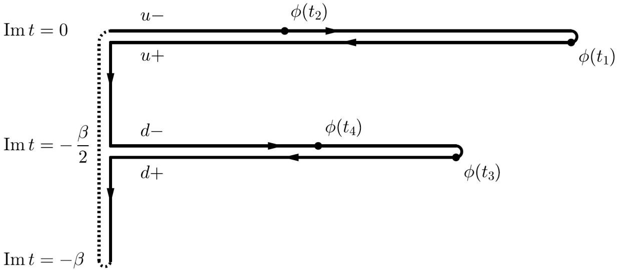
$$S = \int dt \Big[\sum_{i=1}^{N} \left(\frac{1}{2}\dot{\phi}_i^2 - \frac{m^2}{2}\phi_i^2 \right) - \underbrace{\frac{\lambda}{4N} \sum_{i,j=1}^{N} \phi_i^2 \phi_j^2}_{\text{symmetric}} + \underbrace{\alpha^2 \frac{\lambda}{4N} \sum_{i=1}^{N} \phi_i^4}_{\text{nonsymmetric}} \Big]$$

- We also assume the system to be thermal with an inverse temperature β
- We will show that the symmetric model ($\alpha = 0$) is both classically and quantum integrable, whereas the nonsymmetric model ($\alpha \neq 0$) is chaotic

Augmented Schwinger-Keldysh technique

To calculate the regularized OTOCs, we use the augmented Schwinger-Keldysh technique on the **twofold contour** (note that in our notation $C(t) = C_{tt;00}$):

$$\begin{aligned} C_{12;34} = &- \langle \phi_{u+}(t_1)\phi_{u-}(t_3)\phi_{d+}(t_2)\phi_{d-}(t_4)\rangle - \langle \phi_{u-}(t_1)\phi_{u+}(t_3)\phi_{d-}(t_2)\phi_{d+}(t_4)\rangle \\ &+ \langle \phi_{u+}(t_1)\phi_{u-}(t_3)\phi_{d-}(t_2)\phi_{d+}(t_4)\rangle + \langle \phi_{u-}(t_1)\phi_{u+}(t_3)\phi_{d+}(t_2)\phi_{d-}(t_4)\rangle \\ = &- \langle \phi_{uc}(t_1)\phi_{dc}(t_2)\phi_{uq}(t_3)\phi_{dq}(t_4)\rangle . \end{aligned}$$

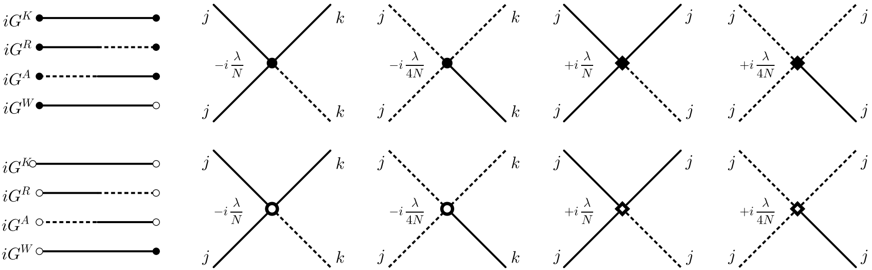


Augmented Schwinger-Keldysh technique

- The **vertices are the same** as in the standard (onefold) technique
- In addition to the standard R/A/K propagators, the augmented technique contains the **W propagator that connects different folds**:

$$iG_0^R(t_1,t_2) = -i\theta(t_{12})\,\frac{\sin(mt_{12})}{m}, \qquad iG_0^A(t_1,t_2) = i\theta(-t_{12})\,\frac{\sin(mt_{12})}{m},$$

$$iG_0^K(t_1,t_2) = \frac{1}{2}\coth\frac{\beta m}{2}\,\frac{\cos(mt_{12})}{m}, \qquad iG_0^W(t_1,t_2) = \frac{e^{\beta m/2}}{e^{\beta m}-1}\,\frac{\cos(mt_{12})}{m}.$$



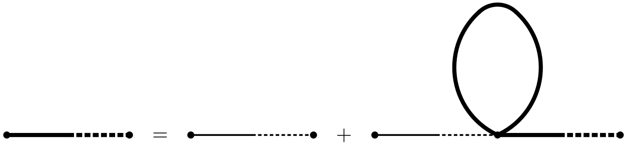
Leading corrections to propagators

- In the leading order in $1/N$, correlation functions in the full ($\alpha \neq 0$) and $O(N)$-symmetric ($\alpha = 0$) models **coincide**
- In this order, loop corrections to propagators result in a simple **mass shift**:

$$\frac{\tilde{m}^2}{m^2} = 1 + \frac{\lambda}{2m^3} \frac{m}{\tilde{m}} \coth\left(\frac{\beta m}{2} \frac{\tilde{m}}{m}\right)$$

- In the high-temperature limit, this transcendental equation is easily solved:

$$\tilde{m} \approx \sqrt[4]{\lambda/\beta}, \quad \text{as} \quad \beta m \ll 1 \quad \text{and} \quad \beta m \ll \lambda/m^3$$



