Worldsheet of a continuous helicity particle

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Work

Link 1

D.S. Kaparulin, S.L. Lyakhovich, and I. A. Retuntsev, Worldsheet of a continuous helicity particle, Phys. Rev. D **105**, 065004 (2022).

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Research object

Classical irreducible spinning particle with continuous helicity in d=3 Minkowski space. Here,

- spinning particle – a particle with internal degrees of freedom,
- irreducibility and continuous helicity – quantization of the classical model corresponds to the irreducible continuous helicity representation of the Poincare group.

World sheet concept

The work

Link 2

S.L. Lyakhovich and D.S. Kaparulin, World sheets of spinning particles, Phys. Rev. D **96**, 105014 (2017).

tells that irrespectively of the model for any classical irreducible spinning particle in Minkowski space

- particle position x belongs to some surface – **worldsheet**,
- the worldsheet position is defined by momentum and total angular momentum of the particle,
- all particle trajectories are connected by gauge transformation.

World sheet concept

The work

S.L. Lyakhovich and D.S. Kaparulin, World sheets of spinning particles, Phys. Rev. D **96**, 105014 (2017).

also

- gives a method to find geometrical equation of motion,
- considers the massive three-dimensional case in detail.

Notations and conventions

We consider three-dimensional Minkowski space with local coordinates $x^\mu, \mu = 0, 1, 2$ with the following metric:

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1)\,. \tag{1}$$

Scalar, vector and scalar triple products are

$$\begin{aligned} (x, y) = \eta_{\mu\nu} x^\mu y^\mu\,, \qquad & x^2 = (x, x)\,, \\ [x, y]^\mu = \varepsilon^{\mu\nu\rho} x_\nu y_\rho\,, \qquad & (x, y, z) = (x, [y, z])\,, \end{aligned} \tag{2}$$

where $\varepsilon^{\mu\nu\rho}$ is levi-Civita symbol with property $\varepsilon^{012} = -1$.

Worldsheet of a continuous helicity particle

For classical irreducible spinning particle with continuous helicity, its momentum p and total angular momentum J are subjected to conditions

$$p^2 = 0\,, \quad (p, J) - \sigma = 0\,, \tag{3}$$

where $\sigma \neq 0$ is helicity. Position of the particle belongs to the worldsheet given by equation

$$(J - [x, p])^2 - \varrho = 0 \tag{4}$$

where ϱ is parameter of the model.

Worldsheet of a continuous helicity particle

After introducing new parameters

$$v = [J, p]\,, \qquad \Delta = J^2 - \varrho \tag{5}$$

the equation of worldsheet takes the following form:

$$(p, x)^2 + 2(v, x) + a = 0\,, \tag{6}$$

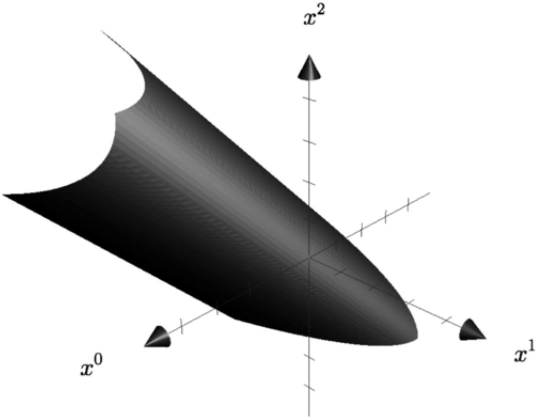
where

$$p^2 = 0\,, \qquad v^2 = \sigma^2\,, \qquad (p, v) = 0\,. \tag{7}$$

The worldsheet is **parabolic cylinder with lightlike axis**:

- p determines the direction of symmetry axis,
- v determines the direction of asymptotes of parabola, being orthogonal section of cylender,
- Δ determines the distance between the vertex of parabolic cylinder and origin,
- σ determines the focal distance of parabolic cylinder.

Worldsheet of a continuous helicity particle



Worldsheet of continuous helicity spinning particle with $p = (1, 0, 1)$, $J = (0, 0, 1)$, $\varrho = 1$. The step equals 1.

Worldsheet of a continuous helicity particle

Since we can express

$$J = \frac{\Delta + \varrho}{2\sigma} p + \sigma e \,, \tag{8}$$

where e is determined by equations

$$e^2 = (e, v) = (e, p) - 1 = 0 \tag{9}$$

we have one to one correspondence between set of particle state p, J and set of worldsheets p, v, a.

Equations of motion

- The world sheet is the only restriction for irreducible particle position.

therefore

- Trajectory of irreducible spinning particle is general curve on its worldsheet.
- All curves on worldsheet are connected by gauge transformations.

We have the known differential geometric problem:

- The surface is given.
- The task is to find the equations of motion describing the general curves lying on this surface.

