Elliptic Feynman integrals and pure functions

Johannes Broedel, Claude Duhr, Falko Dulat, Brenda Penante, Lorenzo Tancredi

arXiv:1809.10698 [hep-th]

Speaker: Maxim Bezuglov

Plan

- Introduction to the general theory of elliptic functions
- Elliptic polylogarithms
- Motivation and examples

Topics that will not be covered

- Hopf algebra for elliptic polylogarithms
- Specific methods for calculating elliptic integrals in terms of elliptic polylogarithms

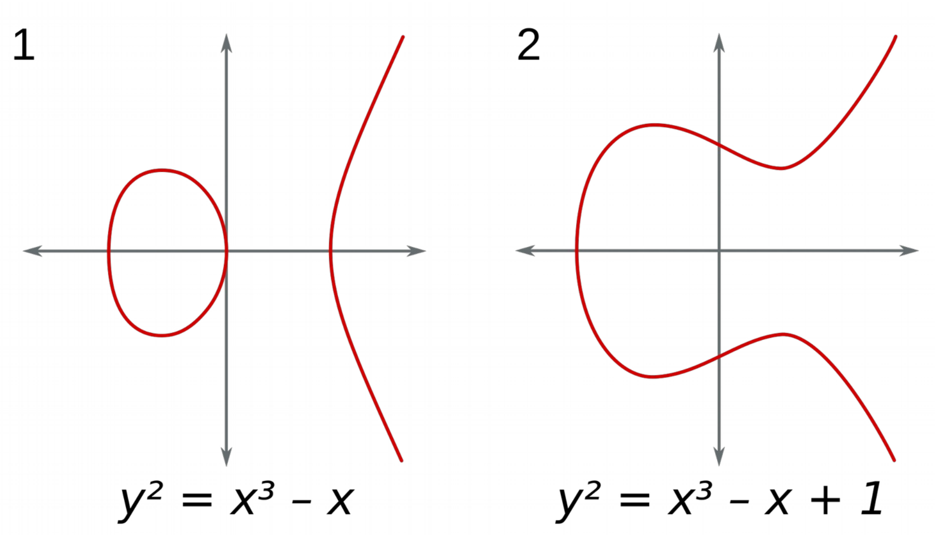
Elliptic Curves

$$y^2 = (x-a_1)(x-a_2)(x-a_3)(x-a_4)$$

$$\omega_1 = 2c_4 \int_{a_2}^{a_3} \frac{dx}{y} = 2\mathrm{K}(\lambda), \quad \omega_2 = 2c_4 \int_{a_1}^{a_2} \frac{dx}{y} = 2i\mathrm{K}(1-\lambda),$$

$$\lambda = \frac{a_{14}a_{23}}{a_{13}a_{24}}, \; c_4 = \sqrt{a_{13}a_{24}}, \; a_{ij} = a_i - a_j$$

$\tau = \dfrac{\omega_2}{\omega_1}$ - modulus of the elliptic curve



modular group: $$x \to \frac{ax-b}{cx-d}, \qquad y \to \frac{y}{(cx-d)^2}, \qquad ad - bc = 1.$$

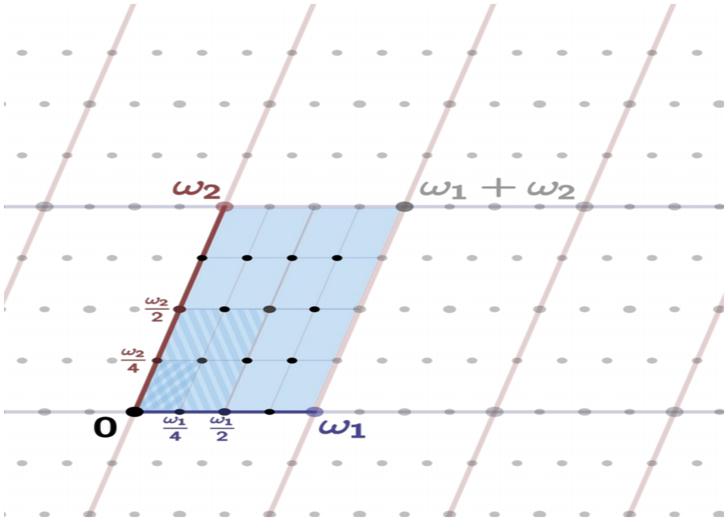
Weierstrass canonical form $$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau), \qquad j(\tau) = 1728\frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2}.$$

