ISSN 1547-4771, Physics of Particles and Nuclei Letters, 2021, Vol. 18, No. 6, pp. 629–639. © Pleiades Publishing, Ltd., 2021.
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Thermodynamically Consistent Description of One-Phonon States Fragmentation in Hot Nuclei

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Received June 4, 2021; revised June 16, 2021; accepted June 16, 2021

Abstract—The fragmentation of one-phonon states in hot nuclei is studied. For this purpose, the quasiparticle-phonon nuclear model is extended to a finite temperature by applying the formalism of thermo field dynamics. It is shown that consistent application of the thermal state condition leads to the realization of the detailed balance principle at each stage of the thermal Hamiltonian diagonalization. The equations describing the coupling between thermal one-phonon and two-phonon states are derived.

DOI: 10.1134/S1547477121060054

1. INTRODUCTION

Collective excitations in hot nuclei have been actively studied since the early 1980s. For a long time, both experimentalist and theorist efforts were focused on elucidating the reasons for the changes in giant isovector dipole resonance properties with increasing nuclear excitation energy (see review articles [1, 2] and reference therein). However, hot nuclei play an important role in numerous astrophysical processes as well. Specifically, in collapsing supernovae, where a temperature of the stellar matter is about 10^{10}K (0.86 MeV), reactions mediated by weak interaction (electron captures, beta-decays, neutrino scattering etc.) and with involving of highly excited nuclei strongly affect both the collapse dynamics and nucleosynthesis occurs during it [3–5].

Progress in theoretical studies of hot nuclei has been due to generalization to finite temperatures the methods which are widely used for "cold" nuclei. For example, nuclear field theory and finite Fermi systems theory have been extended on finite temperatures applying formalism of thermal Green's functions in [6] and [7, 8], respectively. In works [9, 10] an attempt has been made to generalize to finite temperatures the quasiparticle-phonon model [11]. In this case, the formalism of thermo field dynamics (TFD) [12, 13] has been used. Afterwards this approach was partially revised and rethought in [14, 15]. One more approach in the field using the Green's function formalism and time-blocking approximation is actively developed and applied in [16, 17]. Fundamental problem to be solved in these studies is to understand the mechanism of damping of collective state in hot nuclei and to learn how to calculate at $T \neq 0$ a coupling of collective states with complex configurations which is responsible for the fragmentation of resonance strength in "cold" nuclei.

It is worth mentioning that the fragmentation of nuclear excitation strength is very important aspect in nuclear structure theory applications for astrophysical purposes. The shell model calculations have shown that under certain conditions the cross sections and rates of weak processes involving atomic nuclei which take place in hot and dense stellar matter depend significantly on the details of nuclear excitations strength functions [18, 19]. However, although the contemporary shell model calculations well reproduce the experimental strength functions built on the ground states of nuclei with mass numbers $A \simeq 50–70$, such calculations for highly-excited or much heavier nuclei are not feasible. That is why while calculating the cross sections and rates of weak processes with hot nuclei one is forced to use some approximations, such as Axel-Brink hypothesis or so called "back-resonance" method allowing to consider transitions with de-excitation of a hot nucleus.

In [20–23, 27], to calculate cross sections and rates of weak-interaction-mediated processes with hot nuclei in stellar environments, the approach based on the thermal extension of the quasiparticle random phase approximation (TQRPA) within the TFD formalism was developed. The essential advantage of TQRPA consists in allowing to calculate thermal strength functions in a thermodynamically consistent way and guarantees fulfillment of the detailed balance principle (DBP) in contrast with formally similar approaches of Refs. [24–26]. Comparing the TQRPA results with those of shell model calculations as well as with results of just mentioned works [24–26] have shown the enhancement of thermal impact on strength functions of multipole transitions. The consequence is the stronger temperature dependence of weak-process cross sections and rates calculated in works [20–23, 27].

The goal of the present paper is to go beyond TQRPA and include the fragmentation of one-phonon states in the calculations of thermal strength functions of multipole transitions. To this aim we again use the TFD formalism and the ideas of the quasiparticle-phonon model (QPM) [11]. The QPM has been formulated for "cold" nuclei. Its essence is to consider the interaction of simple states, one-quasiparticle, or one-phonon, with more complicated ones. The latter is achieved by including more complex configurations in the wave functions of nuclear excited states. Here we use this idea but for hot nuclei. It should be mentioned that the same trick was used in the previous attempts to generalize the QPM on finite temperatures [9, 14, 15]. However, in the cited works the problem of the DBP fulfillment was missed and its demands were fully or partially violated.

2. DERIVING STRENGTH FUNCTIONS IN THE FORMALISM OF THERMO FIELD DYNAMICS

Let us consider the strength function of arbitrary operator $\mathcal{T}$, describing the influence of an external perturbation on a nucleus. In the case of a hot nucleus the strength function contains averaging over all possible thermally excited initial states

$$S_{\mathcal{T}}(E,T) = \sum_{if} p_i(T) B_{if}(\mathcal{T}) \delta(E - E_{if}). \qquad (1)$$

Here $p_i(T) = e^{-E_i/T} \big/ Z(T)$ is excitation probability of state i with energy E_i; $B_{if}(\mathcal{T}) = |\langle f|\mathcal{T}|i\rangle|^2$ is strength (probability) of transition $i \to f$; $E_{if} = E_f - E_i$ is transition energy. At $T \neq 0$ the strength function is defined for both positive and negative values of E. In the first case, energy is transferred to a hot nucleus, in the second case, a hot nucleus de-excites. Since $B_{if}(\mathcal{T}) = B_{fi}(\mathcal{T}^\dagger)$, the strength function satisfies the detailed balance principle

$$S_{\mathcal{T}^\dagger}(-E,T) = e^{-E/T} S_{\mathcal{T}}(E,T). \qquad (2)$$

Due to completeness of the eigenstates of the nuclear Hamiltonian H, the strength function (1) can be written as the Fourier transform of the time correlation function of the operator $\mathcal{T}$

$$S_{\mathcal{T}}(E,T) = \int \frac{dt}{2\pi} e^{iEt} \langle\langle \mathcal{T}^\dagger(t)\mathcal{T}(0)\rangle\rangle, \qquad (3)$$

where $\langle\langle ... \rangle\rangle$ means the statistical average, and $\mathcal{T}(t) = e^{iHt}\mathcal{T}e^{-iHt}$ is the Heisenberg representation of the operator $\mathcal{T}$.

Thermo field dynamics is based on the postulate that the statistical average of an arbitrary operator A can be represented as a matrix element with respect to a temperature dependent state $|0(T)\rangle$, termed the thermal vacuum [12, 13]

$$\langle\langle A \rangle\rangle = \langle 0(T)|A|0(T)\rangle. \qquad (4)$$

The thermal vacuum is a vector in the Hilbert space of twice the dimension than the Hilbert space of the physical system in question. Space extension occurs due to introduction of a fictitious system identical to the considered one. Rigorous substantiation of the Hilbert space expansion in describing heated and non-equilibrium quantum systems can be done within the superoperator formalism [29–31]. Let $H = H(a^\dagger, a)$ be the Hamiltonian of the system considered. Then the Hamiltonian of the fictitious system has the form $\widetilde{H} = H(\tilde{a}^\dagger, \tilde{a})$, where operators associated with the fictitious system are denoted by a "tilde". The relation between the operators from two Hilbert spaces is given by the rules of tilde-conjugation

$$(AB)^{\tilde{}} = \tilde{A}\tilde{B}, \quad (aA + bB)^{\tilde{}} = a^*\tilde{A} + b^*\tilde{B}, (\tilde{A})^{\tilde{}} = A. \qquad (5)$$

It should be noted that the double tilde-conjugation rule, which is the last in the relation set (5) differs from that has been proposed originally in the works [12, 13]. The necessity of this redefinition and the corresponding consequences are discussed in Refs. [15, 32].

For the vector $|0(T)\rangle$ to have the feature (4), it must satisfy so-called thermal state condition

$$A|0(T)\rangle = \sigma_A e^{\mathcal{H}/2T} \tilde{A}^\dagger |0(T)\rangle. \qquad (6)$$

Here, $\sigma_A = 1(-i)$ for a boson-like (fermion-like) operator A, and operator $\mathcal{H} = H - \tilde{H}$ is termed a thermal Hamiltonian. A thermal Hamiltonian is the time-evolution operator in the extended Hilbert space, i.e., $A(t) = e^{i\mathcal{H}t} A e^{-i\mathcal{H}t}$, and its eigenstates constitute the complete set in this space.

Note that each positive-energy eigenstate

$$\mathcal{H}|\mathbb{O}_k\rangle = \mathcal{E}_k|\mathbb{O}_k\rangle \qquad (7)$$

corresponds to a tilde-conjugate eigenstate with negative energy

$$\mathcal{H}|\tilde{\mathbb{O}}_k\rangle = -\mathcal{E}_k|\tilde{\mathbb{O}}_k\rangle. \qquad (8)$$

The thermal vacuum is the eigenvalue of the thermal Hamiltonian with the zero eigenvalue, i.e. $\mathcal{H}|0(T)\rangle = 0$. By acting on both sides of Eq. (6) on the eigenstates of the thermal Hamiltonian, the thermal state condition can be rewritten in the form

$$\langle \mathbb{O}_k|A|0(T)\rangle = \sigma_A e^{\mathcal{E}_k/2T} \langle 0(T)|A|\tilde{\mathbb{O}}_k\rangle, \qquad (9)$$

where the relation $\langle 0(T)|\tilde{A}|\mathbb{O}_k\rangle^* = \langle 0(T)|A|\tilde{\mathbb{O}}_k\rangle$ is considered.

