

# Sasaki-Einstein Spaces and Isomonodromic Deformations

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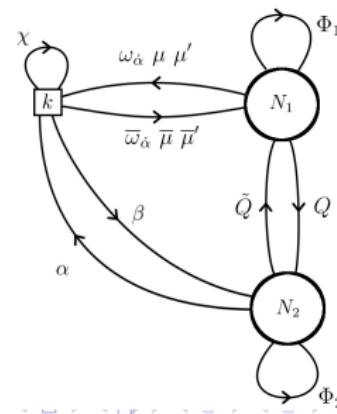
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# Outline

- Sasaki-Einstein spaces -  $Y^{p,q}$
- EoM for a point string
- Isomonodromic deformation
- Flow of parameters
- Singularity coalescence and theory matching
- Outlooks

# Sasaki-Einstein Spaces

- SUGRA in  $\text{AdS}^5 \times S_5 = 4\text{-d boundary SCFT}$
- Dual to quiver theories
- Important properties:
  - Riemannian
  - Symplectic
  - Complex
- Infinite families  $Y^{p,q}$  and  $L^{p,q,r}$



Sasaki-Einstein  $\mathbb{Y}^{p,q}$ 

Metric of  $\mathbb{Y}^{p,q}$  with  $p < q$ :

$$ds^2 = \frac{1-y}{6}(d\theta + \sin^2\theta d\phi^2) + \frac{1}{w(y)q(y)}dy^2 +$$

$$\frac{q(y)}{9}(d\psi - \cos\theta d\phi)^2 + w(y)(d\alpha + f(y)(d\psi - \cos\theta d\phi))^2$$

$$w(y) = \frac{2(b-y^2)}{1-y}, \quad q(y) = \frac{b-3y^2+2y^3}{b-y^2}, \quad f(y) = \frac{b-2y+y^2}{6(b-y^2)}$$

$$b = \frac{1}{2} + \frac{p^2-3q^2}{4p^3} \sqrt{4p^2 - 3q^2}$$

$$y_1 \leq y \leq y_2, \quad 0 < \theta < \pi, \quad 0 < \phi < 2\pi, \quad 0 \leq \psi \leq 2\pi, \quad 0 \leq \alpha \leq 2\pi/$$

# Equations of motion

We obtain them using  $\nabla^2\Phi = -E\Phi$

Solutions take the following form:

$$\Phi(y, \theta, \phi, \psi, \alpha) = \exp[i(P_\phi\phi + P_\psi\psi + \frac{P_\alpha}{I}\alpha)]\Theta(\theta)Y(y),$$

using this ansatz we can separate the variables and obtain:

$$\frac{d^2}{dy^2} Y(y) + \left(\frac{1}{y-y_1} + \frac{1}{y-y_3} + \frac{1}{y-y_5}\right)Y(y) + Q(y)Y(y) = 0$$

$Q(y)$  is just some rational function of  $y$ .

# Heun equation

This equation looks like Heun, but not quite. So we change variables:

$$z = \frac{y - y_1}{y_2 - y_1}$$

and rescale our equation:

$$Y(y) = z^{\alpha_1}(z-1)^{\alpha_2}(z-t)^{\alpha_3}h(z)$$

This way we can obtain the canonical form of the Heun equation:

$$\frac{d^2h(z)}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t} \right) \frac{dh(z)}{dz} + \left( \frac{\alpha\beta z - k}{z(z-1)(z-t)} \right) h(z) = 0$$

# Parameters

The parameters of our Heun are connected to the geometry parameters as follows:

$$\alpha = -\lambda + \sum_i \alpha_i, \beta = 2 + \lambda + \sum_i \alpha_i,$$

$$\alpha_1 = \pm \frac{1}{4} [P_\alpha(p + q - \frac{1}{3l}) - Q_R], \alpha_2 = \pm \frac{1}{4} [P_\alpha(p - q + \frac{1}{3l}) + Q_R]$$

$$\alpha_3 = \pm \frac{1}{4} [P_\alpha(\frac{-2p^2+q^2+p\sqrt{4p^2-3q^2}}{q} - \frac{1}{3l}) - Q_R]$$

$$\gamma = 1 + 2\|\alpha_1\|, \delta = 1 + 2\|\alpha_2\|, \epsilon = 1 + \alpha_3, k =$$

$$(\|\alpha_1\| + \|\alpha_3\|)(\|\alpha_1\| + \|\alpha_3\| + 1) - \|\alpha_2\|^2 + t \left( (\|\alpha_1\| + \|\alpha_2\|)(\|\alpha_1\| + \|\alpha_2\| + 1) - \|\alpha_3\|^2 \right) - \tilde{\mu}$$

$$t = \frac{1}{2} \left( 1 + \frac{\sqrt{4p^2-3q^2}}{q} \right)$$

# Schlesinger system

Let us consider the linearized form of our equation:

$$\frac{dh(z)}{dz} = \left( \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t} \right) h(z)$$

$$Tr(A_\mu) = \theta_\mu, A_\infty = -A_0 - A_1 - A_t,$$

For our system to be m-inv.  $d_{t_i} \mathbf{h}(z) = A_i \mathbf{h}(z) dt_i$ ,

And  $A$  must satisfy  $d_{t_i}(A_j d_{t_j}) = (A_i d_{t_i}) \wedge (A_j d_{t_j})$