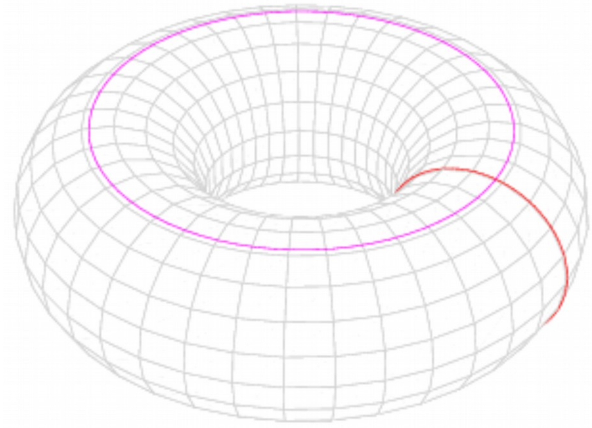
Elliptic functions

Weierstrass elliptic function $$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z+m\omega_1+n\omega_2)^2} - \frac{1}{(m\omega_1+n\omega_2)^2} \right)$$

$$\wp(z+i\omega_1+j\omega_2) = \wp(z), \qquad \wp'(z+i\omega_1+j\omega_2) = \wp'(z), \qquad i,l \in \mathbb{Z}$$

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau)$$

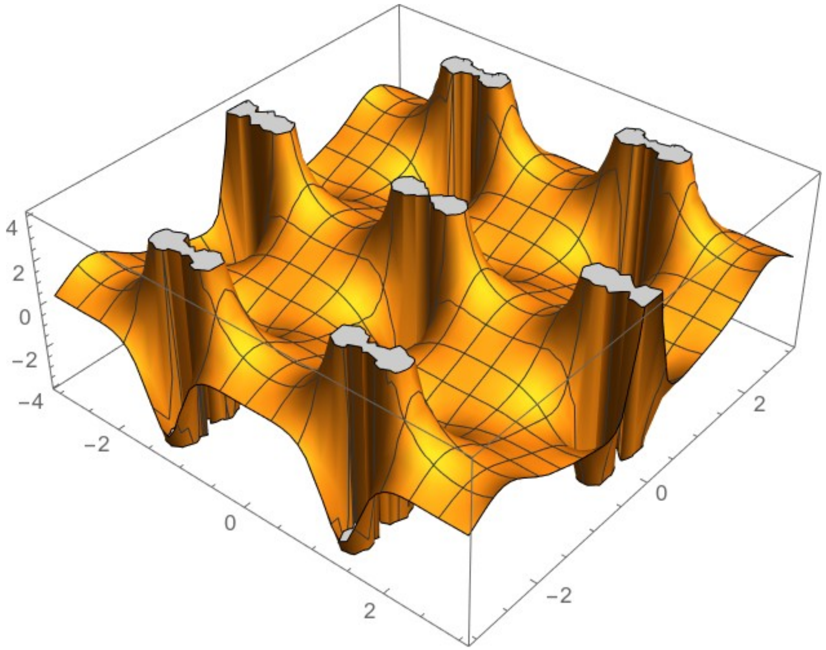




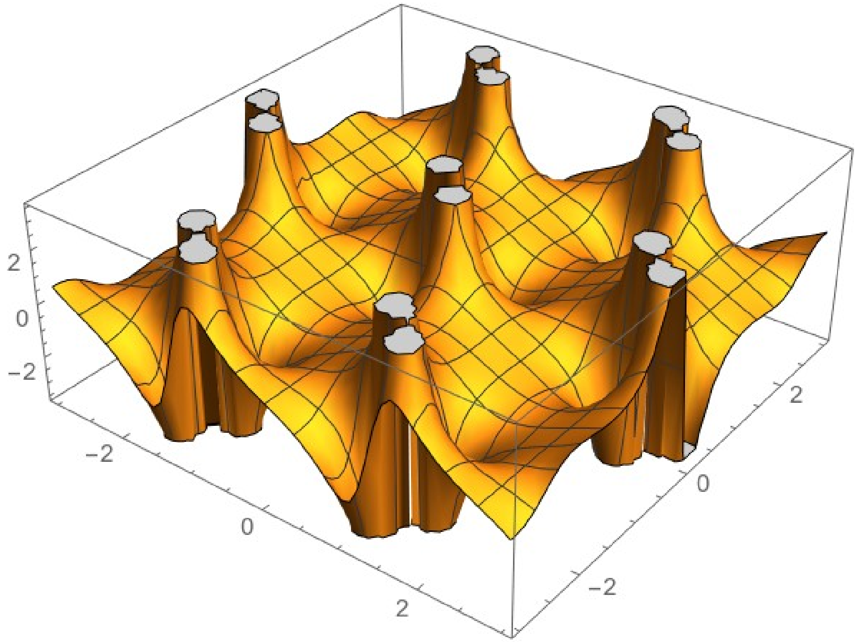
$$\mathbb{C}/\Lambda\ , \qquad \Lambda = \mathbb{Z} + \tau\mathbb{Z} \cong \mathbb{Z}^2$$

Elliptic functions

$\mathrm{Re}\ \wp(z)$



$\mathrm{Im}\ \wp(z)$



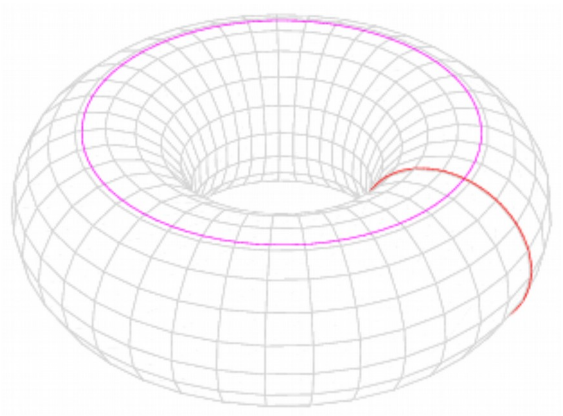
$$\rho_\tau : \mathbb{C}/\Lambda \to \mathrm{E}$$

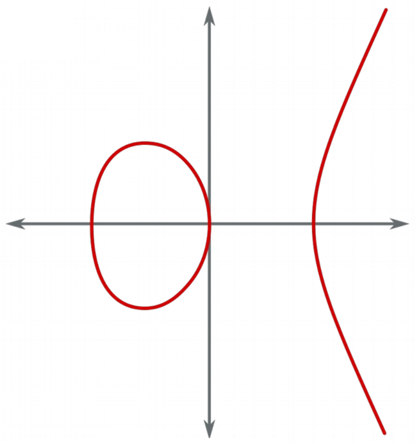
