PERIODS AND FEYNMAN AMPLITUDES

FRANCIS BROWN

arXiv:1512.09265 [math-ph]

References

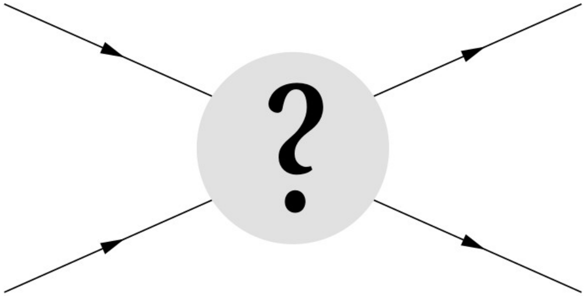
[1] **F. Brown**: Motivic periods and the cosmic Galois group (video lectures at the IHES), May 2015, https://www.youtube.com/watch?v=PlJIECqRZRA

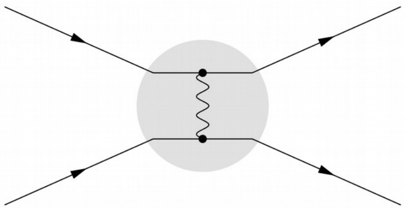
[2] **F. Brown**: Feynman integrals and cosmic Galois group, arXiv:1512.06409 (2015)

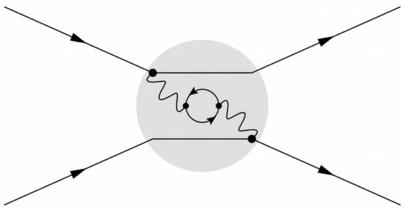
[3] **F. Brown**: Notes on motivic periods, arXiv:1512.06410 (2015)

