# «On explicit cutoff regularization»

Aleksandr V. Ivanov

10.1007/s10958-021-05500-5
10.1088/1361-6471/abb939
10.48550/arXiv.2203.07131
10.48550/arXiv.2209.01783

*St. Petersburg Department*
*of Steklov Mathematical Institute*
*of Russian Academy of Sciences*

The XXVI International Scientific Conference of Young Scientists
and Specialists (AYSS-2022)
October 24 – 28
2022

# Theory of regularization

1) Integral calculus

$$\int_0^1 \frac{ds}{s} \;\rightarrow\; \int_\epsilon^1 \frac{ds}{s} = -\ln\epsilon \;\text{ or }\; \int_0^1 \frac{ds}{s^{1-\epsilon}} = \frac{1}{\epsilon}$$

2) Theory of generalized functions

Sokhotski–Plemelj theorem
(Sokhotski formula)

3) Quantum field theory: Feynman diagrams and perturbative expansions

We need to deform $G \rightarrow G_\Lambda$ near $x \sim y$

| type | deform |
|---|---|
| dimensional | ~~dimension parameter~~ |
| cutoff | ~~gauge invariance~~ |
| high covariant derivative | ~~operator of second order~~ |

# The main properties

1) Spectral representation: let $A\varphi_\lambda = \lambda^2\varphi_\lambda$ and $AG = \mathbf{1}$, then $G_\Lambda = \mathfrak{J}_\Lambda G$, where

$$\mathfrak{J}_\Lambda(x,y) = \int_{\mathbb{R}_+} d\mu(\lambda)\, \phi_\lambda(x)\rho(\lambda/\Lambda)\phi_\lambda^*(y);$$

2) Explicit cutoff in coordinate representation near $x \sim y$ :

$$|x-y|_\Lambda = \begin{cases} |x-y|, & |x-y| > 1/\Lambda; \\ 1/\Lambda, & |x-y| \leqslant 1/\Lambda, \end{cases} \quad \text{and} \quad G_\Lambda(x,y) = \frac{1}{4\pi|x-y|_\Lambda^2}\big(1+o(1)\big) \text{ in } \mathbb{R}^4;$$

3) Homogenization:

$$\frac{1}{4\pi|x|_\Lambda^2} = \frac{1}{S_3}\int_{\mathbb{S}^3} d\tilde{\sigma}(y)\frac{\omega(y)}{4\pi|y/\Lambda + x|^n}, \qquad \frac{1}{S_3}\int_{\mathbb{S}^3} d\tilde{\sigma}(y)\omega(y) = 1;$$

4) Covariance: Does $\mathfrak{J}_\Lambda$ inherit physical symmetries of A?

# The simplest example

Let $d \in \mathbb{N},\ x \in \mathbb{R}^d,\ A(x) = -\partial_{x_\mu}\partial_{x^\mu},$ and

$$G(|x|) = \begin{cases} -|x|/2, & d = 1; \\ -\ln(|x|)/2\pi, & d = 2; \\ |x|^{2-d}/(d-2)S_{d-1}, & d \geqslant 3, \end{cases}$$

then we have

$$G(|x|_\Lambda) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} d^d y\, e^{ix_\mu y^\mu}\hat{G}(|y|)\rho(|y|/\Lambda) = \frac{1}{S_{d-1}}\int_{\mathbb{S}^{d-1}} d\sigma(y)\, G(|x - y/\Lambda|),$$

where

$$\rho(|t|/\Lambda) = \Gamma(d/2)\left(\frac{|t|}{2\Lambda}\right)^{1-d/2} J_{d/2-1}(|t|/\Lambda)\,.$$

# Yang—Mills theory

Yang and Mills (1954)
Trautman (1979)
Babelon and Viallet (1981)

Let $G$ be a compact semisimple Lie group, and $\mathfrak{g}$ is its Lie algebra.

Then, $t_a$ denote the corresponding generators, which satisfy $[t^a, t^b] = f^{abc}t^c$, $\mathrm{tr}\left(t^a t^b\right) = -2\delta^{ab}$.

Yang-Mills connection components: $B_\mu(x) = B^a_\mu(x)t^a$.

Components of the field strength: $F_{\mu\nu}(x) = F^a_{\mu\nu}(x)t^a,\ F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + f^{abc}B^b_\mu B^c_\nu$.

Covariant Laplace operator is $M_{\mu\nu}{}^{ab}(x;\alpha) = -D^{ac}_\sigma(x)D^{cb}_\sigma(x)\delta_{\mu\nu} - 2\alpha f^{acb}F^c_{\mu\nu}(x)$.

