On 4-Dimensional Exceptional Drinfel'd Algebra

Sameer Kumar

Moscow Institute of Physics and Technology

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Advisor: Dr. Edward Musaev

Moscow Institute of Physics and Technology

Outline

Motivation

Prove or disprove the existence of U-dualities in 10 dimensional string theory compactified to 4 dimensions.

T-duality [4]

- Bosonic string action (String sigma or Polyakov) -

$$L = -\frac{T}{2}\int\sqrt{-h}h^{\alpha\beta}\partial\alpha X\partial\beta X\, d^2\sigma \tag{1}$$

- Bosonic string is embedded in 25 dimensions.
- Compactify last dimension - $X^{25}(\sigma+\pi,\tau) = X^{25}(\sigma,\tau) + 2\pi RW$. W - winding number.
- Expanding X^{25} with boundary condition above -

$$X^{25} = x^{25} + 2\alpha' p^{25} r + 2RW\sigma \tag{2}$$

- As evident - X^{25} compact = momentum p^{25} quantized.

T-duality [4]

- Therefore, $p^{25} = \frac{K}{R}, K \in \mathbb{Z}$
- A few calculations show that - (N_L and N_R some numbers)

$$\alpha' M = \alpha' \left[(\frac{K}{R})^2 + (\frac{WR}{\alpha'})^2 \right] + 2N_L + 2N_R - 4 \qquad (3)$$

- T-duality: $W \leftrightarrow K,\ R \leftrightarrow \tilde{R} = \frac{\alpha'}{R}$

The O(n,n) T-duality Group [4]

- For right moving modes of toroidally compactified bosonic string. Then these strings follow the equations -

$$(L_0 - 1)\,|\psi\rangle = (\tilde{L_0} - 1)\,|\psi\rangle \tag{4}$$

- Reduced to mass shell conditions -

$$\frac{M}{8} = \frac{1}{2}G_{IJ}(p_L^I p_L^J) + N_L - 1 = \frac{1}{2}G_{IJ}(p_R^I p_R^J) + N_R - 1 \tag{5}$$

- Here momenta p_L and p_R are given by -

$$p_L^I = W^I + G^{IJ}(\frac{1}{2}K_J - B_{JK}W^K)$$

$$p_R^I = -W^I + G^{IJ}(\frac{1}{2}K_J - B_{JK}W^K)$$

The O(n,n) T-duality Group [4]

- Adding and subtracting -

$$\frac{M_o}{2} = (WK)G^{-1}\begin{pmatrix} W \\ K \end{pmatrix} \tag{6}$$

- $G^{-1} = \begin{pmatrix} 2(G - BG^{-1}B) & BG^{-1} \\ -G^{-1}B & \frac{G^{-1}}{2} \end{pmatrix}$
- For T-duality, $R \leftrightarrow \tilde{R}$ is equivalent to inversion symmetry:

$$W^I \leftrightarrow K_I,\ G \leftrightarrow G^{-1} \tag{7}$$

- Shift symmetry:

$$B_{IJ} \to B_{IJ} + \frac{N_{IJ}}{2},\ K_I \to K_I + N_{IJ}W^J \tag{8}$$

- Inversion + Shift: (Generates O(n,n) group)

$$A:\ A^T\begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} A = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \tag{9}$$

M-theory [4]

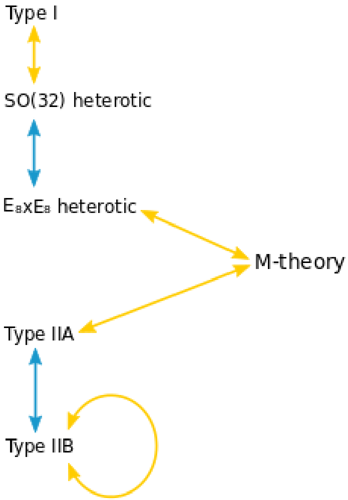

Figure: M-theory (taken from Wikipedia)

U-duality [6] [5] [2] [7] [4]

- Generalise T-duality to Type 2 strings.
- Compactify Type 2 String to n dimensions $E_{n,n}$ group symmetry. (Symmetry of corresponding SUGRA from compactification)
- $E_{n,n}$ - Exceptional Group. For n=4:

$$E_{n,n} = SL(5;\mathbb{R}) \tag{10}$$

- Note that U-duality encapsulates T-duality since:

$$SO(n-1, n-1;\mathbb{Z}) \subset E_n(\mathbb{Z}) \tag{11}$$

  , where $E_n(\mathbb{Z})$ is a maximal discrete subgroup of $E_{n,n}$.
- U-duality = T-duality + S-duality (T-duality type transform but for string coupling factor - g :)).