Speaker: Maxim Bezuglov

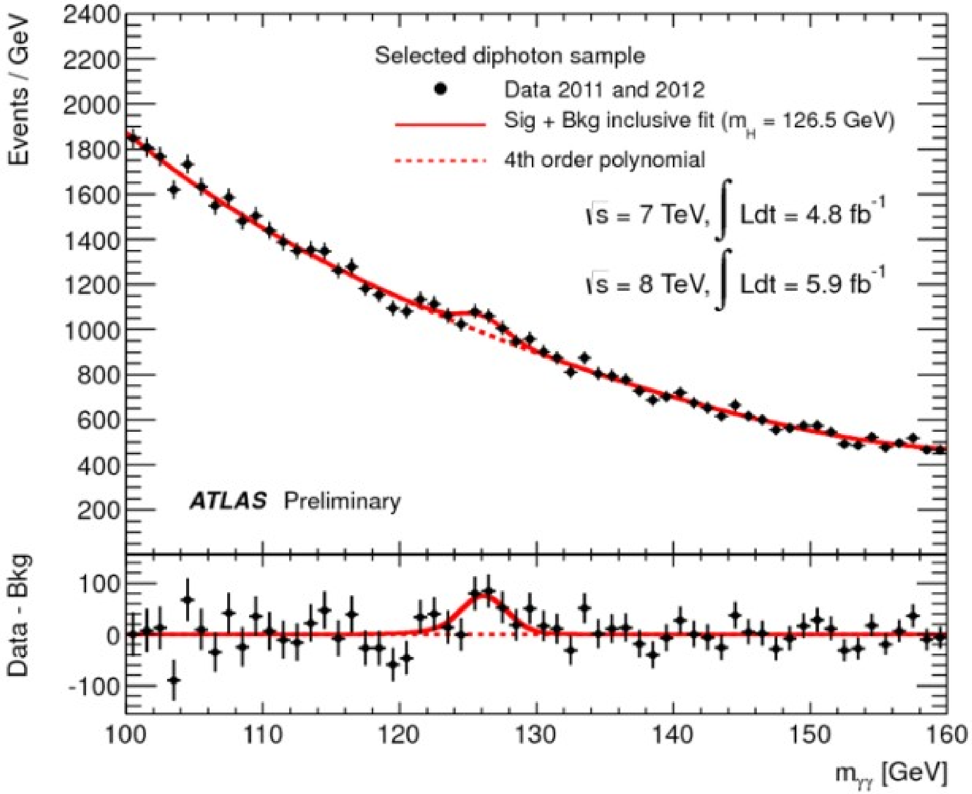
Motivation







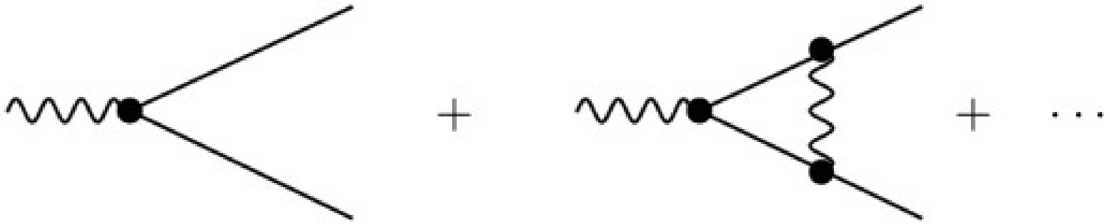
Higgs boson



https://atlas.cern/updates/feature/higgs-boson

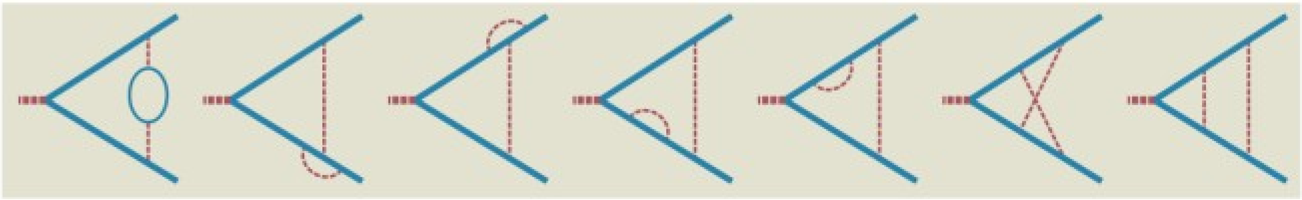
Motivation

Anomalios magnetic moment of the electron



$$\frac{g-2}{2} = 1.00115965218091(\pm 26) \qquad \text{(experiment)}$$

$$\frac{g-2}{2} = 1.00115965218113(\pm 86) \qquad \text{(theoretical)}$$



$$\frac{197}{144} + \frac{1}{2}\zeta(2) - 3\,\zeta(2)\log 2 + \frac{3}{4}\zeta(3)$$

Zeta values

Amplitudes often involve values of the zeta function

$$\zeta(n) = \sum_{k \geq 1} \frac{1}{k^n} \quad , \quad n \geq 2$$

Euler in the 1740's showed that the even zeta values are all rational multiples of powers of π

$$\zeta(2) = \frac{\pi^2}{6} \quad , \quad \zeta(4) = \frac{\pi^4}{90} \quad , \quad \zeta(6) = \frac{\pi^6}{945} \quad , \ldots$$

However, no such formula is known for the odd zeta values and it is conjectured that the ζ(2n + 1), for n ≥ 1, should be algebraically independent of π
but it is not even known at present whether ζ(3)π^−3 is irrational

multiple zeta values (MZV's) $\zeta(n_1, \ldots, n_r) = \sum_{0<k_1<\ldots<k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}} \quad , \quad n_r \geq 2$

Amplitudes in parametric form

Graph: $G = (V, E, E_{ex})$

external edges

vertexes   internal edges

$$\Psi_G = \sum_{T \subset G} \prod_{e \notin E(T)} \alpha_e$$

$$\Phi_G(q) = \sum_{T_1 \cup T_2 \subset G} (q^{T_1})^2 \prod_{e \notin E(T_1) \cup E(T_2)} \alpha_e$$