$$z_c \mapsto (x,y) \equiv (\rho_\tau(z_c), c_4\rho'_\tau(z_c)),$$

$$\left(c_4\rho'_\tau(z_c)\right)^2 = (\rho_\tau(z_c)-a_1)(\rho_\tau(z_c)-a_2)(\rho_\tau(z_c)-a_3)(\rho_\tau(z_c)-a_4).$$

Torus coordinates

$$\rho_\tau(z_c) = R_1(\wp_\tau(z_c)) + R_2(\wp_\tau(z_c))\wp'_\tau(z)$$





$\mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z} + \tau\mathbb{Z} \cong \mathbb{Z}^2$

$$y^2 = (x-a_1)(x-a_2)(x-a_3)(x-a_4)$$

$$\rho_\tau^{-1} : \mathrm{E} \to \mathbb{C}/\Lambda$$

$$z_x = \frac{c_4}{\omega_1}\int_{a_1}^{x} \frac{dx}{y}.$$

Multiple polylogarithms(MPLs)

Definition:

Regularization:

$$G(a_1,...,a_n;x)=\int\limits_0^x \frac{G(a_2,...,a_n;x')}{x'-a_1}dx', \qquad n>0, \qquad G(;x)=1, \qquad G(\vec{0}_n;x)=\frac{\log^n x}{n!}$$