Completeness property of the eigenstate set of the thermal Hamiltonian allows to write the strength function (3) as the following expansion:

$$S_{\mathcal{T}}(E,T) = \sum_k \{B_k(\mathcal{T})\delta(E - \mathcal{E}_k) + \tilde{B}_k(\mathcal{T})\delta(E + \mathcal{E}_k)\}, \qquad (10)$$

where B_k and $\tilde{B}_k$ are probabilities of transitions from the thermal vacuum to the eigenstates of $\mathcal{H}$

$$B_k(\mathcal{T}) = |\langle \mathbb{O}_k | \mathcal{T} | 0(T) \rangle|^2, \quad \tilde{B}_k(\mathcal{T}) = |\langle \tilde{\mathbb{O}}_k | \mathcal{T} | 0(T) \rangle|^2. \tag{11}$$

Transitions from the thermal vacuum to the $|\mathbb{O}_k\rangle$ states of the Hamiltonian correspond energy transfer to the hot nucleus, whereas transitions to tilde-states $|\tilde{\mathbb{O}}_k\rangle$ correspond to de-excitation of the nucleus. Thermal state condition (9) leads to the following relation between probabilities of transitions to tilde-conjugate states:

$$\tilde{B}_k(\mathcal{T}) = e^{-\mathcal{E}_k/T} B_k(\mathcal{T}^\dagger), \tag{12}$$

whence the fulfillment of the DBP follows directly (2). Thus, within TFD the DBP fulfillment is the direct consequence of the thermal state condition (6).

Formally the expression (10) has the form of the strength function at zero temperature, since it contains single summation over eigenstates of (thermal) Hamiltonian, and the role of the ground state is played by the thermal vacuum. The difference is that the thermal Hamiltonian spectrum consists of both the positive and negative eigenvalues, and in principle depends on temperature. The probabilities of transitions from the thermal vacuum state to the thermal Hamiltonian eigenstates are also temperature dependent. Thus, within TFD the problem of calculation of the strength function is reduced to finding the eigenvalues of the thermal Hamiltonian. The TFD advantage is that to diagonalize the thermal Hamiltonian, one can use the same methods as at zero temperature: the independent quasiparticle approximation, the random phase approximation, etc. However, as will be shown below, the demand to satisfy the thermal state condition adds its own specifics to the thermal Hamiltonian diagonalization.

As at $T = 0$, to find the thermal Hamiltonian eigenstates it is convenient to use the equation of motion method [33, 34]

$$\langle 0(T) | [\delta\mathbb{O}, \mathcal{H}, \mathbb{O}_k^\dagger] | 0(T) \rangle = \mathcal{E}_k \langle 0(T) | [\delta\mathbb{O}, \mathbb{O}_k^\dagger] | 0(T) \rangle. \tag{13}$$

Here, $\mathbb{O}_k^\dagger$ is the creation operator of the eigenstate $|\mathbb{O}_k\rangle$, i.e., $|\mathbb{O}_k\rangle = \mathbb{O}_k^\dagger | 0(T)\rangle$. The tilde-conjugate operator $\tilde{\mathbb{O}}_k^\dagger$, such that $|\tilde{\mathbb{O}}_k\rangle = \tilde{\mathbb{O}}_k^\dagger | 0(T)\rangle$, is the solution of (13) corresponding to energy $-\mathcal{E}_k$. Let us define the thermal vacuum as the vacuum for the operators $\mathbb{O}_k$, $\tilde{\mathbb{O}}_k$

$$\mathbb{O}_k | 0(T)\rangle = \tilde{\mathbb{O}}_k | 0(T)\rangle = 0. \tag{14}$$

In contrast with the $T = 0$ case at $T \neq 0$ the solution of equations of motion does not produce the unambiguous determination of the thermal Hamiltonian eigenstates. Indeed, if $\mathbb{O}_k^\dagger$ is a solution of Eq.(13), then, applying the operation of Hermitian and tilde conjugations to both sides of the equation one can show that the operator $\tilde{\mathbb{O}}_k$ is also its solution with the same energy $\mathcal{E}_k$. Hence, an arbitrary linear combination of these operators also satisfies the equation of motion. To determine which solutions are "true" one must select from the whole set of solutions of the equation of motion those which satisfy the thermal state condition (9).

Formally, the equation of motion (13), together with the thermal state condition (9), allows one to find the exact eigenstates of the thermal Hamiltonian. However, in practice, we are looking for a solution to the equation as an expansion in a limited set of some basic operators $\delta\mathbb{O}$. Moreover, the exact solution of Eq. (13) demands to know the thermal vacuum state $|0(T)\rangle$. The latter condition, in principle, can be satisfied solving the equation of motion via the iteration procedure. However, since the double symmetric commutator in the left side of Eq. (13) reduces the sensitivity of the equation solution to the choice of $|0(T)\rangle$ one can use the approximate form of the thermal vacuum instead of the exact one. It is worth recalling that this is exactly how the RPA equations for $T = 0$ are derived when the Hartree–Fock vacuum is used as the ground state instead of the phonon vacuum [34].

Since the solutions of the equation of motion are approximate the thermal state condition (9) can be satisfied for certain operator class A only. Note that if Eq. (9) holds for certain set of operators A_n, then it is also true for their linear combination. Therefore in order for the strength function of one-particle transition operator $\mathcal{T}$ to satisfy the detailed balance principle we will use one- and two-fermion operators $a^\dagger$, a, $a_1^\dagger a_2^\dagger$, $a_1^\dagger a_2$, $a_1 a_2$ as the set A_n.

3. THERMAL QUASIPARTICLES AND PHONONS

Here we use the version of QPM where the nuclear Hamiltonian consists of the proton and neutron mean fields, the pairing BCS interaction, and separable multipole particle-hole interaction

$$H = H_{\rm sp} + H_{\rm pair} + H_{\rm ph}. \tag{15}$$

For spherical nuclei the terms of Hamiltonian H can be written as follows:

$$H_{\rm sp} = \sum_\tau \sum_{jm}^{\tau} (E_j - \lambda_\tau) a_{jm}^\dagger a_{jm}, \tag{16}$$

$$H_{\rm pair} = -\frac{1}{4} \sum_\tau G_\tau \sum_{\substack{j_1 m_1 \\ {}_2 m_2}}^{\tau} a_{j_1 m_1}^\dagger a_{\overline{j_1 m_1}}^\dagger a_{\overline{j_2 m_2}} a_{j_2 m_2} \quad (a_{\overline{jm}} = (-1)^{j-m} a_{j-m}), \tag{17}$$

$$H_{\rm ph} = -\frac{1}{2} \sum_{\lambda\mu} \sum_{\tau\rho=\pm 1} (\kappa_0^{(\lambda)} + \rho\kappa_1^{(\lambda)}) M_{\lambda\mu}^\dagger(\tau) M_{\lambda\mu}(\rho\tau), \tag{18}$$

where $M^{\dagger}_{\lambda\mu}(\tau)$ is the multipole one-particle operator

$$M^{\dagger}_{\lambda\mu}(\tau) = \sum_{\substack{j_1m_1\\ j_2m_2}}^{\tau} \langle j_1m_1 | i^{\lambda} R_{\lambda}(r) Y_{\lambda\mu}(\theta,\varphi) | j_2m_2 \rangle a^{\dagger}_{j_1m_1} a_{j_2m_2}. \quad (19)$$

In Eqs. (16)–(19), the following notations which are standard for QPM are used: the symbol $\sum^{\tau}$ means a summation over proton (neutron) $\tau = p$ ($\tau = n$) single-particle levels with quantum numbers $nljm \equiv jm$ and energy E_j; changing of the sign of τ corresponds to replacing $n \leftrightarrow p$; $\lambda_{n,p}$ are chemical potentials; $G_{n,p}$ is pairing interaction constants; $\kappa^{(\lambda)}_{0,1}$ is constants of isoscalar and isovector interactions of multipolarity λ; the functions $R_{\lambda}(r)$ are the radial formfactors of the separable particle-hole forces. It is assumed that the Hamiltonian parameters are independent on temperature, which is justified at $T \lesssim 5$ MeV [35, 36].