Symanzik polynomials

T- spanning tree and $T_{1,2}$-spanning 2-trees

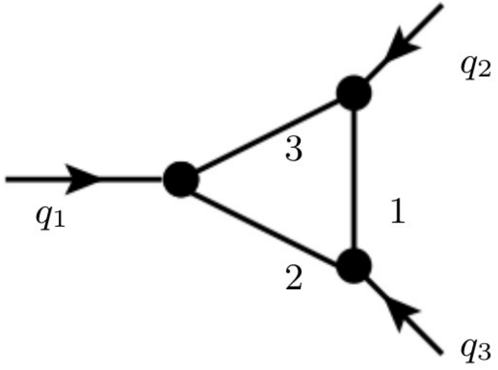
$$\Xi_G(q, m) = \Phi_G(q) + \Big(\sum_{e \in E(G)} m_e^2 \alpha_e \Big) \Psi_G$$

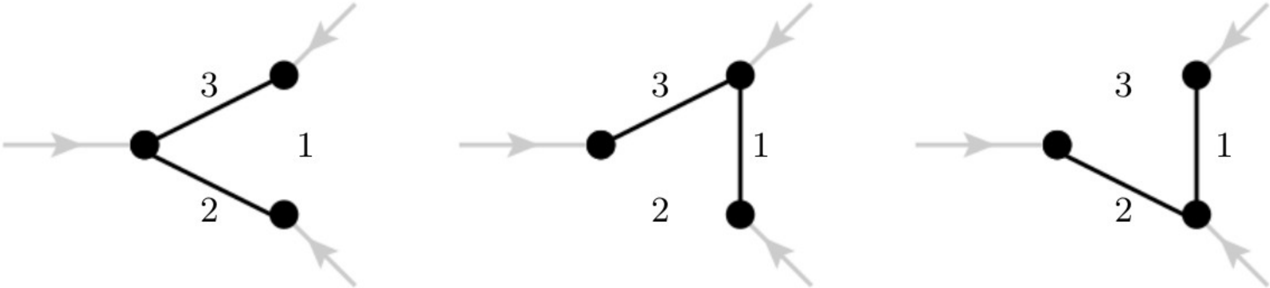
Generalized Feynman amplitude

$$\int_{0 < \alpha_e < \infty} \frac{P(\alpha_e)}{(\Psi_G)^A (\Xi_G(q, m))^B} \Omega_G$$

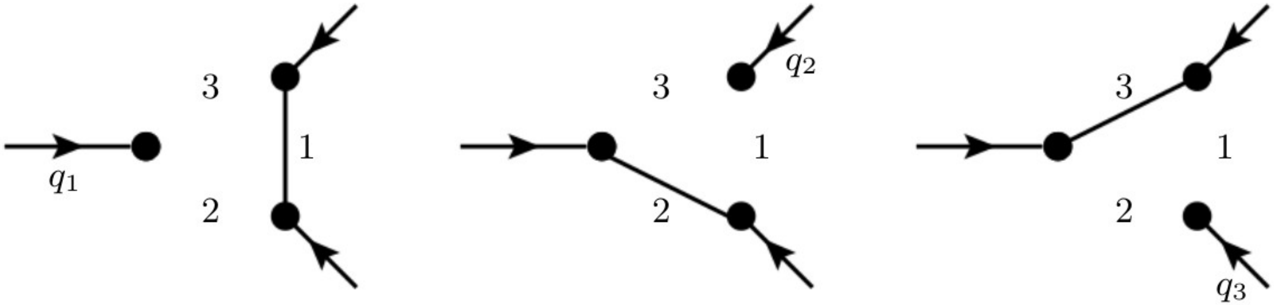
$$\Omega_G = \sum (-1)^e \alpha_e d\alpha_1 \ldots \widehat{d\alpha_e} \ldots d\alpha_{N_G}$$

