Large charge expansion meets epsilon expansion at six loops

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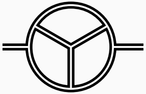
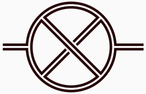
Looking for QFT simplifications

- Expansion in $1/N_c$, non-planar diagrams suppressed

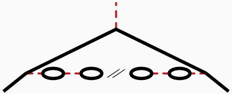


$\sim (g^2 N_c)^3$

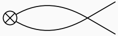


$\sim \frac{(g^2 N_c)^3}{N_c^2}$

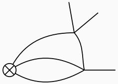
- Bubble chains contributions dominate in $n_f \to \infty$ limit



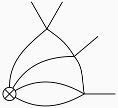
- Large charge expansion in powers of $1/Q$ for diagrams with ϕ^Q operator insertion



$\sim Q(\lambda Q)$



$\sim Q(\lambda Q)^2$



$\sim Q(\lambda Q)^3$

Available predictions

- Large charge expansion, exact in $(g \cdot Q)$ 't Hooft coupling
  - $O(2)$ model, leading Δ_{-1} and subleading Δ_0 [Badel,Cuomo,Monin,Rattazzi'19]

$$\Delta_{\phi^n} = \sum_{k=-1}^{\infty} \lambda^k \Delta_k(\lambda n)$$

  - $O(N)$ model, leading Δ_{-1} and subleading Δ_0 [Antipin,Bersini,Sannino,Wang,Zhang'20]

$$\Delta_Q(N) = \sum_{k=-1}^{\infty} \lambda^k \Delta_k(g \cdot Q, N)$$

- Critical dimension of operator ϕ^Q in $1/N$ expansion [Derkachov,Manashov'98]

$$\Delta_Q(\varepsilon) = \frac{u_1(Q,\varepsilon)}{N} + \frac{u_2(Q,\varepsilon)}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right)$$

- ε-expansion results
  - Four-loop calculation of the leading $Q \to \infty$ part [Jack, Jones'21]
  - Direct five-loop calculation of $\gamma_1 \ldots \gamma_6$ and fit for γ_Q [Jin,Li'22]

Specifying the model

- We consider $O(N)$ symmetric ϕ^4 theory, with N component field ϕ_i, $i = 1 \dots N$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_i \partial^\mu \phi_i + \frac{g}{4!}(\vec{\phi}\cdot\vec{\phi})^2$$

- Anomalous dimensions of the operator $O_{i_1 \dots i_Q}$, which is traceless fully symmetric with Q indices

$$O_{i_1 i_2} \equiv \phi_{i_1}\phi_{i_2} - \frac{1}{N}\delta_{i_1 i_2}\phi^2, \quad O_{i_1 i_2 i_3} \equiv \phi_{i_1}\phi_{i_2}\phi_{i_3} - \frac{1}{N+2}\phi^2(\phi_{i_1}\delta_{i_2 i_3} + \phi_{i_2}\delta_{i_1 i_3} + \phi_{i_3}\delta_{i_1 i_2}), \quad \dots$$

- Six-loop $O(N)$ theory renormalization
  - $Q = 1$ known from γ_ϕ [Kompaniets,Panzer'17]
  - $Q = 2$ known from crossover exponent [Kompaniets,Wiese'19]
- Six-loop beta-functions for general scalar theory [Bednyakov,Pikelner'21]
  - "Dummy fields" method provides anomalous dimensions for the $Q = 3$ and $Q = 4$ cases

General form of the result

- Anomalous dimensions of operators ϕ^Q are polynomial in g, Q, N

$$\gamma_Q = Q \sum_{l=1}^{\infty} g^l \sum_{r=0}^{l} Q^r \sum_{s=0}^{l-1} N^s \gamma_{l,r,s}$$

- Critical dimension, due to specific form of the $O(N)$ fixed point $g^*, \beta(g^*) = 0$

$$\Delta_Q = Q(1-\varepsilon) + \gamma_Q(g^*) = Q(1-\varepsilon) + \sum_{l=1}^{\infty} \varepsilon^l \sum_{k=1}^{2l-1} \frac{P_k(Q)}{[N+8]^k} = Q(1-\varepsilon) + \sum_{l=1}^{\infty} \varepsilon^l \sum_{k=1}^{l+1} Q^k f_k(N)$$