Since the Hamiltonian (15) contains only multipole particle-hole forces, its eigenstates of the multipolarity J correspond to nuclear excited states of normal parity, i.e., $\pi = (-1)^J$. The one-body transition operator to these states written in terms of nucleonic operators of creation and annihilation has the following form:

$$\begin{aligned}\mathcal{T}_{JM} &= \sum_{\tau}\sum_{\substack{j_1m_1\\ j_2m_2}}^{\tau} \langle j_1m_1 | \mathcal{T}_{JM} | j_2m_2 \rangle a^{\dagger}_{j_1m_1} a_{j_2m_2} \\ &= -\hat{J}^{-1} \sum_{\tau}\sum_{j_1j_2}^{\tau} t^{(J)}_{j_1j_2} [a^{\dagger}_{j_1} a_{\bar{j}_2}]_{JM}.\end{aligned} \quad (20)$$

In Eq. (20), $t^{(J)}_{j_1j_2}$ is a reduced matrix element of the transition operator, $\hat{J} = \sqrt{2J+1}$, and $[...]_{JM}$ means a coupling of two angular momenta on the total angular momentum J with the projection M. In what follows, it will be assumed that under Hermitian conjugation the operator $\mathcal{T}_{JM}$ transforms as a multipole operator $\mathcal{T}^{\dagger}_{JM} = (-1)^{J-M}\mathcal{T}_{J-M}$, so that $t^{(J)}_{j_2j_1} = (-1)^{j_1-j_2+J} t^{(J)}_{j_1j_2}$. Then, according to the detailed balance principle (2), the strength function of the operator (20) satisfies the condition

$$S_{\mathcal{T}_J}(-E,T) = e^{-E/T} S_{\mathcal{T}_J}(E,T), \quad (21)$$

connecting probabilities of excitation and deexcitation of a hot nucleus.

The thermal Hamiltonian corresponding to the above QPM Hamiltonian has a form

$$\mathcal{H} = \mathcal{H}_{\rm sp} + \mathcal{H}_{\rm pair} + \mathcal{H}_{\rm ph}. \quad (22)$$

As in the case of zero temperature, the determination of eigenstates of the thermal Hamiltonian (22) starts with the allowance for pair correlations. To this aim the creation operators $\beta^{\dagger}, \tilde{\beta}^{\dagger}$ and annihilation operators $\beta, \tilde{\beta}$ of thermal quasiparticles are introduced. The thermal quasiparticle operators diagonalize the one-body part of the thermal Hamiltonian

$$\begin{aligned}\mathcal{H}_{\rm BCS} &= \mathcal{H}_{\rm sp} + \mathcal{H}_{\rm pair} \\ &\approx \sum_{\tau}\sum_{jm}^{\tau} \varepsilon_j(T)(\beta^{\dagger}_{jm}\beta_{jm} - \tilde{\beta}^{\dagger}_{jm}\tilde{\beta}_{jm}).\end{aligned} \quad (23)$$

The sign of approximate equality in (23) means that the terms for the monopole interaction between thermal quasiparticles are not indicated in the expression for $\mathcal{H}_{\rm BCS}$. The vacuum of thermal quasiparticles $|\varphi_0(T)\rangle$ under the additional condition (9) is a thermal vacuum in the BCS approximation, the values $\pm\varepsilon_j(T)$ are the thermal quasiparticle energies.

The operators of creation and annihilation of thermal quasiparticles relate to the nucleonic operators $a^{\dagger}, a, \tilde{a}^{\dagger}$ and $\tilde{a}$ from the original thermal Hamiltonian (22) by means of two unitary transformations. The first one is the well-known (u, v) Bogoliubov transformation from the nucleonic operators to the quasiparticle operators $(u_j^2 + v_j^2 = 1)$

$$a^{\dagger}_{jm} = u_j\alpha^{\dagger}_{jm} + v_j\alpha_{\overline{jm}}, \quad a_{jm} = u_j\alpha_{jm} + v_j\alpha^{\dagger}_{\overline{jm}}. \quad (24)$$

The similar transformation is performed with the creation and annihilation operators of tilde-particles (nucleons), thereby introducing into consideration the operators of tilde-quasiparticles $\tilde{\alpha}^{\dagger}_{jm}$ and $\tilde{\alpha}_{jm}$. The second, so-called the thermal (x, y)-transformation, mixes the operators of thermal quasiparticles $\beta^{\dagger}, \beta$ and tilde-quasiparticles $\tilde{\beta}^{\dagger}$, $\tilde{\beta}$ $(x_j^2 + y_j^2 = 1)$

$$\alpha^{\dagger}_{jm} = x_j\beta^{\dagger}_{jm} + iy_j\tilde{\beta}_{jm}, \quad \tilde{\alpha}^{\dagger}_{jm} = x_j\tilde{\beta}^{\dagger}_{jm} - iy_j\beta_{jm}. \quad (25)$$

Since the two transformations (24), (25) are unitary, the thermal quasiparticle operators obey the standard anticommutation relations. Note that the use of the double tilde operation in the form (5) leads to a complex thermal transformation (25) [15].

Solution of the corresponding equation of motion

$$\begin{aligned}&\langle \varphi_0(T) | \{\delta\mathbb{O}, \mathcal{H}_{\rm BCS}, \beta^{\dagger}_{jm}\} | \varphi_0(T) \rangle \\ &= \varepsilon_{jm}(T) \langle \varphi_0(T) | \{\delta\mathbb{O}, \beta^{\dagger}_{jm}\} | \varphi_0(T) \rangle,\end{aligned} \quad (26)$$

where $\delta O = a^{\dagger}_{jm}, a^{\dagger}_{jm}, \tilde{a}^{\dagger}_{jm}, \tilde{a}_{jm}$, with additional condition (9), where $A = \alpha^{\dagger}_j, \alpha_j$, leads to well- known equations of the thermal BCS (TBCS), which describe the pair correlations in a hot nucleus [37, 38]. Within TFD the equations were derived in Refs. [10, 39], exploiting the method similar with that was presented above, i.e., diagonalizing the one-body part of $\mathcal{H}_{\rm BCS}$. However, in Refs. [10, 39] the coefficients of the thermal (x, y)-transformation were found by the minimization of the thermodynamical potential. For the first time the diagonalization of $\mathcal{H}_{\rm BCS}$ using the thermal state condition were performed in [14].

Having solved the TBCS equations, one can calculate the energies of thermal quasiparticles and the transformation coefficients (24) and (25) as a function of temperature. In particular, the coefficients of thermal transformation are related to the thermal Fermi–Dirac occupation numbers of Bogoliubov quasiparticles

$$y_j^2 = [1 + e^{\varepsilon_j/T}]^{-1} = \langle \varphi_0(T) | \alpha^\dagger_{jm}\alpha_{jm} | \varphi_0(T) \rangle. \qquad (27)$$

Due to the presence of thermally excited quasiparticles in the thermal vacuum, both the process of creation and the process of annihilation of Bogoliubov quasiparticles are allowed. In the TFD formalism, the first process corresponds to the production of a thermal quasiparticle with positive energy, and the second one to the process of creation of a thermal tilde-quasiparticle with negative energy [15].

In the independent thermal quasiparticle approximation, the simplest excitation modes of a hot nuclei under an external one-body perturbation are the following states $|[\beta^\dagger_{j_1},\beta^\dagger_{j_2}]_{JM}\rangle$, $|[\tilde\beta^\dagger_{j_1},\tilde\beta^\dagger_{j_2}]_{JM}\rangle$, $|[\beta^\dagger_{j_1},\tilde\beta^\dagger_{j_2}]_{JM}\rangle$. Transitions from the thermal vacuum to these states describe, correspondingly, the processes of two quasiparticles creation, two thermally excited quasi-particles annihilation and a transition of an excited quasiparticle from one state to another.

It has been shown [15], that probabilities of transitions to tilde-conjugate states constructed, as written out just above, from two thermal quasiparticles, are related by the principle of detailed balance (12).

At the next stage, the thermal Hamiltonian eigenstates are found considering the interaction between thermal quasiparticles due to the separable particle-hole interaction. For this, the so-called thermal quasiparticle random phase approximation (TQRPA) is used, according to which thermal Hamiltonian (22) is diagonalized approximately within the space of phonon operators constructed as a linear combination of various operators of two thermal quasiparticle [10, 15]

$$\begin{aligned} Q^\dagger_{JMi} &= \frac{1}{2}\sum_\tau \sum_{j_1j_2}^{\tau} (\psi^{Ji}_{j_1j_2}[\beta^\dagger_{j_1}\beta^\dagger_{j_2}]_{JM} + \tilde\psi^{Ji}_{j_1j_2}[\tilde\beta^\dagger_{j_1}\tilde\beta^\dagger_{j_2}]_{JM} \\ &+ i\eta^{Ji}_{j_1j_2}[\tilde\beta^\dagger_{j_1}\tilde\beta^\dagger_{j_2}]_{JM} + i\tilde\eta^{Ji}_{j_1j_2}[\tilde\beta^\dagger_{\bar{j}_1}\beta^\dagger_{j_2}]_{JM} - \varphi^{Ji}_{j_1j_2}[\beta^\dagger_{j_1}\beta^\dagger_{j_2}]^\dagger_{\overline{JM}} \\ &- \tilde\varphi^{Ji}_{j_1j_2}[\tilde\beta^\dagger_{j_1}\tilde\beta^\dagger_{j_2}]^\dagger_{\overline{JM}} + i\xi^{Ji}_{j_1j_2}[\beta^\dagger_{j_1}\tilde\beta^\dagger_{j_2}]^\dagger_{\overline{JM}} + i\tilde\xi^{Ji}_{j_1j_2}[\tilde\beta^\dagger_{j_1}\beta^\dagger_{j_2}]^\dagger_{\overline{JM}}). \end{aligned} \qquad (28)$$

An imaginary unit appeared in the above definition of thermal phonon operator since we use the complex thermal transformation (25) (see also [15]). Other phonon operators $\tilde{Q}^\dagger_{JMi}$, Q_{JMi}, $\widetilde{Q}_{JMi}$ can be evaluated from (28) applying the Hermitian and tilde conjugation operations. A vacuum of thermal phonons $|\psi_0(T)\rangle$ is the true thermal vacuum of TQRPA if it satisfies the thermal state condition (9). Assuming that $|\psi_0(T)\rangle$ is close to the thermal vacuum of the TBCS approximation $|\varphi_0(T)\rangle$, we get the quasi-boson approximation, according to which the thermal phonon operators obey boson commutation relations.