Example





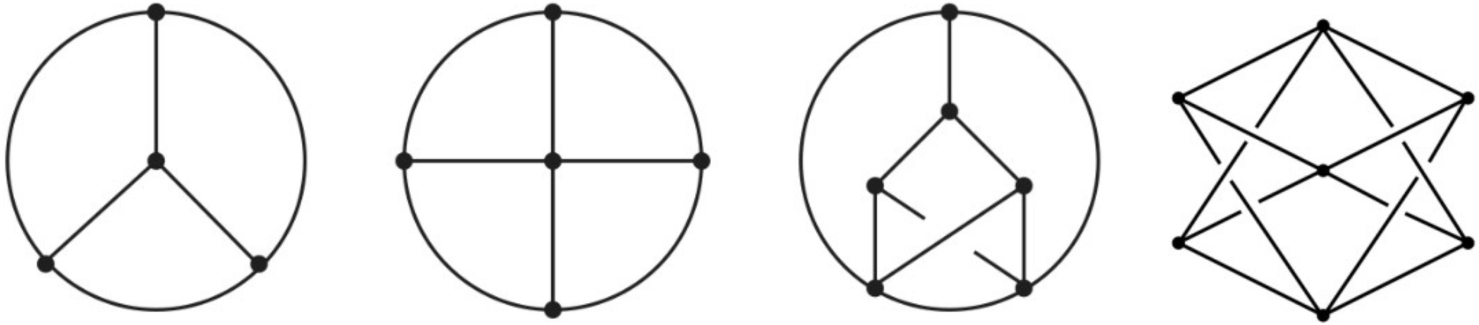
and so $\Psi_G = \alpha_1 + \alpha_2 + \alpha_3$. It has the following spanning 2-trees:



which gives $\Phi_G(q) = q_1^2\alpha_2\alpha_3 + q_2^2\alpha_1\alpha_3 + q_3^2\alpha_1\alpha_2$. Therefore in this case,

$$\Xi_G(q,m) = q_1^2\alpha_2\alpha_3 + q_2^2\alpha_1\alpha_3 + q_3^2\alpha_1\alpha_2 + (m_1^2\alpha_1 + m_2^2\alpha_2 + m_3^2\alpha_2)(\alpha_1 + \alpha_2 + \alpha_3) \ .$$

Example



$I_G:\quad 6\zeta(3) \qquad 20\zeta(5) \qquad 36\zeta(3)^2 \qquad 32\, P_{3,5}$

$$P_{3,5} = -\frac{216}{5}\zeta(3,5) - 81\zeta(5)\zeta(3) + \frac{522}{5}\zeta(8)$$

Where do we go from here?

Are there any organising principles which are valid for all Feynman amplitudes?

There should be a very large group of deeply hidden symmetries which acts on the space of all generalised Feynman amplitudes in a rather subtle way. This group will propagate information between amplitudes of different loop orders.

Periods and families of periods

$$I = \int_\sigma \frac{P}{Q}\, dx_1 \ldots dx_n \ , \qquad P, Q \in \mathbb{Q}[x_1, \ldots, x_n]$$

we call I a family of periods when P, Q or σ depend algebraically on parameters.

Examples of periods

- All algebraic numbers, such as

$$\sqrt{2} = \int_{x^2 \leq 2} \frac{dx}{2} \ .$$

- Many classical transcendental numbers such as

$$\zeta(2) = \int_{0 \leq x, y \leq 1} \frac{dxdy}{1 - xy} \ .$$

- Many classical functions such as

$$\log y = \int_{1 \leq x \leq y} \frac{dx}{x} \ .$$

- Convergent generalised Feynman amplitudes

Galois theory

There is a symmetry group $\mathrm{Gal}\left(\bar{\mathbb{Q}}/\mathbb{Q}\right)$ which is the group of symmetries of the ring of algebraic numbers which respects the operations of addition and multiplication

Y. André has suggested that Galois theory can be generalized to all periods

arXiv:0805.2569 [math.NT]

arXiv:0805.2568 [math.GM]

A fundamental problem with this programme is that it is impossibly conjectural.

Author claim that we can circumvent all conjectures in certain specific situations using a notion called motivic periods, and that this works in the case of Feynman amplitudes.