A. B. Goncharov, Mathematical Research Letters 5, 497 (1998).
A. B. Goncharov, arXiv preprint math/0103059 (2001).

MPLs include usual logs $G(a;b)=\log\left(1-\frac{b}{a}\right), \qquad a\neq 0$

MPLs include "classical" polylogs $\mathrm{Li}_n(x)=-G\left(\vec{0}_{n-1},\frac{1}{x};1\right)=\int\limits_0^x \frac{dx'}{x'}\mathrm{Li}_{n-1}(x')$

Closed space under derivatives $dG(a_1,...,a_n;x)=\sum_{i=1}^{n} G(a_1,...,a_{i-1},a_{i+1},...,a_n;x)d\log\left(\frac{a_{i-1}-a_i}{a_{i+1}-a_i}\right)$

Closed space under primitives $R(x)G(\vec{a};x)$

MPLs form a shuffle algebra: $G(\vec{v},x)G(\vec{u},x)=\sum_{\vec{c}=\vec{v}\sqcup\vec{u}} G(\vec{c},x)$

MPLs form a Hopf algebra

eMPLs as iterated integrals on a torus

$$\Gamma\left(\begin{smallmatrix} n_1,\dots,n_k \\ z_1,\dots,z_k \end{smallmatrix};z;\tau\right)=\int_0^z dz' f^{(n_1)}(z'-z_1,\tau)\Gamma\left(\begin{smallmatrix} n_2,\dots,n_k \\ z_2,\dots,z_k \end{smallmatrix};z';\tau\right),$$

z - torus coordinates

f kernels properties

Complete elliptic function with two periods

<u>Non-elliptic function but with only one simple pole</u>

eMPLs as iterated integrals on a torus

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1,\ldots,n_k \\ z_1,\ldots,z_k \end{smallmatrix};z;\tau\right)=\int_0^z dz' g^{(n_1)}(z'-z_1,\tau)\tilde{\Gamma}\left(\begin{smallmatrix} n_2,\ldots,n_k \\ z_2,\ldots,z_k \end{smallmatrix};z';\tau\right), \quad k-\text{length and } \sum_i n_i - \text{the weight}$$

Eisenstein-Kronecker series: $F(z,\alpha,\tau)=\frac{1}{\alpha}\sum_{n\geq 0} g^{(n)}(z,\tau)\alpha^n = \frac{\theta_1'(0,\tau)\theta_1(z+\alpha,\tau)}{\theta_1(z,\tau)\theta_1(\alpha,\tau)}$

$$\theta_1(z,q)=\sum_{n=-\infty}^{\infty}(-1)^{n-\frac{1}{2}}q^{\left(n+\frac{1}{2}\right)^2}e^{i(2n+1)z} \quad - \text{ odd Jacobi theta function}$$

$$g^{(n)}(-z,\tau)=(-1)^n g^{(n)}(z,\tau),$$

$$g^{(n)}(z+1,\tau)=g^{(n)}(z,\tau), \qquad g^{(n)}(z+\tau,\tau)=\sum_{k=0}^{n}\frac{(-2\pi i)^k}{k!}g^{(n-k)}(z,\tau).$$

$g^{(1)}(z,\tau)$ - has a simple pole at a point $z=0$ with a residue 1

Pure eMPLs as iterated integrals

$$\mathcal{E}_4\left(\begin{smallmatrix} n1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix};x;\vec{a}\right)=\int_0^x dx'\Psi_{n_1}(c_1,x',\vec{a})\mathcal{E}_4\left(\begin{smallmatrix} n2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix};x';\vec{a}\right), \quad k-\text{length and } \sum_i |n_i| - \text{the weight}$$

$$dx\Psi_{\pm n}(c,x,\vec{a}) = dz_x\left[g^{(n)}(z_x-z_c,\tau) \pm g^{(n)}(z_x+z_c,\tau) - \delta_{\pm n,1}\left(g^{(1)}(z_x-z_*,\tau)+g^{(1)}(z_x+z_*,\tau)\right)\right]$$

the same class of functions but with a different basis, more conveniently expressed in coordinates x and y