To find the thermal phonon structure, i.e., the amplitudes ψ, ϕ etc., as well as the phonon energies, the following equation of motion must be solved:

$$\begin{aligned} &\langle \varphi_0(T) | [\delta\mathbb{O}, \mathcal{H}, Q^\dagger_{JMi}] | \varphi_0(T) \rangle \\ &= \omega_{Ji}(T) \langle \varphi_0(T) | [\delta\mathbb{O}, Q^\dagger_{JMi}] | \varphi_0(T) \rangle, \end{aligned} \qquad (29)$$

where the creation and annihilation operators of thermal quasiparticles pairs are used as $\delta\mathbb{O}$. We emphasize once again that, as in the standard QRPA, the equation of motion (29) includes the thermal quasiparticle vacuum, and not a thermal phonon vacuum. Equation (29) leads to a system of linear homogeneous equations for the amplitudes ψ, φetc. Since we use the separable particle-hole interaction, the condition for the solvability of this system of equations has the form of a secular equation for the energy of thermal phonons ω_{Ji} [10, 15]. The requirement that the thermal phonon vacuum satisfies the thermal state condition (9) ($A = \alpha^\dagger_{j_1}\alpha^\dagger_{j_2}, \alpha^\dagger_{j_1}\alpha_{j_2}, \alpha_{j_1}\alpha_{j_2}$), leads to the following relations for the amplitudes [22]:

$$\begin{aligned} \begin{pmatrix}\Psi\\ \Phi\end{pmatrix}^{Ji}_{j_1j_2} &= [X_{Ji}x_{j_1}x_{j_2} - Y_{Ji}y_{j_1}y_{j_2}]^{-1}\begin{pmatrix}\psi\\ \varphi\end{pmatrix}^{Ji}_{j_1j_2} \\ &= [Y_{Ji}x_{j_1}x_{j_2} - X_{Ji}y_{j_1}y_{j_2}]^{-1}\begin{pmatrix}\tilde\varphi\\ \tilde\psi\end{pmatrix}^{Ji}_{j_1j_2} \\ \begin{pmatrix}\mathrm{H}\\ \Xi\end{pmatrix}^{Ji}_{j_1j_2} &= [X_{Ji}x_{j_1}y_{j_2} - Y_{Ji}y_{j_1}x_{j_2}]^{-1}\begin{pmatrix}\eta\\ \xi\end{pmatrix}^{Ji}_{j_1j_2} \\ &= [Y_{Ji}x_{j_1}y_{j_2} - X_{Ji}y_{j_1}x_{j_2}]^{-1}\begin{pmatrix}\tilde\xi\\ \tilde\eta\end{pmatrix}^{Ji}_{j_1j_2}, \end{aligned} \qquad (30)$$

where the so-called effective amplitudes Ψ, Φ, H and Ξ are introduced. Temperature dependent coefficients Y_{Ji} and X_{Ji} are related with the Bose–Einstein distribution function

$$Y_{Ji} = (e^{\omega_{Ji}/T} - 1)^{-1/2}, \quad X_{Ji} = (1 + Y_{Ji}^2)^{1/2}. \qquad (31)$$

Thus, the thermal state condition (9) for the thermal phonon vacuum leads to the emergence of thermal occupation numbers for bosons (phonons). The phonon amplitudes depend on both fermionic and bosonic occupation numbers. The forward-going and their backward-going tilde-conjugate amplitudes (i.e. ψ and $\tilde\phi$, $\widetilde{\psi}$ and ϕ, η and $\tilde\xi$, $\tilde\eta$ and ξ) are expressed through each other by means of the effective amplitudes.

Due to the separable form of the residual particle-hole interaction the following analytical expressions for the effective amplitudes are valid [15]:

$$\begin{aligned} \begin{pmatrix} \Psi \\ \Phi \end{pmatrix}^{Ji}_{j_1j_2} &= \frac{1}{\sqrt{\mathcal{N}^{Ji}_\tau}} \frac{f^{(J)}_{j_1j_2} u^{(+)}_{j_1j_2}}{\varepsilon^{(+)}_{j_1j_2} \mp \omega_{Ji}}, \\ \begin{pmatrix} \mathrm{H} \\ \Xi \end{pmatrix}^{Ji}_{j_1j_2} &= \frac{1}{\sqrt{\mathcal{N}^{Ji}_\tau}} \frac{f^{(J)}_{j_1j_2} v^{(-)}_{j_1j_2}}{\varepsilon^{(-)}_{j_1j_2} \mp \omega_{Ji}}. \end{aligned} \tag{32}$$

Here $f^{(J)}_{j_1j_2}$ is the reduced matrix element of a multipole operator, $\varepsilon^{(\pm)}_{j_1j_2} = \varepsilon_{j_1} \pm \varepsilon_{j_2}$. Besides, the following notations are used for bilinear combinations of the Bogoliubov transformation coefficients: $u^{(+)}_{j_1j_2} = u_{j_1}v_{j_2} + v_{j_1}u_{j_2}$, $v^{(-)}_{j_1j_2} = u_{j_1}u_{j_2} - v_{j_1}v_{j_2}$. The effective amplitudes depend on temperature because T-dependence of energies of thermal quasiparticles and phonons and moreover because of $\mathcal{N}^{Ji}_\tau$ coefficients which can be obtained from the normalization condition [15]

$$\begin{aligned} &\langle Q_{JMi} | Q_{JMi} \rangle \\ &= \frac{1}{2}\sum_\tau \sum_{j_1j_2}^{\tau} \{ (\Psi^{Ji}_{j_1j_2}\Psi^{Ji}_{j_1j_2} - \Phi^{Ji}_{j_1j_2}\Phi^{Ji}_{j_1j_2})(1 - y^2_{j_1} - y^2_{j_2}) \\ &\quad + (\mathrm{H}^{Ji}_{j_1j_2}\mathrm{H}^{Ji}_{j_1j_2} - \Xi^{Ji}_{j_1j_2}\Xi^{Ji}_{j_1j_2})(y^2_{j_2} - y^2_{j_1}) \} = 1. \end{aligned} \tag{33}$$

Note that the above expressions for the effective amplitudes coincide with the expressions for the phonon amplitudes given in [40], where they have been obtained within the formalism of the temperature Green's functions.

Just as the operators of thermal and Bogoliubov quasiparticles are related by thermal trans-formation (25), the operators of thermal phonons can be represented as a result of certain thermal transformation

$$Q^\dagger_{JMi} = X_{Ji} q^\dagger_{JMi} - Y_{Ji}\tilde{q}_{JMi}, \quad \tilde{Q}^\dagger_{JMi} = X_{Ji}\tilde{q}^\dagger_{JMi} - Y_{Ji} q_{JMi}, \tag{34}$$

where the q-phonon operators correspond to the values of coefficients in Eq. (30) $X_{Ji} = 1$, $Y_{Ji} = 0$. By means of transformation inverse to (34) it can be shown that the thermal phonon vacuum $|\psi_0(T)\rangle$ contains Y^2_{Ji} thermally excited q-phonons with the energies ω_{Ji}

$$\langle \psi_0(T) | q^\dagger_{JMi} q_{JMi} | \psi_0(T) \rangle = Y^2_{Ji}. \tag{35}$$

Therefore, in a certain sense, q-phonons can be considered as "cold" phonons, and thermal transformation (34), as it were, heats them up, providing transition to new phonon operators, for which the vacuum state is just the thermal vacuum. In this case, the one-phonon part of the thermal Hamiltonian is diagonal both in terms of q-phonons and in terms of thermal phonons, since the transformation (34) is unitary

$$\begin{aligned} \mathcal{H}_{\mathrm{TQRPA}} &= \sum_{JMi} \omega_{Ji} (Q^\dagger_{JMi} Q_{JMi} - \tilde{Q}^\dagger_{JMi}\tilde{Q}_{JMi}) \\ &= \sum_{JMi} \omega_{Ji} (q^\dagger_{JMi} q_{JMi} - \tilde{q}^\dagger_{JMi}\tilde{q}_{JMi}). \end{aligned} \tag{36}$$

The q-phonon vacuum $|\psi_0\rangle$ relates to the thermal vacuum $|\psi_0(T)\rangle$ through a unitary transformation [13]. It should be noted, that in Refs. [9, 10] it was q-phonons that were considered as thermal, and their vacuum played the role of thermal vacuum. In this case, the bosonic occupation numbers do not appear in the theory.