The motivic period corresponding to 2πi

$$2\pi i = \int_\gamma \frac{dx}{x}$$

Up to an exact differential

$$\left[\frac{dx}{x}\right] \in \frac{\text{Closed } algebraic \text{ 1-forms}}{\text{Exact } algebraic \text{ 1-forms}} = H^1_{dR} \cong \mathbb{Q}[\tfrac{dx}{x}]$$

$$[\gamma] \in \frac{\text{Closed 1-chains}}{\text{Boundaries of 2-chains}} = (H^1_B)^\vee \cong \mathbb{Q}[\gamma]$$

motivic version of 2πi

$$(2\pi i)^{\mathfrak{m}} := \left[(H^1_B, H^1_{dR}, \text{comp}), [\gamma], [\tfrac{dx}{x}]\right)\right] \qquad \text{per}((2\pi i)^{\mathfrak{m}}) = [\gamma]\big(\text{comp}\,[\tfrac{dx}{x}]\big) = 2\pi i$$

Galois group $\mathcal{G}$ of motivic periods as the group of symmetries of their defining data

$$(2\pi i)^{\mathfrak{m}} \mapsto \lambda(2\pi i)^{\mathfrak{m}} \qquad \text{for some } \lambda \in \mathbb{Q}^\times$$

Motivic periods corresponding to Feynman amplitudes

Theorem 5.1. *For every Feynman graph G, and any convergent generalised amplitude of G, we can* ***canonically*** *define the corresponding motivic period. Let $\mathcal{FP}^{\mathfrak{m}}_{Q,M}$ be the vector space of all motivic periods*[26] *of Feynman graphs of type (Q, M). It admits an action of a group $\mathcal{C}_{Q,M}$, the* cosmic Galois group. *For every convergent Feynman amplitude I_G we have a motivic version*

$$I^{\mathfrak{m}}_G \in \mathcal{FP}^{\mathfrak{m}}_{Q,M} \quad \textit{satisfying} \quad \mathrm{per}\,(I^{\mathfrak{m}}_G) = I_G \ .$$

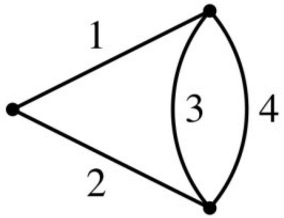
$$W_k \mathcal{FP}^{\mathfrak{m}}_{Q,M} = \langle \text{motivic periods of weight } \leq k \rangle$$

Theorem 5.2. *For every (Q, M), the space $W_k \mathcal{FP}^{\mathfrak{m}}_{Q,M}$ is finite-dimensional.*

Partial factorization of graph polynomials

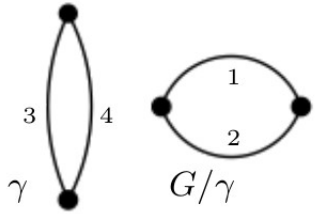
$$\gamma \subset G \qquad\qquad \Psi_G = \Psi_\gamma \Psi_{G/\gamma} + R_{\gamma,G}$$

Polinomial of higher degree in the edge variables corresponding to the subgraph

$G =$ 

$$\Psi_G = \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4$$

The following choice of subgraph gives one partial factorisation:



$$\Psi_G = \underbrace{(\alpha_3 + \alpha_4)}_{\Psi_\gamma}\underbrace{(\alpha_1 + \alpha_2)}_{\Psi_{G/\gamma}} + \underbrace{\alpha_3\alpha_4}_{R_{\gamma,G}}$$

Conclusion

We saw that Feynman integrals and amplitudes form the basis for most predictions in high-energy physics experiments. They are very far from being understood mathematically, and there are numerous challenges with practical applications. Some problems which were not touched upon here are questions about resummability and existence of renormalisable quantum field theories.

Grothendieck's deep ideas on motives suggest that there exists a huge symmetry group (motivic Galois group) acting on period integrals. This is still highly conjectural, but we can define motivic periods of graphs unconditionally.

As a result, we can define a group of hidden symmetries (called the cosmic Galois group) which acts on motivic periods of graphs. It provides an organising principle for the structure of amplitudes and leads to powerful constraints, via the stability theorem, to all orders in perturbation theory. We are only just beginning to scratch the surface of this structure.

Thank you for your attention!