- At L-loop order we need $L+1$ independent predictions to fix all $f_k(N)$

Fixed Q results

- For $Q = 1, 2, 3, 4$ anomalous dimensions derived from general scalar theory six-loop result, e.g. $\lambda_{abcd} \rightarrow \frac{g}{3}(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + d_{abcd},\ d_{abca} = 0$ [Bednyakov,Pikelner'21]
- With additional two terms of $1/Q$ expansion enough to fix general Q result at five loops
- At six loops need more input $\rightarrow$ **six-loop calculation of $\gamma_{Q=5}$**
- Calculated from renormalization constant for five-point function with $O_{i_1 i_2 i_3 i_4 i_5}$ insertion
- Z_5 from $\mathscr{K}\mathscr{R}'$ applied to individual diagrams - possible generalisation beyond $O(N)$

$$Z_\phi^5 Z_5 = 1 - \sum_i \mathscr{K}\mathscr{R}' G_i$$

- Anomalous dimension checked to be free of poles

$$\gamma_{Q=5} = -\frac{\partial \log Z_5}{\partial \log \mu} = -\beta \frac{\partial \log Z_5}{\partial g}$$

Calculation details

- We use $\mathcal{KR}'$ operation as the main tool to extract each diagram contribution to Z_Q

$$\mathcal{KR}'G = \mathcal{K}G + \sum_{\{\gamma\}} \mathcal{K}\Big[\prod_{\gamma_i \in \{\gamma\}} (-\mathcal{KR}'\gamma_i) * G/\{\gamma\}\Big]$$

  - G is origanal diagram, $\mathcal{K}G$ its $\mathcal{O}(1/\varepsilon)$ part
  - γ_i are UV subgraphs, $\mathcal{KR}'\gamma_i$ is known from lower-loop calculations
  - $G/\{\gamma\}$ is a co-graph after shrinking all γ_i, with the same momenta routing as G

- Results for $\mathcal{KR}'\gamma_i$ calculated in the bottom-up way from lower-loop order

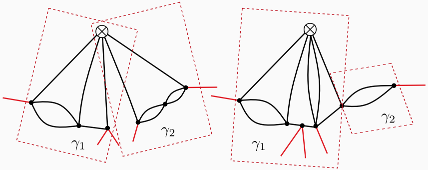
- All integrals,but one entering $\mathcal{K}G$ and $G/\{\gamma\}$ are calculated with IR safe non-exceptional external momentum routing with `HyperlogProcedures` [Schnetz'2022]

- UV subgraph identification for $\mathcal{KR}'$ operation implemented in private `C++` code

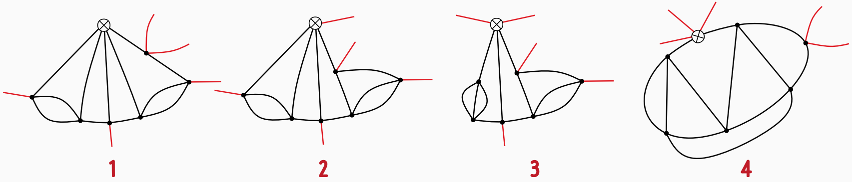
- Single diagram with exceptional external momentum routing calculated with $\mathcal{KR}^*$ operation

Classification of diagrams

- Factorizable, loop order reduced $\mathcal{KR}'(\Gamma) = \mathcal{KR}'(\gamma_1) \cdot \mathcal{KR}'(\gamma_2)$