Let us obtain expressions for the reduced probability (strength) of transitions from thermal vacuum to thermal one-phonon states. For this we express the one-body transition operator $\mathcal{T}_{JM}$ (20) via the thermal phonon operators [15]

$$\begin{aligned} &\mathcal{T}_{JM} \\ &= \hat{J}^{-1} \sum_i \Gamma_{Ji} \{ X_{Ji} (Q^\dagger_{JMi} + Q_{\overline{JMi}}) + Y_{Ji} (\tilde{Q}^\dagger_{\overline{JMi}} + \tilde{Q}_{JMi}) \}, \end{aligned} \tag{37}$$

where

$$\begin{aligned} \Gamma_{Ji} = \sum_\tau \sum_{j_1j_2}^{\tau} t^{(J)}_{j_1j_2} &\{ (1 - y^2_{j_1} - y^2_{j_2}) u^{(+)}_{j_1j_2} (\Psi^{Ji}_{j_1j_2} + \Phi^{Ji}_{j_1j_2}) \\ &+ (y^2_{j_2} - y^2_{j_1}) v^{(-)}_{j_1j_2} (H^{Ji}_{j_1j_2} + \Xi^{Ji}_{j_1j_2}) \}. \end{aligned} \tag{38}$$

The temperature dependent coefficients Γ_{Ji} determine the reduced probabilities of transitions to the thermal one-phonon states

$$\begin{aligned} B_{Ji}(\mathcal{T}_J) &= |\langle Q_{Ji} \| \mathcal{T}_J \| \psi_0(T) \rangle|^2 = X^2_{Ji}\Gamma^2_{Ji}, \\ \tilde{B}_{Ji}(\mathcal{T}_J) &= |\langle \tilde{Q}_{Ji} \| \mathcal{T}_J \| \psi_0(T) \rangle|^2 = Y^2_{Ji}\Gamma^2_{Ji}. \end{aligned} \tag{39}$$

The above expressions together with the transition energies $\pm\omega_{Ji}$ unambiguously determine the strength functions of the operator $\mathcal{T}_{JM}$ (20) within TQRPA. Since the thermal vacuum TQRPA satisfies the thermal state condition (9), for the strength function of the transitions to thermal one-phonon states the detailed balance principle is valid (21).

The expressions (38), (39) agree with the interpretation of q-phonons as "cold". Indeed, let us find the transition strengths to one-phonon q-states

$$\begin{aligned} b_{Ji}(\mathcal{T}_J) &= |\langle q_{Ji} \| \mathcal{T}_J \| \psi_0 \rangle|^2 = \Gamma^2_{Ji}, \\ \tilde{b}_{Ji}(\mathcal{T}_J) &= |\langle \tilde{q}_{Ji} \| \mathcal{T}_J \| \psi_0 \rangle|^2 = 0. \end{aligned} \tag{40}$$

The vanishing of the transition probabilities to states with negative energies means that in the system of q-phonons, as in the nucleus in its ground state, only the excitation process is possible. This circumstance was not considered in [9, 10], which led to the violation of the detailed balance principle.

It was shown above that a hot nucleus contains Y_{Ji}^2 thermally excited q-phonons with energies ω_{Ji}. Therefore, the probability of the de-excitation process, when a phonon is removed from the system, is proportional to Y_{Ji}^2, and the probability of the reverse process—excitation,—when a phonon is added to the system, is proportional to $X_{Ji}^2 = 1 + Y_{Ji}^2$.

4. FRAGMENTATION OF ONE-PHONON STATES IN HOT NUCLEI

The concepts of "thermal quasiparticles" and "thermal phonons" as elementary excitation modes of a hot nucleus, as well as the structure of the thermal Hamiltonian inheriting the structure of the original nuclear Hamiltonian, make it possible to go beyond the framework of the one-phonon approximation by the traditional for QPM method—adding two-phonon components to the wave function [11]. To this aim, as at $T = 0$, we express the thermal Hamiltonian (22) in terms of the creation and annihilation operators of thermal quasiparticles and phonons

$$\mathcal{H} = \mathcal{H}_{\mathrm{TQRPA}} + \mathcal{H}_{\mathrm{qph}}. \tag{41}$$

The term $\mathcal{H}_{\mathrm{qph}}$ describes the coupling of thermal phonons and thermal quasiparticles

$$\begin{aligned}\mathcal{H}_{\mathrm{qph}} = -\frac{1}{2}\sum_{JMi}\sum_{\tau}\sum_{j_1j_2}^{\tau}\frac{f_{j_1j_2}^{(J)}}{\sqrt{\mathcal{N}_{\tau}^{Ji}}}&[\{X_{Ji}(Q_{\overline{JMi}}^{\dagger} + Q_{JMi}) \\ &+ Y_{Ji}(\tilde{Q}_{JMi}^{\dagger} + \tilde{Q}_{\overline{JMi}})\}B_{JM}(j_1j_2) + (\mathrm{h.c.}) - (\mathrm{t.c.})],\end{aligned} \tag{42}$$

where, to shorten the notation, the terms which are Hermitian- and tilde- conjugates to those given in Eq. (42) are denoted as (h.c.) and (t.c.), respectively. The operator $B_{JM}(j_1j_2)$ is expressed in terms of creation and annihilation operators of thermal quasiparticles as follows:

$$\begin{aligned}B_{JM}(j_1j_2) = &-v_{j_1j_2}^{(-)}(x_{j_1}x_{j_2}[\beta_{j_1}^{\dagger}\beta_{\overline{j_2}}]_{JM} + y_{j_1}y_{j_2}[\tilde{\beta}_{j_1}^{\dagger}\tilde{\beta}_{j_2}]_{JM}) \\ &+ iu_{j_1j_2}^{(+)}(x_{j_1}y_{j_2}[\beta_{j_1}^{\dagger}\tilde{\beta}_{j_2}]_{JM} + y_{j_1}x_{j_2}[\tilde{\beta}_{j_1}^{\dagger}\beta_{j_2}]_{JM}).\end{aligned} \tag{43}$$

As in the case of a cold nucleus the $\mathcal{H}_{\mathrm{qph}}$ term mixes states with different number of phonons resulting in fragmentation of the transition strengths, which in TQRPA are concentrated in one-phonon states.

Strictly speaking, the thermal Hamiltonian $\mathcal{H}$, contains terms of the $B^{\dagger}B$ type which also contribute to the interaction of thermal quasiparticles but was not considered in TQRPA. We discard these terms, as this is done in QPM at zero temperature [11, 28]. In addition, considering the interaction of thermal quasiparticles and phonons, we will neglect the Pauli principle, i.e., we will regard the thermal phonon operators as "true" bosons. Besides, we will use one more approximation, namely, we will assume that

$$[B_{JM}(j_1j_2), \tilde{q}_{J'M'}^{\dagger}] = [B_{JM}(j_1j_2), \tilde{q}_{J'M'}] = 0. \tag{44}$$

To clarify the meaning of this approximation, consider the operator $\mathcal{H}_{\mathrm{qph}}$, written in terms of q-phonons

$$\begin{aligned}\mathcal{H}_{\mathrm{qph}} = -\frac{1}{2}\sum_{JMi}\sum_{\tau}\sum_{j_1j_2}^{\tau}\frac{f_{j_1j_2}^{(J)}}{\sqrt{\mathcal{N}_{\tau}^{Ji}}} \\ \times [(q_{\overline{JMi}}^{\dagger} + q_{JMi})B_{JM}(j_1j_2) + (\mathrm{h.c.}) - (\mathrm{t.c.})].\end{aligned} \tag{45}$$

Comparison of Eq. (45) with (42) for $\mathcal{H}_{\mathrm{qph}}$ shows that approximation (44) turns off the interaction of "cold" $q^{\dagger}, q$ and $\tilde{q}^{\dagger}, \tilde{q}$ phonons. Interaction occurs only after "heating".

To satisfy the thermal state condition (9), we will seek the eigenstates of the thermal Hamiltonian (41) in the following form:

$$\begin{aligned}|\mathfrak{Q}_{JM\nu}\rangle = \mathfrak{Q}_{JM\nu}^{\dagger}|\Psi_0(T)\rangle = \Big[\sum_i \{R_i(J\nu)Q_{JMi}^{\dagger} \\ + \tilde{R}_i(J\nu)\tilde{Q}_{\overline{JMi}}^{\dagger} - N_i(J\nu)Q_{\overline{JMi}} - \tilde{N}_i(J\nu)\tilde{Q}_{JMi}\} \\ + \sum_{\substack{\lambda_1 i_1 \\ \lambda_2 i_2}}\{P_{\lambda_2 i_2}^{\lambda_1 i_1}(J\nu)[Q_{\lambda_1 i_1}^{\dagger}Q_{\lambda_2 i_2}^{\dagger}]_{JM} + \tilde{P}_{\lambda_2 i_2}^{\lambda_1 i_1}(J\nu)[\tilde{Q}_{\overline{\lambda_1 i_1}}^{\dagger}\tilde{Q}_{\overline{\lambda_2 i_2}}^{\dagger}]_{JM} \\ + 2S_{\lambda_2 i_2}^{\lambda_1 i_1}(J\nu)[Q_{\lambda_1 i_1}^{\dagger}\tilde{Q}_{\overline{\lambda_2 i_2}}^{\dagger}]_{JM} - T_{\lambda_2 i_2}^{\lambda_1 i_1}(J\nu)[Q_{\overline{\lambda_1 i_1}}Q_{\overline{\lambda_2 i_2}}]_{JM} \\ - \tilde{T}_{\lambda_2 i_2}^{\lambda_1 i_1}(J\nu)[\tilde{Q}_{\lambda_1 i_1}\tilde{Q}_{\lambda_2 i_2}]_{JM} \\ - 2Z_{\lambda_2 i_2}^{\lambda_1 i_1}(J\nu)[Q_{\overline{\lambda_1 i_1}}\tilde{Q}_{\lambda_2 i_2}]_{JM}\}\Big]|\Psi_0(T)\rangle.\end{aligned} \tag{46}$$