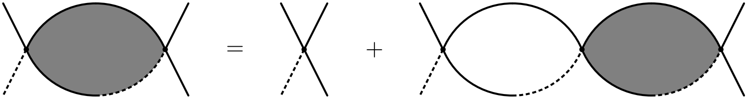
Leading corrections to vertices

In the leading order in $1/N$, loop corrections to vertices are also easily summed:

$$B(t_1, t_2) = \delta(t_1 - t_2) + 2i\lambda \int_{t_0}^{\infty} dt_3 \, G^R(t_1, t_3) G^K(t_1, t_3) B(t_3, t_2)$$
$$= \delta(t_{12}) - \nu \, \tilde{m} \, \theta\left(t_{12}\right) \sin\left(\mu \tilde{m} t_{12}\right),$$

where (the approximate identities hold in the high-temperature limit)

$$\mu^2 = 6 - 2\,\frac{m^2}{\tilde{m}^2} \approx 6,$$
$$\nu = \mu - \frac{4}{\mu} \approx \sqrt{\frac{2}{3}}$$



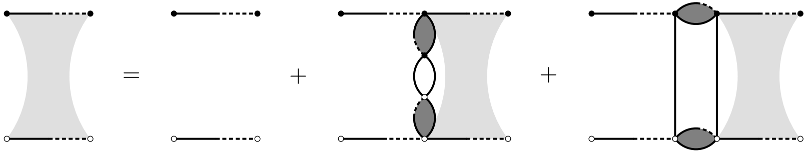
No chaos in the symmetric model

- The leading corrections to the averaged OTOC in the $O(N)$-symmetric model are described by the so-called **"ladder" diagrams**:
- Substituting the ansatz $C_{12;34} \sim e^{2\kappa t}$, where $t = \frac{1}{2}(t_1 + t_2 - t_3 - t_4)$, into the Bethe-Salpeter equation, we get the equation on κ:

$$1 \approx \frac{64}{N}\frac{w^2\lambda^2}{\tilde{m}^6}\frac{1}{\mu^4}\frac{1}{\left(1+\frac{\kappa^2}{\tilde{m}^2}\right)^2} + \frac{4}{N}\frac{w^2\lambda^2}{\tilde{m}^6}\frac{5+\frac{\kappa^2}{\tilde{m}^2}}{\left((\mu+1)^2+\frac{\kappa^2}{\tilde{m}^2}\right)\left((\mu-1)^2+\frac{\kappa^2}{\tilde{m}^2}\right)\left(1+\frac{\kappa^2}{\tilde{m}^2}\right)},$$

where $w = e^{\beta\tilde{m}/2} / \left(e^{\beta\tilde{m}} - 1\right)$

- All solutions to this equation are purely inaginary; hence, there is **no quantum chaos in the $O(N)$-symmetric model**



Chaos in the full nonsymmetric model

- Keeping in mind the leading contributions from nonsymmetric vertices and using the same ansatz for $C_{12;34}$, we get the equation on κ in the full model:

$$\frac{1}{\alpha^2} \approx -\frac{1536}{N^2}\frac{w^2\lambda^2}{\tilde{m}^6}\frac{1}{\mu^6}\frac{1}{\left(1+\frac{\kappa^2}{\tilde{m}^2}\right)^2} - \frac{24}{N^2}\frac{w^2\lambda^2}{\tilde{m}^6}\frac{\left(5+\frac{\kappa^2}{\tilde{m}^2}\right)\left(3\mu^2-3+(\mu^2+6)\kappa^2+\kappa^4\right)}{\left(1+\frac{\kappa^2}{\tilde{m}^2}\right)\left((\mu+1)^2+\frac{\kappa^2}{\tilde{m}^2}\right)^2\left((\mu-1)^2+\frac{\kappa^2}{\tilde{m}^2}\right)^2}$$

- The solutions to this equation **has a positive real part**
- So, the maximal quantum Lyapunov exponent is as follows:

$$\kappa_q = \alpha\frac{8\sqrt{6}}{\mu^3}\frac{w\lambda}{\tilde{m}^3}\frac{\tilde{m}}{N}$$

- The exponent scales as $\kappa_q \sim \sqrt[4]{\lambda/\beta}$ in the high-temperature limit and is exponentially suppressed in the low-temperature limit:

$$\kappa_q^{\text{high}} \approx \frac{4}{3}\frac{\alpha}{N}\sqrt[4]{\frac{\lambda}{\beta}}, \qquad \kappa_q^{\text{low}} \approx \sqrt{6}\frac{\alpha}{N}\frac{\lambda}{m^3} m \exp\left(-\frac{\beta m}{2}\right)$$

Classical chaos

- The $O(N)$-symmetric model has exactly N independent conserved quantities, so it is **classically integrable**
- Let us show that the **full model has a positive Lyapunov exponent**
- To do this, we **numerically solve** the following system of differential equations with fixed N and total energy E, but arbitrary initial conditions:

$$\dot{z}_I = \pi_{IJ} \frac{\partial H}{\partial z_J},$$

$$\dot{\Phi}_{IJ} = \pi_{IK} \frac{\partial^2 H}{\partial z_K \partial z_L} \Phi_{LJ},$$

where $z_I = (x_i, p_i)$ and $\pi = \begin{pmatrix} 0 & \mathbb{1}_{N\times N} \\ -\mathbb{1}_{N\times N} & 0 \end{pmatrix}$