Mathematical Formulation [1]

- T-duality - O(d,d) , U-duality - SL(5) (for d=4)
- $\sigma = \text{span}(T_{AB}) - \textit{generators}$
- We let $\sigma \subset sl(5) \oplus \mathbb{R}^{+}$ - we need SL(5) symmetry
- ϵ_{ABCDE} - 5d SL invariant: $\epsilon^{ABCDE} T_{AB} T_{CD} = 0$
- T_{a5}, a = 1,...,4, antisymmetrisation of T_{ab} - 6 more generators: 10d Algebra
- Assume $[T_{AB}, T_{CD}] = \frac{i}{2} F_{AB,CD}{}^{GH} T_{GH}$ (7d max-gauged SUGRA analogy)
- Structure constants - $F_{AB,CD}{}^{GH} = 4 F_{AB,[C}{}^{[G} \delta^{H]}_{D]}$

$$F_{AB,C}{}^{D} = \frac{1}{2}\epsilon_{ABCGH} Z^{GHD} + \frac{1}{2}\delta^{D}_{[A} S_{B]C} + \frac{1}{3}\delta^{D}_{[A}\tau_{B]C} + \frac{1}{6}\delta^{D}_{C}\tau_{AB}$$

- τ -antisymmetric, S - symmetric, $Z^{[ABC]} = 0$

Mathematical Formulation [1] [3]

- Leibniz Identity:
  $2F_{AB[C}^{G}F_{GD],H}^{I} - F_{ABG}^{I}F_{CDH}^{G} + F_{ABH}^{G}F_{CDG}^{I} = 0$
- Decompose f into trace and traceless part -
  $f_{ab}^{c} = \overline{f}_{d}^{abc} = -4\epsilon^{abce}(S_{de} - 2\tau_{de}) + \frac{2}{3}\delta_{[b}^{c}I_{a]},\ I_a = f_{ab}^{b}$
  $S_{a5} = \frac{4}{3}I_a + \frac{2}{3}\tau_{a5},\ T_{ab5}^{c} = \overline{f}_{ab}^{c},\ S_{55} = 0$
- Choose - $T_{abc}^{5} = 0,\ T_{ab5}^{5} = -\frac{2}{3}\tau_{ab},\ T_{abc}^{d} = \tau_{[ab}\delta_{c]}^{d}$
- Now we can write the final structure constraints explicitly.

Structure Constraints for EDA d=4

$$6f_{f[a}^{[c}\tilde{f}_{b]}^{de]f} + f_{ab}^{f}\tilde{f}_{f}^{cde} + \frac{2}{3}\tilde{f}_{[a}^{cde}L_{b]} = 0 \tag{12}$$

$$\tilde{f}_{c}^{abc}I_b = 0 \tag{13}$$

$$\tilde{f}_{c}^{abc}\tau_{b5} = 0 \tag{14}$$

$$f_{de}^{a}\tilde{f}_{c}^{bde} + \frac{2}{3}\tilde{f}_{c}^{abd}L_d = 0 \tag{15}$$

$$\tilde{f}_{c}^{abg}\tilde{f}_{g}^{def} - 3\tilde{f}_{c}^{g[de}\tilde{f}_{g}^{f]ab} = 0 \tag{16}$$

Put $L_b = Z_b = \frac{\tau_{a5} - I_a}{3}$ ...

Structure Constraints for EDA d=4

$$6f_{f[a}^{[c}\tilde{f}_{b]}^{de]f} + f_{ab}^{f}\tilde{f}_{f}^{cde} + 2\tilde{f}_{[a}^{cde}Z_{b]} = 0 \tag{17}$$

$$\tilde{f}_{c}^{abc}f_{bd}^{d} = 0 \tag{18}$$

$$\tilde{f}_{c}^{abc}(3Z_b + f_{bd}^{d}) = 0 \tag{19}$$

$$f_{de}^{a}\tilde{f}_{c}^{bde} + 2\tilde{f}_{c}^{abd}Z_d = 0 \tag{20}$$

$$\tilde{f}_{c}^{abg}\tilde{f}_{g}^{def} - 3\tilde{f}_{c}^{g[de}\tilde{f}_{g}^{f]ab} = 0 \tag{21}$$

These are the final equations used for classification.