The wave function (46) should be normalized

$$\begin{aligned}\langle\mathfrak{Q}_{JM\nu}|\mathfrak{Q}_{JM\nu}\rangle = \sum_i \{[R_i(J\nu)]^2 + [\tilde{R}_i(J\nu)]^2 \\ - [N_i(J\nu)]^2 - [\tilde{N}_i(J\nu)]^2\} + 2\sum_{\substack{\lambda_1 i_1 \\ \lambda_2 i_2}}\{[P_{\lambda_2 i_2}^{\lambda_1 i_1}(J\nu)]^2 \\ + [\tilde{P}_{\lambda_2 i_2}^{\lambda_1 i_1}(J\nu)]^2 + 2[S_{\lambda_2 i_2}^{\lambda_1 i_1}(J\nu)]^2 \\ - [T_{\lambda_2 i_2}^{\lambda_1 i_1}(J\nu)]^2 - [\tilde{T}_{\lambda_2 i_2}^{\lambda_1 i_1}(J\nu)]^2 - 2[Z_{\lambda_2 i_2}^{\lambda_1 i_1}(J\nu)]^2]^2\} = 1.\end{aligned} \tag{47}$$

As before, we assume that the "tilde-less" state corresponds to positive energy $E_{J\nu}$. Then tilde-conjugate state $|\tilde{\mathfrak{Q}}_{JM\nu}\rangle = \tilde{\mathfrak{Q}}_{JM\nu}^{\dagger}|\Psi_0(T)\rangle$ corresponds the negative energy $-E_{J\nu}$. The new thermal vacuum is defined as the vacuum for the corresponding annihilation operators

$$\mathfrak{Q}_{JM\nu}|\Psi_0(T)\rangle = \tilde{\mathfrak{Q}}_{JM\nu}|\Psi_0(T)\rangle = 0. \tag{48}$$

The presence of the thermal phonon annihilation operators in (46) indicates on redefinition of the thermal vacuum because of appearance the coupling of one- and two-phonon configurations. It is worth recalling that in the standard QPM [11], when the phonon interaction was switched on, the ground state wave function was not redefined—it was assumed that it coincides with the phonon vacuum. The same assumption at $T \neq 0$ was implied in Refs. [9, 14, 15].

Additional constraints on the structure of the $\mathfrak{Q}^{\dagger}_{JM\nu}$ operator follow from requirement that the new vacuum must satisfy the thermal state condition (9). These constraints can be obtained by using in (9) as the operator A the two-quasiparticle operators $[\alpha^{\dagger}_{j_1}\alpha^{\dagger}_{j_2}]_{JM}$, $[\alpha^{\dagger}_{j_1}\alpha^{-}_{j_2}]_{JM}$ and their Hermitian conjugates. Then, expressing the two-quasiparticle operators in terms of thermal phonon operators, we obtain the condition, which should be satisfied by the amplitudes one-phonon terms in (46)

$$\begin{pmatrix} \widetilde{N} \\ \widetilde{R} \end{pmatrix}_i (J\nu) = \frac{X_{Ji}e^{-E_{J\nu}/2T} - Y_{Ji}}{X_{Ji} - Y_{Ji}e^{-E_{J\nu}/2T}} \begin{pmatrix} R \\ N \end{pmatrix}_i (J\nu), \tag{49}$$

where $E_{J\nu}$ is the eigenvalue of the thermal Hamiltonian (41), corresponding the wave function (46).

Thus, we have obtained an important result regarding the structure of the wave function (46): if we require the fulfillment of the thermal state condition for the vacuum of operators $\mathfrak{Q}_{JM\nu}$, then the wave function (46) should contain both the forward-going one-phonon terms, and the backward-going tilde-conjugate terms, i.e., the terms consisting of the phonon creation operator. In this regard, it seems logical to include in the wave function (46) inverse two-phonon terms as well.

As in the TQRPA, in the definition of the wave function (46) it is convenient to use the effective amplitudes

$$\begin{aligned} \begin{pmatrix} \mathbb{R} \\ \mathbb{N} \end{pmatrix}_i (J\nu) &= [X_{J\nu}X_{Ji} - Y_{J\nu}Y_{Ji}]^{-1} \begin{pmatrix} R \\ N \end{pmatrix}_i (J\nu) \\ &= [Y_{J\nu}X_{Ji} - X_{J\nu}Y_{Ji}]^{-1} \begin{pmatrix} \widetilde{N} \\ \widetilde{R} \end{pmatrix}_i (J\nu), \end{aligned} \tag{50}$$

where $X^2_{J\nu} - Y^2_{J\nu} = 1$ and $X_{J\nu}/Y_{J_\nu} = e^{-E_{J\nu}/2T}$. Using equality

$$\begin{aligned} &[R_i(J\nu)]^2 + [\tilde{R}_i(J\nu)]^2 - [N_i(J\nu)]^2 - [\tilde{N}_i(J\nu)]^2 \\ &= [\mathbb{R}_i(J\nu)]^2 - [\mathbb{N}_i(J\nu)]^2 \end{aligned} \tag{51}$$

normalization condition (47) can be written via the effective amplitudes.

To find the eigenstates and eigenvalues of the thermal Hamiltonian (41) we use again the equation of motion (13) in the form

$$\begin{aligned} &\langle \psi_0(T) | [\delta O, \mathcal{H}, \mathfrak{Q}^{\dagger}_{JM\nu}] | \psi_0(T) \rangle \\ &= E_{J\nu} \langle \psi_0(T) | [\delta O, \mathfrak{Q}^{\dagger}_{JM\nu}] | \psi_0(T) \rangle, \end{aligned} \tag{52}$$

where $|\psi_0(T)\rangle$ is the TQRPA phonon vacuum, and as the operator δO we consider the operators which are Hermitian conjugate to those in the right hand side (46), i.e. $\delta O = Q_{JM}, [Q_{\lambda_1 i_1}, Q_{\lambda_2 i_2}]_{JM}$ etc. The result is the following linear equation system:

$$\begin{aligned}
&R_i(J\nu)(\omega_{Ji} - E_{J\nu}) + \sum_{\substack{\lambda_1 i_1 \\ \lambda_2 i_2}} [P^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) \\
&+ \tilde{P}^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) \tilde{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) + 2S^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) W^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) \\
&- T^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) - \tilde{T}^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) \tilde{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) \\
&- 2Z^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) G^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)] = 0, \\
&\tilde{R}_i(J\nu)(\omega_{Ji} + E_{J\nu}) + \sum_{\substack{\lambda_1 i_1 \\ \lambda_2 i_2}} [P^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) \tilde{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) \\
&+ \tilde{P}^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) + 2S^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) \tilde{W}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) \\
&- T^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) \tilde{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) - \tilde{T}^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) \\
&- 2Z^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) \tilde{G}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)] = 0, \\
&N_i(J\nu)(\omega_{Ji} + E_{J\nu}) + \sum_{\substack{\lambda_1 i_1 \\ \lambda_2 i_2}} [P^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) \\
&+ \tilde{P}^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) \tilde{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) + 2S^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) G^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) \\
&- T^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) - \tilde{T}^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) \tilde{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) \\
&- 2Z^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) W^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)] = 0, \\
&\tilde{N}_i(J\nu)(\omega_{Ji} - E_{J\nu}) + \sum_{\substack{\lambda_1 i_1 \\ \lambda_2 i_2}} [P^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) \tilde{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) \\
&+ \tilde{P}^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) + 2S^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) \tilde{G}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) \\
&- T^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) \tilde{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) - \tilde{T}^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) \\
&- 2Z^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu) \tilde{W}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)] = 0, \\
&2P^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)(\omega_{\lambda_1 i_1} + \omega_{\lambda_2 i_2} - E_{J\nu}) \\
&+ \sum_i [R_i(J\nu) U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) - \tilde{R}_i(J\nu) \tilde{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) \\
&- N_i(J\nu) V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) + \tilde{N}_i(J\nu) \tilde{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)] = 0, \\
&2\tilde{P}^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)(\omega_{\lambda_1 i_1} + \omega_{\lambda_2 i_2} + E_{J\nu}) \\
&- \sum_i [R_i(J\nu) \tilde{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) - \tilde{R}_i(J\nu) U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) \\
&- N_i(J\nu) \tilde{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) + \tilde{N}_i(J\nu) V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)] = 0, \\
&2S^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)(\omega_{\lambda_1 i_1} - \omega_{\lambda_2 i_2} - E_{J\nu}) \\
&+ \sum_i [R_i(J\nu) W^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) - \tilde{R}_i(J\nu) \tilde{W}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) \\
&- N_i(J\nu) G^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) + \tilde{N}_i(J\nu) \tilde{G}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)] = 0, \\
&2T^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)(\omega_{\lambda_1 i_1} + \omega_{\lambda_2 i_2} + E_{J\nu}) \\
&+ \sum_i [R_i(J\nu) V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) - \tilde{R}_i(J\nu) \tilde{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) \\
&- N_i(J\nu) U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) + \tilde{N}_i(J\nu) \tilde{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)] = 0,
\end{aligned} \tag{53}$$