To find a solution $x(\tau)$, we supplement the system with differential consequences of the worldsheet equation. As the result we have the following system ($k = 0, 1, 2, 3, 4$):

$$\begin{gathered} \frac{d^k}{d\tau^k}\left(\big(p, x(\tau)\big)^2 + 2\big(v, x(\tau)\big) + \Delta \right) = 0\,, \\ p^2 = 0\,, \qquad v^2 = \sigma^2\,, \qquad (p, v) = 0\,. \end{gathered} \tag{10}$$

Equations of motion

As the solution of

$$\frac{d^k}{d\tau^k}\left(\big(p, x(\tau)\big)^2 + 2\big(v, x(\tau)\big) + \Delta\right) = 0\,,$$
$$p^2 = 0\,, \qquad v^2 = \sigma^2\,, \qquad (p, v) = 0\,, \tag{11}$$

where $k = 0, 1, 2, 3, 4,$ we have

- integrals of motion p, J (or equivalently p, v, a) in terms of characteristics of trajectory $x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}$,
- equation of motion for general curve on worldsheet.

$$F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}) = 0\,. \tag{12}$$

Equations of motion

We consider two types of trajectories:

- timelike curves on the worldsheet, for physical reason,

$$(\dot{x}, \dot{x}) < 0\,, \tag{13}$$

- lightlike curves on the worldsheet, to simplify calculation,

$$(\dot{x}, \dot{x}) = 0\,. \tag{14}$$

We also exclude straight lines from consideration because they don't define their world sheet in a unique way.

In timelike case

$$(\dot{x}, \dot{x}) < 0\,, \tag{15}$$

equation of motion and integrals of motion have the following form:

$$\begin{aligned}
&81F^2G^2H - 18FG^2H^2 - 15C^4F^2H + 18C^2F^3H - C^2F^2H^2 + 486CFG^3 - \\
&-324CG^3H + 270C^5FG - 162C^3F^2G + 1458C^2FG^2 + 18C^3GH + \\
&+189C^2G^2H + 4C^4FH - 1134C^3FG + 15C^2G^2H^2 + 2C^3GH^2 + 30C^5GH - \\
&-486CF^3G + G^2H^3 - 120C^3FGH + 90C^2FG^2H - 6CFGH^2 + 108CF^2GH - \\
&-216C^3G + 972C^2G^2 - 36C^4F + 504C^5G + 540C^4G^2 - 216C^4F^2 + \\
&+36C^6F + 540C^3G^3 - 81C^2F^4 - 90C^4F^3 - 1458CG^3 + \\
&+C^6H - 7C^6 + 729G^4 + 15C^8 = 0\,.
\end{aligned} \tag{16}$$

Timelike curves on worldsheet

$$p = \sqrt{-\mathrm{sign}(\sigma)\sigma\varkappa_1\alpha}\left(\dot{x} - \frac{B\alpha}{\varkappa_1(3\alpha - C)}\ddot{x} + \frac{\alpha}{\varkappa_1}[\dot{x},\ddot{x}]\right),$$

$$J = \left[\frac{(x,p)^2\varkappa_1\alpha - \sigma}{2\sqrt{-\mathrm{sign}(\sigma)\sigma\varkappa_1\alpha}} + \frac{\Delta+\varrho}{2}\sqrt{-\frac{\alpha\varkappa_1}{\sigma}}\right]\dot{x}$$

$$+\frac{\alpha}{\varkappa_1}\left[\left((x,p) + \frac{B}{C-3\alpha}\frac{(x,p)^2\varkappa_1\alpha+\sigma}{2\sqrt{-\mathrm{sign}(\sigma)\sigma\varkappa_1\alpha}}\right) + \frac{B}{(C-3\alpha)}\frac{\Delta+\varrho}{2}\sqrt{-\frac{\alpha\varkappa_1}{\sigma}}\right]\ddot{x}$$

$$+\frac{\alpha}{\varkappa_1}\left[\left(\frac{(x,p)^2\varkappa_1\alpha+\sigma}{2\sqrt{-\mathrm{sign}(\sigma)\sigma\varkappa_1\alpha}} - \frac{B(x,p)}{C-3\alpha}\right) + \sqrt{-\frac{\alpha\varkappa_1}{\sigma}}\right][\dot{x},\ddot{x}], \tag{17}$$

where

$$\Delta = \frac{\sigma\alpha}{\varkappa_1}\left[\left((x,\dot{x})\varkappa_1 + (x,\dot{x},\ddot{x})\alpha + \frac{(x,\ddot{x})B\alpha}{C-3\alpha}\right)^2 + \frac{(x,\dot{x},\ddot{x})B}{C-3\alpha} - 2(x,\ddot{x})\right],$$

$$\alpha = -\frac{U}{V}. \tag{18}$$

Here, the following notations are used:

$$U = 225FC^2 - 6CGH + 15C^2FH - 216GFG - 90C^3G + 90C^2F^2$$

$$-75C^4 - 5C^2H + 81G^2 - 108CG + 36C^2 + 81F^3 + FH^2 - 18F^2H\,,$$

$$V = C^3H + GH^2 - 18FGH + 81F^2G - 135CG^2 - 6C^3 + 15C^2GH$$

$$+15C^5 - 24C^3F + 90FC^2G + 99C^2G\,, \tag{19}$$

$$F = -\frac{1}{9}(4B^2 + 4C^2 - 3E + 21)\,,$$

$$G = \frac{1}{9}(BD - EC + 7C)\,,$$

$$H = B^2 + C^2 - 9\,,$$

Where B, C, DE are functions of curve invariants: curvature $\varkappa_1$ and torsion $\varkappa_2$ and their derivative up to second order,

$$B = {\varkappa_1}^{-2}\dot{\varkappa}_1\,, \quad C = {\varkappa_1}^{-1}\varkappa_2\,,$$

$$D = {\varkappa_1}^{-3}(2\dot{\varkappa}_1\varkappa_2 + \varkappa_1\dot{\varkappa}_2)\,, \tag{20}$$

$$E = {\varkappa_1}^{-3}(\ddot{\varkappa}_1 + \varkappa_1^3 - \varkappa_1\varkappa_2^2)\,.$$

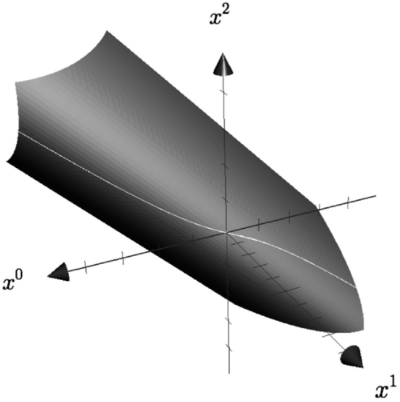
Since $\varkappa_1$ and $\varkappa_2$ are defined as

$$\varkappa_1 = \sqrt{(\ddot{x},\ddot{x})}\,, \quad \varkappa_2 = \frac{(\dot{x},\ddot{x},\dddot{x})}{\sqrt{(\ddot{x},\ddot{x})}}\,, \tag{21}$$

the equation of motion has derivatives up to 4th order. One also has the following gauge symmetries:

$$\delta_\xi x = \dot{x}\xi\,, \qquad \delta_\eta x = p\eta. \tag{22}$$

Timelike curves on worldsheet



Example of timelike trajectory of irreducible continuius helicity spinning particle with $p = (1, 0, 1)$, $J = (0, 0, 1)$, $\varrho = 1$. The step equals 1.

Lightlike curves on worldsheet

In lightlike case

$$(\dot{x}, \dot{x}) = 0\,, \tag{23}$$

integrals of motion read

$$p = \sqrt{\operatorname{sign}(\sigma)\sigma}\,\dddot{x}\,,$$

$$J = -\sqrt{\operatorname{sign}(\sigma)\sigma}\left[\dot{x} + (x, \dddot{x})\ddot{x} - \left((x, \ddot{x}) + \frac{\varrho}{2\sigma}\right)\dddot{x}\right]. \tag{24}$$

Equations of motion have derivatives ut to 3rd order

$$(\dddot{x}, \dddot{x}) = 0\,, \qquad (\dot{x}, \dot{x}) = 0 \tag{25}$$

and one gauge symmetry – reparametrization,

$$\delta_\xi x = \dot{x}\xi\,. \tag{26}$$

Solving equations

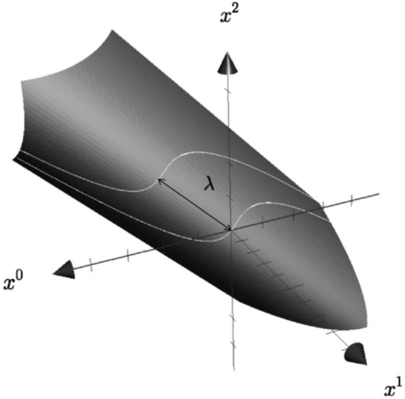
$$(\dddot{x}, \dddot{x}) = 0\,, \qquad (\dot{x}, \dot{x}) = 0 \tag{27}$$

we get

$$\begin{aligned} x(\tau) = \frac{1}{6}\frac{\tau^3}{\sqrt{\mathrm{sign}(\sigma)\sigma}}\,p + \frac{\tau^2}{2\sigma}\,[p, J] - \tau\sqrt{\mathrm{sign}(\sigma)\sigma}\,e \\ + \frac{1}{2\sigma^2}((J, J) - \varrho)\,[p, J] + \frac{\lambda}{p^0}\,p\,, \end{aligned} \tag{28}$$

where λ can't be expressed in terms of p, J, so lightlike curves on parabolic cylinder with lightlike axis can't be considered as world-paths of continuous helicity spinning particle.

Lightlike curves on worldsheet



Lightlike curves ($\lambda = 0$ and $\lambda = 2$) on the parabolic cylinder with $p = (1, 0, 1)$, $J = (0, 0, 1)$, $\varrho = 1$. The step equals 1.

Other results

In work

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we also

- check our results by establishing a correspondence with the known anion model,
- give some comments about worldsheet and equations of motion in truly massless case $(\Delta = 0)$.

Thanks for your attention!