Pure eMPLs as iterated integrals

in x,y coordinates:

$$\Psi_0(0,x,\vec{a}) = \frac{c_4}{\omega_1 y}$$

$$\Psi_1(c,x,\vec{a}) = \frac{1}{x-c}, \qquad \Psi_{-1}(c,x,\vec{a}) = \frac{y(c)}{y(x-c)} + Z_4(c,\vec{a}), \qquad c \neq \infty,$$

$$\Psi_1(\infty,x,\vec{a}) = -Z_4(x,\vec{a})\frac{c_4}{y}, \qquad \Psi_{-1}(\infty,x,\vec{a}) = \frac{x}{y} - \frac{a_1 + 2c_4 G_*(\vec{a})}{y}.$$

$$Z_4(x,\vec{a}) = -\frac{1}{\omega_1}\left(g^{(1)}(z_x - z_*,\tau) + g^{(1)}(z_x + z_*,\tau)\right), \qquad G_*(\vec{a}) = \frac{g^{(1)}(z_*,\tau)}{\omega_1}$$

Properties of pure eMPLs

Purity, function is called pure if it is unipotent and its total differential involves only pure functions and one-forms with at most logarithmic singularities

Ordinary MPLs are a subset of eMPLs $\mathcal{E}_4\left(\begin{smallmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_n \end{smallmatrix}; x; \vec{a}\right) = G(a_1, ..., a_n; x)$

Rescaling of arguments $\mathcal{E}_4\left(\begin{smallmatrix} n_1, \dots, n_k \\ pc_1, \dots, pc_k \end{smallmatrix}; px; p\vec{a}\right) = \mathcal{E}_4\left(\begin{smallmatrix} n_1, \dots, n_k \\ c_1, \dots, c_k \end{smallmatrix}; x; \vec{a}\right)$

eMPLs form a shuffle algebra $\mathcal{E}_4(\vec{V}; x; a)\mathcal{E}_4(\vec{U}; x; a) = \sum_{\vec{C}=\vec{V} ⧢ \vec{U}} \mathcal{E}_4(\vec{C}; x; a).$

eMPLs form a Hopf algebra

Motivation, Feynman integrals calculs

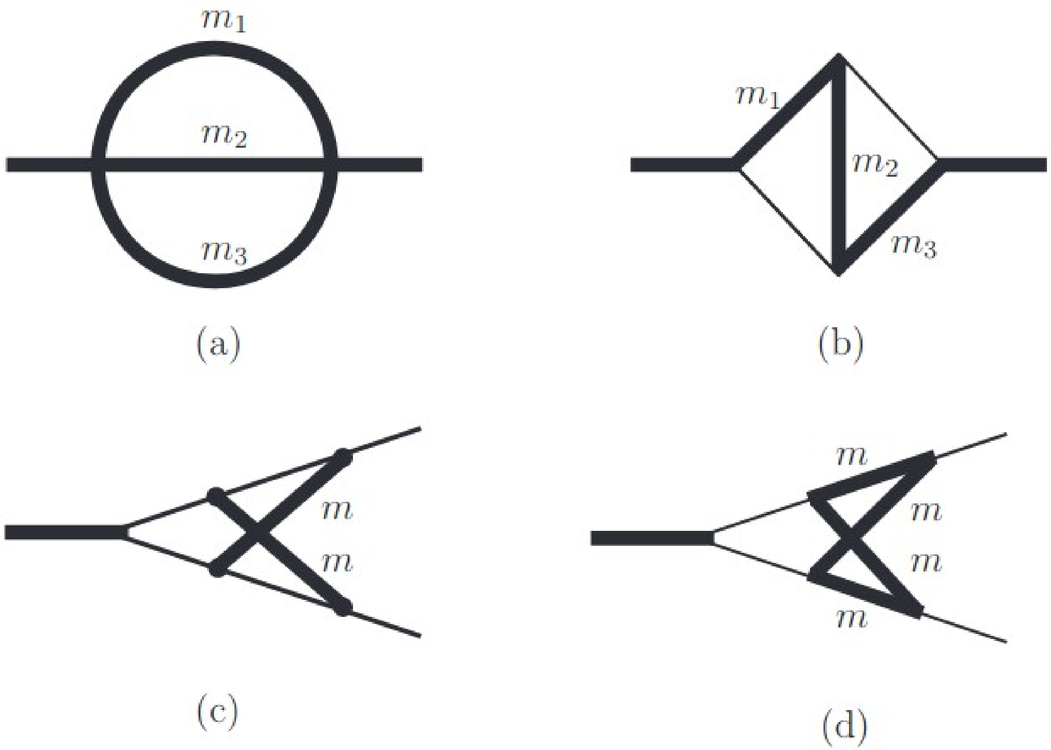
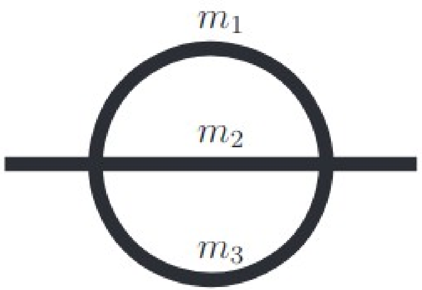