$$2\tilde{T}^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)(\omega_{\lambda_1 i_1} + \omega_{\lambda_2 i_2} - E_{J\nu})$$
$$- \sum_i [R_i(J\nu)\tilde{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) - \tilde{R}_i(J\nu)V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)$$
$$- N_i(J\nu)\tilde{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) + \tilde{N}_i(J\nu)U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)] = 0,$$
$$2Z^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)(\omega_{\lambda_1 i_1} - \omega_{\lambda_2 i_2} + E_{J\nu})$$
$$+ \sum_i [R_i(J\nu)G^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) - \tilde{R}_i(J\nu)\tilde{G}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)$$
$$- N_i(J\nu)W^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) + \tilde{N}_i(J\nu)\tilde{W}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)] = 0,$$

where the following notations of the matrix elements of the operator $\mathcal{H}_{\rm qph}$ are introduced:

$$\begin{aligned}
U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= \left\langle \psi_0(T) \left\| [Q_{JMi}, \mathcal{H}_{\rm qph}, [Q^\dagger_{\lambda_1 i_1} Q^\dagger_{\lambda_2 i_2}]_{JM}] \right\| \psi_0(T) \right\rangle,\\
\tilde{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= \left\langle \psi_0(T) \left\| [Q_{JMi}, \mathcal{H}_{\rm qph}, [\tilde{Q}^\dagger_{\overline{\lambda_1 i_1}} \tilde{Q}^\dagger_{\overline{\lambda_2 i_2}}]_{JM}] \right\| \psi_0(T) \right\rangle,\\
V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= \left\langle \psi_0(T) \left\| [Q_{JMi}, \mathcal{H}_{\rm qph}, [Q_{\overline{\lambda_1 i_1}} Q_{\overline{\lambda_2 i_2}}]_{JM}] \right\| \psi_0(T) \right\rangle,\\
\tilde{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= \left\langle \psi_0(T) \left\| [Q_{JMi}, \mathcal{H}_{\rm qph}, [\tilde{Q}_{\lambda_1 i_1} \tilde{Q}_{\lambda_2 i_2}]_{JM}] \right\| \psi_0(T) \right\rangle,\\
W^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= \left\langle \psi_0(T) \left\| [Q_{JMi}, \mathcal{H}_{\rm qph}, [Q^\dagger_{\lambda_1 i_1} \tilde{Q}^\dagger_{\overline{\lambda_2 i_2}}]_{JM}] \right\| \psi_0(T) \right\rangle,\\
\tilde{W}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= \left\langle \psi_0(T) \left\| [Q_{JMi}, \mathcal{H}_{\rm qph}, [\tilde{Q}^\dagger_{\overline{\lambda_1 i_1}} Q^\dagger_{\lambda_2 i_2}]_{JM}] \right\| \psi_0(T) \right\rangle,\\
G^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= \left\langle \psi_0(T) \left\| [Q_{JMi}, \mathcal{H}_{\rm qph}, [Q_{\overline{\lambda_1 i_1}} \tilde{Q}_{\lambda_2 i_2}]_{JM}] \right\| \psi_0(T) \right\rangle,\\
\tilde{G}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= \left\langle \psi_0(T) \left\| [Q_{JMi}, \mathcal{H}_{\rm qph}, [\tilde{Q}_{\lambda_1 i_1} Q_{\overline{\lambda_2 i_2}}]_{JM}] \right\| \psi_0(T) \right\rangle.
\end{aligned} \tag{54}$$

It can be shown, that within the approximation (44) the above matrix elements fulfill the equalities

$$\begin{aligned}
X_{Ji}U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) - Y_{Ji}\tilde{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= X_{\lambda_1 i_1}X_{\lambda_2 i_2}\mathbb{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji),\\
X_{Ji}\tilde{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) - Y_{Ji}U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= -Y_{\lambda_1 i_1}Y_{\lambda_2 i_2}\mathbb{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji),\\
X_{Ji}V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) - Y_{Ji}\tilde{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= X_{\lambda_1 i_1}X_{\lambda_2 i_2}\mathbb{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji),\\
X_{Ji}\tilde{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) - Y_{Ji}V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= -Y_{\lambda_1 i_1}Y_{\lambda_2 i_2}\mathbb{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji),\\
X_{Ji}W^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) - Y_{Ji}\tilde{G}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= X_{\lambda_1 i_1}Y_{\lambda_2 i_2}\mathbb{W}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji),\\
X_{Ji}\tilde{G}^{\lambda_1 i_1}_{\lambda_2 i_2}Ji) - Y_{Ji}W^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= -Y_{\lambda_1 i_1}X_{\lambda_2 i_2}\mathbb{W}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji),\\
X_{Ji}G^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) - Y_{Ji}\tilde{W}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= X_{\lambda_1 i_1}Y_{\lambda_2 i_2}\mathbb{G}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji),\\
X_{Ji}\tilde{W}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) - Y_{Ji}G^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= -Y_{\lambda_1 i_1}X_{\lambda_2 i_2}\mathbb{G}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji).
\end{aligned} \tag{55}$$

In (55) the functions $\mathbb{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)$, $\mathbb{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)$, $\mathbb{W}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)$ and $\mathbb{G}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)$ have the following forms:

$$\begin{aligned}
&\mathbb{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) = \hat{\lambda}_1\hat{\lambda}_2\\
&\times [\mathbb{B}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) + (-1)^{\lambda_1+\lambda_2+J}\mathbb{A}^{\lambda_1 i_1}_{Ji}(\lambda_2 i_2) + \mathbb{A}^{\lambda_2 i_2}_{Ji}(\lambda_1 i_1)],\\
&\mathbb{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) = -\hat{\lambda}_1\hat{\lambda}_2\\
&\times [\mathbb{B}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) + (-1)^{\lambda_1+\lambda_2+J}\mathbb{B}^{\lambda_1 i_1}_{Ji}(\lambda_2 i_2) + \mathbb{B}^{\lambda_2 i_2}_{Ji}(\lambda_1 i_1)],\\
&\mathbb{W}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) = \hat{\lambda}_1\hat{\lambda}_2\\
&\times [\mathbb{A}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) + (-1)^{\lambda_1+\lambda_2+J}\mathbb{B}^{\lambda_1 i_1}_{Ji}(\lambda_2 i_2) + \mathbb{A}^{\lambda_2 i_2}_{Ji}(\lambda_1 i_1)],\\
&\mathbb{G}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) = -\hat{\lambda}_1\hat{\lambda}_2\\
&\times [\mathbb{A}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) + (-1)^{\lambda_1+\lambda_2+J}\mathbb{A}^{\lambda_1 i_1}_{Ji}(\lambda_2 i_2) + \mathbb{B}^{\lambda_2 i_2}_{Ji}(\lambda_1 i_1)],
\end{aligned} \tag{56}$$

whereas in their turn the coefficients $\mathbb{A}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)$ and $\mathbb{B}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)$ are expressed through the effective TQRPA amplitudes (30) and the fermionic thermal occupation numbers

$$\begin{aligned}
\mathbb{A}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= \sum_\tau \sum_{j_1 j_2 j_3}^{\tau} \frac{f^{(J)}_{j_1 j_2}}{\sqrt{\mathcal{N}^{Ji}_\tau}} \begin{Bmatrix} J & \lambda_1 & \lambda_2 \\ j_3 & j_1 & j_2 \end{Bmatrix}\\
&\times \{v^{(-)}_{j_1 j_2}[(\Psi^{\lambda_1 i_1}_{j_2 j_3}\Psi^{\lambda_2 i_2}_{j_3 j_1} + \Phi^{\lambda_1 i_1}_{j_2 j_3}\Phi^{\lambda_2 i_2}_{j_3 j_1})\\
&\times (x^2_{j_1}x^2_{j_2}x^2_{j_3} - y^2_{j_1}y^2_{j_2}y^2_{j_3})\\
&- (\mathrm{H}^{\lambda_1 i_1}_{j_2 j_3}\Xi^{\lambda_2 i_2}_{j_3 j_1} + \Xi^{\lambda_1 i_1}_{j_2 j_3}\mathrm{H}^{\lambda_2 i_2}_{j_3 j_1})(x^2_{j_1}x^2_{j_2}y^2_{j_3} - y^2_{j_1}y^2_{j_2}x^2_{j_3})]\\
&+ u^{(+)}_{j_1 j_2}[(\Psi^{\lambda_1 i_1}_{j_2 j_3}\mathrm{H}^{\lambda_2 i_2}_{j_3 j_1} + \Phi^{\lambda_1 i_1}_{j_2 j_3}\Xi^{\lambda_2 i_2}_{j_3 j_1})(x^2_{j_1}y^2_{j_2}y^2_{j_3} - y^2_{j_1}x^2_{j_2}x^2_{j_3})\\
&+ (\mathrm{H}^{\lambda_1 i_1}_{j_2 j_3}\Phi^{\lambda_2 i_2}_{j_3 j_1} + \Xi^{\lambda_1 i_1}_{j_2 j_3}\Psi^{\lambda_2 i_2}_{j_3 j_1})(x^2_{j_1}y^2_{j_2}x^2_{j_3} - y^2_{j_1}x^2_{j_2}x^2_{j_3})]\},
\end{aligned} \tag{57}$$