This allows us to derive the Schlesinger equations:

$$\frac{\partial A_i}{\partial t_j} = [A_i, (1 - \delta_{ij}) \frac{A_j}{t_i - t_j} + \delta_{ij} \sum_{j \neq i} \frac{A_i}{a_i - a_j}]$$

# Painleve VI

We effectively have a symplectic structure on each submanifold:

$$\frac{d\lambda}{dt} = \{H, \lambda\}, \quad \frac{d\mu}{dt} = \{H, \mu\}, \quad \{f, g\} = \frac{df}{d\lambda} \frac{dg}{d\mu} - \frac{df}{d\mu} \frac{dg}{d\lambda}$$

Deformed Heun has the form:

$$h_1''(z) - (\text{Tr} A + \partial_z \ln A_{12}) h_1'(z) + (\det A - A'_{11} + A_{11} \partial_z \ln A_{12}) h_1(z) = 0$$

Hamiltonian and  $\mu$  can be expressed as:

$$H_i = -\text{Res}_{z \rightarrow t_i} q(z), \quad \mu_i = A_{11}(\lambda_i) = \text{Res}_{z \rightarrow \lambda_i} q(z)$$

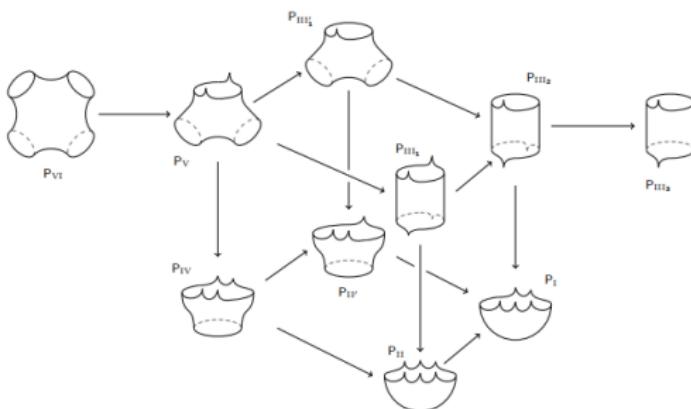
Using these we can obtain the Painleve VI:

$$\begin{aligned} \lambda''(t) &= \frac{1}{2} \left( \frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t} \right) \lambda'^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t} \right) \lambda' \\ &+ \frac{\lambda(\lambda-1)(\lambda-t)}{2t^2(t-1)^2} \left( (\theta_\infty - 1)^2 - \frac{\theta_0^2 t}{\lambda^2} - \frac{\theta_1^2(t-1)}{(\lambda-1)^2} - \frac{(\theta_t^2 - 1)t(t-1)}{(\lambda-t)^2} \right) \end{aligned}$$

# Some geometric intuition

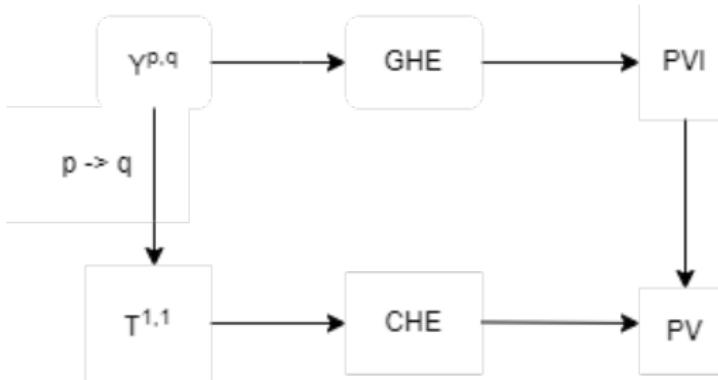


Degeneration of Painlevé equations [Chekhov, Mazzocco, Rubtsov, '15]



## Theory "matching"

$$\lambda'' = \left( \frac{1}{2\lambda} + \frac{1}{\lambda-1} \right) \lambda'^2 - \frac{1}{t} \lambda' + \frac{(\lambda-1)^2}{t^2} \left( \alpha\lambda + \frac{\beta}{\lambda} \right) + \gamma \frac{\lambda}{t} + \delta \frac{\lambda(\lambda+1)}{\lambda-1}$$



By comparing the coefficients of PV, we can establish what are the conditions for  $Y^{p,q}$  and  $T^{1,1}$  to be equivalent

# Results

- PVI describes the flow geometric parameters
- Non-integrability of  $Y^{p,q}$  and PVI non-integrability?
- Cfluent Heun/PVI encode changes in geometry
- Conditions for algebraic equivalence

# Outlooks

- Black hole backgrounds - equivalence/embedding of theories
- Scattering, S-matrices and Quasinormal modes
- Spectral curves and Seiberg-Witten curves

# Thank you!

# Sources

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