- Then we extract the classical Lyapunov exponent from the maximal singular value σ_{max} of Φ:

$$\kappa_{cl} = \lim_{t\to\infty} \frac{1}{t} \log \sigma_{\mathsf{max}}(t),$$

repeat this calculation a 1000 times and average κ over all initial conditions

Classical chaos in the nonsymmetric model

- Calculating the average classical Lyapunov exponent for different N and E and **fitting these points** with a line, we establish the approximate behavior:

$$\bar{\kappa}_{cl} \approx (1.3 \pm 0.2) \frac{1}{N^{1.18 \pm 0.05}} \left(\frac{\lambda}{\beta}\right)^{0.28 \pm 0.02},$$

where we assume $\alpha = 1$ and use the relation $\beta \sim N/E$

- Note that the average exponent is slightly less than the maximal one, although their qualitative behaviors are the same

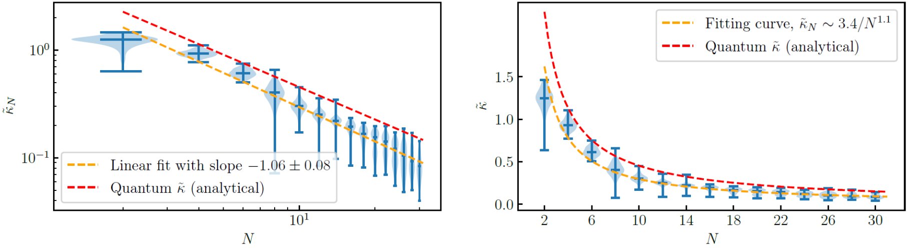


Figure: The comparison of $\bar{\kappa}_{cl}$ and κ_q for $\tilde{\beta} = 0.01$

Qualitative analysis

- In fact, the high-temperature behavior of classical and quantum Lyapunov exponents can be **deduced from dimensional grounds**
- In the limit $\beta m \ll 1$ and $\beta m \ll \lambda/m^3$, the quadratic part of the potential energy is negligible, so the Hamiltonian acquires the following form ($\alpha = 1$):

$$H^{\text{high}} \approx \sum_{i=1}^{N} \frac{1}{2}\pi_i^2 + \frac{\lambda}{4N} \sum_{i \neq j} \phi_i^2 \phi_j^2$$

- This "pruned" Hamiltonian is **invariant under the scale transformations**:

$$t \to \gamma^{-1} t, \qquad \phi_i \to \gamma \phi_i, \qquad H \to \gamma^4 H$$

- Since the Lyapunov exponent has the dimension of inverse time, this invariance implies the high-temperature dependence $\kappa \sim \sqrt[4]{E} \sim \sqrt[4]{\lambda/\beta}$

Analogy to billiards

- Furthermore, we can compare the **constant potential energy surface** (CPE surface) with a wall of a **Sinai billiard**
- It is known that Sinai billiards exhibit a chaotic behavior in the presence of **concave walls**
- In the nonsymmetric model, the CPE surface becomes concave at energies $E > E_{\text{con}} = 3Nm^4/2\lambda$, which **agrees with the emergence of chaos**

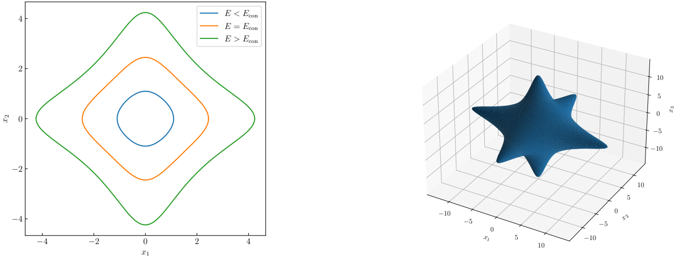


Figure: [Left] CPE curve for $N = 2$ and $E < E_{\text{con}}$ (blue line), $E = E_{\text{con}}$ (orange line), $E > E_{\text{con}}$ (green line). [Right] CPE surface for $N = 3$ and $E \gg E_{\text{con}}$.

Conclusion

- We suggest a tractable chaotic model — the nonlinear vector mechanics with a quartic interaction and thermal initial state
- In the $O(N)$-symmetric case, both classical and quantum Lyapunov exponents are zero
- In the nonsymmetric case, both exponents emerge in the high-temperature limit, approximately coincide, and scale as $\kappa_q \sim \kappa_{cl} \sim \frac{1}{N} \sqrt[4]{\lambda/\beta}$
- This calculation supports the use of OTOCs as a diagnostic of quantum chaos

Further research

Possible future directions:

- Nonthermal initial states — e.g., study the relationship between the scrambling and delocalization of coherent states
- Other diagnostics of quantum chaos — e.g., Krylov complexity
- Nonstationary and dissipative generalizations