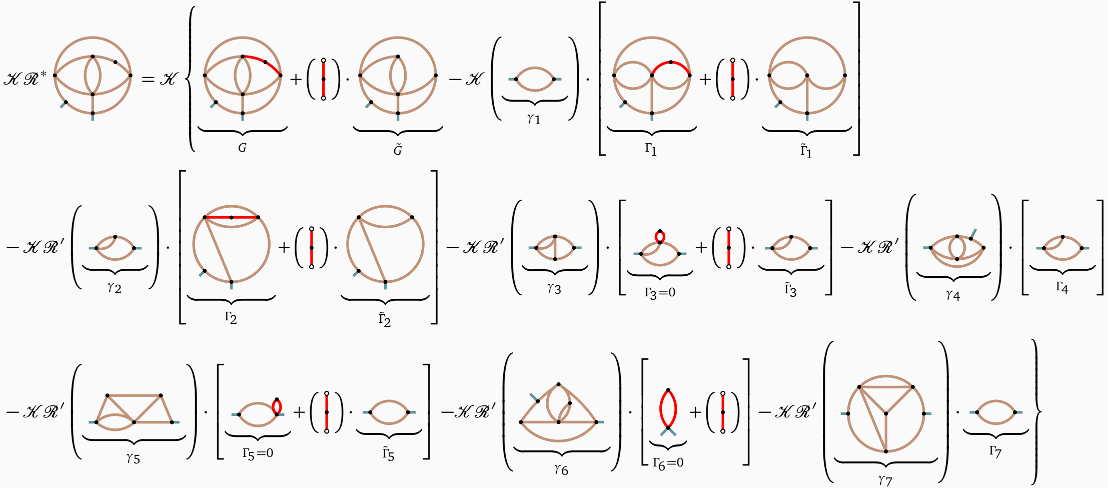


- Non-factorizable:



  - Only diagrams similar to (1) need special treatment
  - Diagrams (2),(3),(4) known from six-loop ϕ^4 theory renormalization [Kompaniets,Panzer'17]

IR operation for diagram with exceptional routing



Anomalous dimension ansatz for arbitrary Q

$$
\begin{aligned}
\gamma_Q^{6\text{-loop}} = Q(&\gamma_{6,0,0} + N\gamma_{6,0,1} + N^2\gamma_{6,0,2} + N^3\gamma_{6,0,3} + N^4 \boxed{\gamma_{6,0,4}} + N^5 \boxed{\gamma_{6,0,5}}) \\
+ Q^2(&\gamma_{6,1,0} + N\gamma_{6,1,1} + N^2\gamma_{6,1,2} + N^3\gamma_{6,1,3} + N^4 \boxed{\gamma_{6,1,4}} + N^5 \boxed{\boxed{\gamma_{6,1,5}}}) \\
+ Q^3(&\gamma_{6,2,0} + N\gamma_{6,2,1} + N^2\gamma_{6,2,2} + N^3\gamma_{6,2,3} + N^4 \boxed{\boxed{\gamma_{6,2,4}}}) \\
+ Q^4(&\gamma_{6,3,0} + N\gamma_{6,3,1} + N^2\gamma_{6,3,2} + N^3 \boxed{\gamma_{6,3,3}}) \\
+ Q^5(&\gamma_{6,4,0} + N\gamma_{6,4,1} + N^2 \boxed{\gamma_{6,4,2}}) \\
+ Q^6(&\boxed{\gamma_{6,5,0}} + N \boxed{\boxed{\gamma_{6,5,1}}}) \\
+ Q^7 &\boxed{\boxed{\gamma_{6,6,0}}}
\end{aligned}
$$

$\boxed{Q \to \infty}$ $\boxed{N \to \infty}$ $\boxed{Q \to \infty, N \to \infty}$

Critical dimension and checks

- Substituting IR fixed point $g^* \simeq \frac{6\varepsilon}{N+8}$ in $d = 4 - 2\varepsilon$ into γ_Q we obtain critical dimension Δ_Q

$$\Delta_Q = Q(1-\varepsilon) + \gamma_Q(g^*)$$

- Reexpanding Δ_Q in $1/N$ with $J = Q/N$ fixed

$$\frac{\Delta_Q}{Q} \simeq 1 - \epsilon + \sum_{l=1}^{\infty} (6\epsilon)^l \sum_{k=1}^{l} J^k \gamma_{l,k,l-k} + \mathcal{O}\left(\frac{1}{N}\right)$$

- Compare with predictions for $h_i(d)$, e.g.: $h_2(d) = -\frac{2^{d-3} d \sin\frac{\pi d}{2} \Gamma\left(\frac{d-1}{2}\right)}{\pi^{3/2} \Gamma\left(\frac{d}{2}+1\right)}$ [Giombi,Hyman'21]

$$\frac{\Delta_Q}{Q} = \left(\frac{d}{2} - 1\right) + h_2(d) J + h_3(d) J^2 + h_4(d) J^3 + \dots$$

Conclusion

- Calculated six-loop anomalous dimensoin of the ϕ^5 operator
- Derived six-loop expression for ϕ^Q operator anomalous dimension with arbitrary Q
- Result checked with available $1/N$ and fixed $J = Q/N$ expansions
- Calculated individual diagrams $\mathscr{K}\mathscr{R}'$ results allow extenstion to more general theories

Thank you for attention!