DeWitt(1965)
Seeley(1967)
Luscher(1982)

$$G_{\mu\nu}(x,y) = R_0(x-y)a_{0\mu\nu}(x,y) + R_1(x-y)a_{1\mu\nu}(x,y) + R_2(x-y)a_{2\mu\nu}(x,y) + PS(x,y),$$

$$R_0(x) = \frac{1}{4\pi^2|x|^2},\ R_1(x) = -\frac{\ln(|x|^2\mu^2)}{16\pi^2},\ R_2(x) = \frac{|x|^2\left(\ln(|x|^2\mu^2) - 1\right)}{64\pi^2}.$$

# Problem statement

Additional restrictions:

$B^a_\mu(x) = \frac{s}{2} x^\nu \xi^a_{\nu\mu}$, where $\left(\xi^a\right)_{\mu\nu} = \frac{1}{\sqrt{8 \dim \mathfrak{g}}} \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$ for all $a \in \{1, \ldots, \dim \mathfrak{g}\}$,

then, with $z = x - y$, we have

$$G_{\mu\nu}(x, y) = R_0(z)\delta_{\mu\nu} + 2\alpha s \xi_{\mu\nu} R_1(z) - \frac{\alpha^2 s^2 \delta_{\mu\nu} \xi_{\sigma\beta} \xi_{\sigma\beta}}{2} R_2(z) - \frac{s^2 \delta_{\mu\nu} \xi_{\sigma\beta} \xi_{\sigma\beta} |z|^2 (1 - 2\alpha^2)}{2^9 \pi^2} + o\left(s^2\right).$$

We need to calculate $\rho\left(\sqrt{\hat{M}(x, y)}/\Lambda\right) G(x, y)$, where $\rho(r) = \frac{2J_1(r)}{r}$.

# Results

$$\tilde{R}_0(z)\delta_{\mu\nu} + 2\alpha s\xi_{\mu\nu}\Big(\bar{R}_0(z) + \tilde{R}_1(z)\Big)$$
$$+s^2\delta_{\mu\nu}\xi_{\sigma\beta}\xi_{\sigma\beta}\Bigg(\frac{3\,|z|^2\Lambda^2 - 2\alpha^2}{2^4 3\Lambda^2}\bar{R}_0(z) + \frac{1-\alpha^2}{2^5\Lambda^2}\hat{R}_0(z) - \alpha^2\bar{R}_1(z)$$
$$\frac{\alpha^2}{2}\tilde{R}_2(z) - \frac{\big(|z|^2 + \Lambda^{-2}\big)(1-2\alpha^2)}{2^9\pi^2}\Bigg) + \mathcal{O}\big(s^3\big)\,.$$

$$\tilde{R}_0(x) = \frac{1}{4\pi^2}\begin{cases} |x|^{-2}, & |x| > 1/\Lambda; \\ \Lambda^2, & |x| \leqslant 1/\Lambda, \end{cases}$$

$$\tilde{R}_1(x) = \frac{1}{4\pi^2}\begin{cases} -\frac{1}{4}\ln(|x|^2\mu^2) - \frac{1}{8}|x|^{-2}\Lambda^{-2}, & |x| > 1/\Lambda; \\ \frac{1}{2}L - \frac{1}{8}|x|^2\Lambda^2, & |x| \leqslant 1/\Lambda, \end{cases}$$

$$\tilde{R}_2(x) = \frac{1}{4\pi^2}\begin{cases} \frac{1}{16}|x|^2\big(\ln(|x|^2\mu^2) - 1\big) + \frac{1}{16}\Lambda^{-2}\ln(|x|^2\mu^2) + \frac{1}{96}|x|^{-2}\Lambda^{-4} + \frac{1}{32}\Lambda^{-2}, & |x| > 1/\Lambda; \\ -\frac{1}{8}\Lambda^{-2}L - \frac{1}{8}|x|^2 L + \frac{1}{96}|x|^4\Lambda^2 + \frac{1}{32}|x|^2 - \frac{1}{16}\Lambda^{-2}, & |x| \leqslant 1/\Lambda\,. \end{cases}$$

# Result for 2-D Sigma-model

Operator is $M^{ab}(x) = -D^{ab}_{x_\mu}\partial_{x^\mu}$. Then, after some transformations and simplifications we get

$$G(x-y) = R_0(x-y) + s^2 m R_1(x-y) + o\left(s^2\right)$$

with $R_0(x) = -\dfrac{1}{4\pi}\ln(|x|^2\mu^2)$, $R_1(x) = \dfrac{|x|^2\left(\ln(|x|^2\mu^2) - 2\right)}{16\pi}$.

$$\tilde{R}_0(x) = \frac{1}{4\pi}\begin{cases} -\ln(|x|^2\mu^2), & |x| > 1/\Lambda; \\ 2L, & |x| \leqslant 1/\Lambda, \end{cases}$$

$$\tilde{R}_1(x) = \frac{1}{4\pi}\begin{cases} \frac{1}{4}|x|^2\left(\ln(|x|^2\mu^2) - 2\right) + \frac{1}{4}\Lambda^{-2}\ln(|x|^2\mu^2), & |x| > 1/\Lambda; \\ -\frac{1}{2}(L+1)\Lambda^{-2} - \frac{1}{2}L|x|^2, & |x| \leqslant 1/\Lambda, \end{cases}$$

Many thanks!