Classification Algorithm

- Start with Lie Algebras classified into non-isomorphic classes.
- Write down structure constants of $\boldsymbol{g}$ (the base Lie Algebra)
- Solve the above set of constraints to find structure constants of $\tilde{\boldsymbol{g}}$
- All calculations done in the ecosystem of Wolfram Mathematica

Some Results

Base Lie Algebra (g)	Non-zero Structure Constants of g	Non-zero Structure Constants of $\tilde{g}$
1) $[T_2, T_4] = T_1$ $[T_3, T_4] = T_2$	${f_{24}}^1 = 1$ ${f_{34}}^2 = 1$	a) $\tilde{f}^{123}{}_2 = \tilde{f}^{134}{}_4$, $\tilde{f}^{124}{}_2 = \tilde{f}^{134}{}_3$ $\tilde{f}^{123}{}_4 = \frac{\tilde{f}^{123}{}_4\tilde{f}^{134}{}_4 - \tilde{f}^{124}{}_4\tilde{f}^{134}{}_4}{2\tilde{f}^{134}{}_3}$, $\tilde{f}^{124}{}_3 = \frac{(\tilde{f}^{124}{}_4 - \tilde{f}^{123}{}_3)\tilde{f}^{134}{}_3}{2\tilde{f}^{134}{}_4}$ b) $\tilde{f}^{123}{}_2 = -\tilde{f}^{134}{}_4$, $\tilde{f}^{124}{}_4 = \tilde{f}^{123}{}_3$, $\tilde{f}^{234}{}_4 = \tilde{f}^{123}{}_1$ c) $\tilde{f}^{124}{}_2 = \tilde{f}^{134}{}_3$, $\tilde{f}^{124}{}_4 = \tilde{f}^{123}{}_3$ d) $\tilde{f}^{124}{}_4 = \tilde{f}^{123}{}_3$, $\tilde{f}^{234}{}_4 = \tilde{f}^{123}{}_1$
2) $[T_2, T_4] = \beta T_1$ $[T_2, T_4] = T_2$ $[T_3, T_4] = T_2 + T_3$	${f_{14}}^1 = \beta$, $f_{24}^2 = 1$, $f_{34}^2 = 1$, $f_{34}^3 = 1$	a) $\tilde{f}^{124}{}_4 = \frac{1}{3}\left(-\tilde{f}^{123}{}_3 - 2\beta\tilde{f}^{123}{}_3\right)$ b) $\tilde{f}^{234}{}_4 = \frac{(\beta-4)\tilde{f}^{123}{}_1}{3\beta}$
3) $[T_1, T_4] = T_1$ $[T_3, T_4] = T_2$	${f_{14}}^1 = 1$, ${f_{34}}^2 = 1$	a) $\tilde{f}^{234}{}_4 = \frac{1}{3}\tilde{f}^{123}{}_1$
4) $[T_1, T_4] = T_1$ $[T_2, T_4] = T_1 + T_2$ $[T_3, T_4] = T_2 + T_3$	${f_{14}}^1 = 1$, ${f_{24}}^1 = 1$, ${f_{24}}^2 = 1$, ${f_{34}}^2 = 1$, $f_{34}^3 = 1$	a) $\tilde{f}^{124}{}_4 = -\tilde{f}_3^{123}$
5) $[T_1, T_4] = AT_1$ $[T_2, T_4] = BT_2$ $[T_3, T_4] = CT_3$ $ABC \neq 0$	${f_{14}}^1 = A$, ${f_{23}}^2 = B$, ${f_{34}}^3 = C$	a) $\tilde{f}^{124}{}_4 = \frac{(-2A-2B+C)\tilde{f}^{123}{}_3}{3C}$ b) $\tilde{f}^{134}{}_4 = \frac{-(-2A+B-2C)\tilde{f}^{123}{}_2}{3B}$ c) $\tilde{f}^{234}{}_4 = \frac{(A-2V-2C)\tilde{f}^{123}{}_1}{3A}$

Figure: Some of the 4d EDA

Finding U-dualities (Current Work)

- Finding Dualities = Finding Classes isomorphic under SL(5):

$$T^A_B = M^A_C \circ T^C_E \circ (M^{-1})^E_B \tag{22}$$

- Brute Force: Very big computational space $\rightarrow$ Use invariants.
- Killing Form to rescue: Generalizes to EDA as : (in form of a 2-matrix)

$$\kappa = F^D_{AE} F^E_{BD} \tag{23}$$

- We seek the signature of this form to divide all algebras into smaller subclasses.
- Within these subclasses $\rightarrow$ find SL(5) isomorphisms by brute force.

МФТИ

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Thank you!