Figure 1. The collection of two-loop Feynman integrals of uniform weight that we have evaluated analytically. Thick lines denote massive propagators.

Motivation, Feynman integrals calculs

Sunset Integral



$$m = m_1 = m_2 = m_3$$

$$S_1(p^2, m^2) = -\frac{\omega_1}{(p^2+m^2)\,c_4}\,T_1(p^2,m^2)\,,$$

with

$$T_1(p^2,m^2) = \left(\frac{m^2}{-p^2}\right)^{-2\epsilon}\left[T_1^{(0)} + \epsilon\,T_1^{(1)} + \mathcal{O}(\epsilon^2)\right]\,,$$

and

$$\begin{aligned} T_1^{(0)} &= 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & \infty \end{smallmatrix};1,\vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix};1,\vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 \\ 0 & 1 \end{smallmatrix};1,\vec{a}\right)\,, \\ T_1^{(1)} &= -4\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_3 & \infty \end{smallmatrix};1,\vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_1 & \infty \end{smallmatrix};1,\vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_4 & \infty \end{smallmatrix};1,\vec{a}\right) - 4\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_2 & \infty \end{smallmatrix};1,\vec{a}\right) \\ &\quad - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_3 & 0 \end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_3 & 1 \end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_1 & 0 \end{smallmatrix};1,\vec{a}\right) - 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & a_1 & 1 \end{smallmatrix};1,\vec{a}\right) + \ldots \end{aligned}$$

where the vector of branch points $\vec{a}$ is

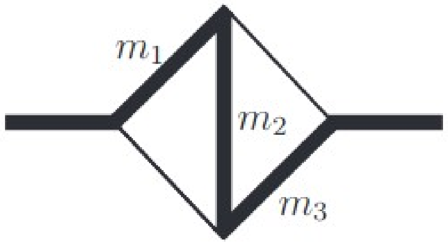
$$\vec{a} = \left(\frac{1}{2}(1+\sqrt{1+\rho}), \frac{1}{2}(1+\sqrt{1+\overline{\rho}}), \frac{1}{2}(1-\sqrt{1+\rho}), \frac{1}{2}(1-\sqrt{1+\overline{\rho}})\right)\,.$$

with

$$\rho = -\frac{4m^2}{(m+\sqrt{-p^2})^2} \quad\text{and}\quad \overline{\rho} = -\frac{4m^2}{(m-\sqrt{-p^2})^2}\,.$$

Motivation, Feynman integrals calculs

Kite Integral



$$m = m_1 = m_2 = m_3$$

$$K(p^2, m^2) = \frac{1}{m^4}\frac{1}{z}\left[K_0(z) + \mathcal{O}(\epsilon)\right]$$