$$\begin{aligned}
\mathbb{B}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= \sum_\tau \sum_{j_1 j_2 j_3}^{\tau} \frac{f^{(J)}_{j_1 j_2}}{\sqrt{\mathcal{N}^{Ji}_\tau}} \begin{Bmatrix} J & \lambda_1 & \lambda_2 \\ j_3 & j_1 & j_2 \end{Bmatrix}\\
&\times \{v^{(-)}_{j_1 j_2}[(\Psi^{\lambda_1 i_1}_{j_2 j_3}\Phi^{\lambda_2 i_2}_{j_3 j_1} + \Phi^{\lambda_1 i_1}_{j_2 j_3}\Psi^{\lambda_2 i_2}_{j_3 j_1})(x^2_{j_1}x^2_{j_2}x^2_{j_3} - y^2_{j_1}y^2_{j_2}y^2_{j_3})\\
&- (\mathrm{H}^{\lambda_1 i_1}_{j_2 j_3}\mathrm{H}^{\lambda_2 i_2}_{j_3 j_1} + \Xi^{\lambda_1 i_1}_{j_2 j_3}\Xi^{\lambda_2 i_2}_{j_3 j_1})(x^2_{j_1}x^2_{j_2}y^2_{j_3} - y^2_{j_1}y^2_{j_2}x^2_{j_3})]\\
&+ u^{(+)}_{j_1 j_2}[(\Psi^{\lambda_1 i_1}_{j_2 j_3}\Xi^{\lambda_2 i_2}_{j_3 j_1} + \Phi^{\lambda_1 i_1}_{j_2 j_3}\mathrm{H}^{\lambda_2 i_2}_{j_3 j_1})(x^2_{j_1}y^2_{j_2}y^2_{j_3} - y^2_{j_1}x^2_{j_2}x^2_{j_3})\\
&+ (\mathrm{H}^{\lambda_1 i_1}_{j_2 j_3}\Psi^{\lambda_2 i_2}_{j_3 j_1} + \Xi^{\lambda_1 i_1}_{j_2 j_3}\Phi^{\lambda_2 i_2}_{j_3 j_1})(x^2_{j_1}y^2_{j_2}x^2_{j_3} - y^2_{j_1}x^2_{j_2}x^2_{j_3})]\}.
\end{aligned} \tag{58}$$

The dimension of the system (53) can be sizably reduced eliminating the two-phonon amplitudes. After that using the definition (50) one gets the system of linear homogeneous equations for the effective amplitudes $\mathbb{R}_i$ and $\mathbb{N}_i$

$$\begin{pmatrix} \mathbb{M}^{(1)}(E) & \mathbb{M}^{(2)}(E) \\ \mathbb{M}^{(2)}(-E) & \mathbb{M}^{(1)}(-E) \end{pmatrix} \begin{pmatrix} \mathbb{R} \\ \mathbb{N} \end{pmatrix} = 0. \tag{59}$$

The expressions for the matrix elements $\mathbb{M}^{(1,2)}_{ii'}$ are the following:

$$\begin{aligned}
\mathbb{M}^{(1)}_{ii'}(E) &= \delta_{ii'}(\omega_{Ji} - E)\\
&-\frac{1}{2}\sum_{\substack{\lambda_1 i_1\\ \lambda_2 i_2}}\left\{\left(\frac{\mathbb{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)\mathbb{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji')}{\omega_{\lambda_1 i_1} + \omega_{\lambda_2 i_2} - E} + \frac{\mathbb{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)\mathbb{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji')}{\omega_{\lambda_1 i_1} + \omega_{\lambda_2 i_2} + E}\right)\right.\\
&\times (1 + Y^2_{\lambda_1 i_1} + Y^2_{\lambda_2 i_2}) + \left(\frac{\mathbb{W}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)\mathbb{W}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji')}{\omega_{\lambda_1 i_1} - \omega_{\lambda_2 i_2} - E}\right.\\
&\left.\left. + \frac{\mathbb{G}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)\mathbb{G}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji')}{\omega_{\lambda_1 i_1} - \omega_{\lambda_2 i_2} + E}\right)(Y^2_{\lambda_2 i_2} - Y^2_{\lambda_1 i_1})\right\},
\end{aligned} \tag{60}$$

$$\begin{aligned}
\mathbb{M}^{(2)}_{ii'}(E) &= \frac{1}{2}\sum_{\substack{\lambda_1 i_1\\ \lambda_2 i_2}}\left\{\left(\frac{\mathbb{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)\mathbb{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji')}{\omega_{\lambda_1 i_1} + \omega_{\lambda_2 i_2} - E} + \frac{\mathbb{V}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)\mathbb{U}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji')}{\omega_{\lambda_1 i_1} + \omega_{\lambda_2 i_2} + E}\right)\right.\\
&\times (1 + Y^2_{\lambda_1 i_1} + Y^2_{\lambda_2 i_2}) + \left(\frac{\mathbb{W}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)\mathbb{G}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji')}{\omega_{\lambda_1 i_1} - \omega_{\lambda_2 i_2} - E}\right.\\
&\left.\left. + \frac{\mathbb{G}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)\mathbb{W}^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji')}{\omega_{\lambda_1 i_1} - \omega_{\lambda_2 i_2} + E}\right)(Y^2_{\lambda_2 i_2} - Y^2_{\lambda_1 i_1})\right\}.
\end{aligned} \tag{61}$$

Note that the matrix elements $\mathbb{M}_{ii'}^{(1,2)}$ depend not only on the quasiparticle thermal quantum numbers, but also the phonon ones. In this respect our results have something in common with those of Ref. [6], where the bosonic occupation numbers arise due to the special properties of the temperature Green's functions.

The existence condition of a nontrivial solution to system (63) leads to the secular equation $\det\|\mathbb{M}\| = 0$ for the eigenvalues $E_{J\nu}$ of the thermal Hamiltonian (41). Solving the system (63) for each positive eigenvalue $E_{J\nu}$, we find the unnormalized effective amplitudes $\mathbb{R}_i(J\nu)$ and $\mathbb{N}_i(J\nu)$. Normalization is carried out using the equations (53) and conditions (47), (51). Thus, we completely determine the operator structure of $\mathfrak{Q}^{\dagger}_{JM\nu}$. The structure of the operator $\tilde{\mathfrak{Q}}^{\dagger}_{JM\nu}$, corresponding to the negative eigenvalue $-E_{J\nu}$, can be found with the tilde-conjugation operation applying to the operator $\mathfrak{Q}^{\dagger}_{JM\nu}$(46).

After the structure of the eigenstates of the thermal Hamiltonian (41) has been determined one can calculate the reduced transition probabilities for the one-body multipole operator $\mathcal{T}_{JM}$

$$\begin{aligned}
\mathcal{B}_{J\nu}(\mathcal{T}_J) &= \left|\langle \mathfrak{Q}_{J\nu}\|\mathcal{T}_J\|\Psi_0(T)\rangle\right|^2 \\
&= X^2_{J\nu}\left|\sum_i \Gamma_{Ji}[\mathbb{R}_i(J\nu) + \mathbb{N}_i(J\nu)]\right|^2, \\
\tilde{\mathcal{B}}_{J\nu}(\mathcal{T}_J) &= \left|\langle \tilde{\mathfrak{Q}}_{J\nu}\|\mathcal{T}_J\|\Psi_0(T)\rangle\right|^2 \\
&= Y^2_{J\nu}\left|\sum_i \Gamma_{Ji}[\mathbb{R}_i(J\nu) + \mathbb{N}_i(J\nu)]\right|^2.
\end{aligned} \tag{62}$$

The obtained expressions of the reduced probabilities satisfy the detailed balance principle (12).

Thus, requiring for the thermal vacuum the fulfillment of the thermal state condition at each stage of diagonalization, we succeeded in constructing a thermodynamically consistent method to describe the fragmentation of one-phonon states in hot nuclei. In Ref. [9], the thermal state condition was valid in the TBCS approximation only whereas in TQRPA the detailed balance principle was already violated because the strength function had not the term allowing the transitions with negative energy $E < 0$ (de-excitation process). Later, in [14, 15], the consistent construction of the thermal vacuum and thermal phonons have ensured the fulfillment of the detailed balance principle within the TQRPA. However, when considering the coupling of one- and two-phonon states the TQRPA vacuum state was used as the thermal vacuum, which has led to violation of the DBP. It was fulfilled on average only, i.e., after averaging the strength function over a certain energy interval.

5. CONCLUSIONS

Taking the quasiparticle-phonon nuclear model as an example, applying the formalism of thermo field dynamics we built the consistent procedure of deriving the thermal strength function for hot nuclei. At every stage of the procedure the detailed balance principle is fulfilled. It is shown that thermodynamically consistent consideration of the one- and two-phonon coupling in a hot nucleus demands the consistent re-definition of the thermal vacuum state. This is the main result of the present work. According with the obtained equations, the matrix elements of effective phonon-phonon interaction in a hot nucleus are depended on both the quasiparticle (fermion) and phonon (boson) thermal occupation numbers. This procedure can be extended easily on the spin-isospin excitations (magnetic dipole, Gamow–Teller etc.) in hot nuclei which seems to be relevant for the astrophysical applications.

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