we find for the first order in the ϵ-expansion

$$\begin{aligned}
K_0(z) = \frac{1}{6}\Big[&- 9\mathcal{E}_4\left(\begin{smallmatrix} -1 & -1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}; 1, \vec{a}\right) - 9\mathcal{E}_4\left(\begin{smallmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \end{smallmatrix}; 1, \vec{a}\right) - 18\mathcal{E}_4\left(\begin{smallmatrix} -1 & -1 & 1 \\ 0 & \infty & 1 \end{smallmatrix}; 1, \vec{a}\right) - 9\mathcal{E}_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \end{smallmatrix}; 1, \vec{a}\right) \\
&- 9\mathcal{E}_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ 0 & 1 & 1 \end{smallmatrix}; 1, \vec{a}\right) - 18\mathcal{E}_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ 0 & 1 & \infty \end{smallmatrix}; 1, \vec{a}\right) + 3\mathcal{E}_4\left(\begin{smallmatrix} -1 & -1 & 1 \\ \infty & 0 & 1 \end{smallmatrix}; 1, \vec{a}\right) + 3\mathcal{E}_4\left(\begin{smallmatrix} -1 & -1 & 1 \\ \infty & 1 & 1 \end{smallmatrix}; 1, \vec{a}\right) \\
&+ 6\mathcal{E}_4\left(\begin{smallmatrix} -1 & -1 & 1 \\ \infty & \infty & 1 \end{smallmatrix}; 1, \vec{a}\right) + 3\mathcal{E}_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ \infty & 1 & 0 \end{smallmatrix}; 1, \vec{a}\right) + 3\mathcal{E}_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ \infty & 1 & 1 \end{smallmatrix}; 1, \vec{a}\right) + 6\mathcal{E}_4\left(\begin{smallmatrix} -1 & 1 & -1 \\ \infty & 1 & \infty \end{smallmatrix}; 1, \vec{a}\right) \\
&+ 3\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \end{smallmatrix}; 1, \vec{a}\right) + 3\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \end{smallmatrix}; 1, \vec{a}\right) + 6\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & 0 & \infty \end{smallmatrix}; 1, \vec{a}\right) - 6\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \end{smallmatrix}; 1, \vec{a}\right) \\
&- 6\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \end{smallmatrix}; 1, \vec{a}\right) - 12\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & 1 & \infty \end{smallmatrix}; 1, \vec{a}\right) - 3\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & \infty & 0 \end{smallmatrix}; 1, \vec{a}\right) - 3\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & \infty & 1 \end{smallmatrix}; 1, \vec{a}\right) \\
&- 6\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ 0 & \infty & \infty \end{smallmatrix}; 1, \vec{a}\right) - 6\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ \xi & 0 & 0 \end{smallmatrix}; 1, \vec{a}\right) - 6\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ \xi & 0 & 1 \end{smallmatrix}; 1, \vec{a}\right) - 12\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ \xi & 0 & \infty \end{smallmatrix}; 1, \vec{a}\right) \\
&+ 3\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ \xi & 1 & 0 \end{smallmatrix}; 1, \vec{a}\right) + 3\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ \xi & 1 & 1 \end{smallmatrix}; 1, \vec{a}\right) + 6\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ \xi & 1 & \infty \end{smallmatrix}; 1, \vec{a}\right) \\
&+ 9\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ \xi & \infty & 0 \end{smallmatrix}; 1, \vec{a}\right) + 9\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ \xi & \infty & 1 \end{smallmatrix}; 1, \vec{a}\right) + 18\mathcal{E}_4\left(\begin{smallmatrix} 1 & -1 & -1 \\ \xi & \infty & \infty \end{smallmatrix}; 1, \vec{a}\right) \\
&+ 6\mathcal{E}_4\left(\begin{smallmatrix} 1 & 1 & 1 \\ \xi & 0 & 0 \end{smallmatrix}; 1, \vec{a}\right) - 2\pi^2 G(0; z) + \pi^2 G(1; z) - 3G(0,0,0; z) + 6G(0,1,0; z) \\
&- 12G(0,1,1; z) + 3G(1,0,0; z) + 6G(1,0,1; z) + 27\zeta_3\Big] \\
&+ 2\pi i\Big[2\mathcal{E}_4\left(\begin{smallmatrix} 1 & 0 & -1 \\ \xi & 0 & \infty \end{smallmatrix}; 1, \vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & 0 & -1 \\ \xi & 0 & 0 \end{smallmatrix}; 1, \vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix} 1 & 0 & -1 \\ \xi & 0 & 1 \end{smallmatrix}; 1, \vec{a}\right) + 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & \infty & 1 \end{smallmatrix}; 1, \vec{a}\right) \\
&\qquad + 2\mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 1 & \infty \end{smallmatrix}; 1, \vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}; 1, \vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \end{smallmatrix}; 1, \vec{a}\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \end{smallmatrix}; 1, \vec{a}\right) \\
&\qquad + \mathcal{E}_4\left(\begin{smallmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \end{smallmatrix}; 1, \vec{a}\right)\Big]
\end{aligned}$$

Thank you for your attention!