International Journal of Modern Physics E
Vol. 18, No. 7 (2009) 1535–1560

ON THE TFD TREATMENT OF COLLECTIVE VIBRATIONS IN HOT NUCLEI

ALAN A. DZHIOEV* and A. I. VDOVIN†

Bogoliubov Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research, Dubna 141980, Russia
** dzhioev@theor.jinr.ru*
† vdovin@theor.jinr.ru

Accepted 8 September 2008

The approach in a theory of collective excitations in hot nuclei exploring the formalism of thermo field dynamics and the model Hamiltonian consisting of a mean field, the BCS paring interaction, and long-range particle–hole effective forces is reexamined. In contrast with earlier studies, it is found that a wave function of a thermal phonon is depended not only on the Fermi–Dirac thermal occupation numbers of the Bogoliubov quasiparticles consisting the phonon but also on the Bose thermal occupation numbers of the phonon. This strongly affects a thermal phonon coupling due to the renormalization of a phonon–phonon interaction and enlarging the number of thermal two-phonon configurations coupled with one-phonon ones. Moreover, it is shown that the formulation of the double tilde conjugation rule for fermions proposed by I. Ojima is more appropriate in the context of the present study than the original one by H. Umezawa and co-workers.

Keywords: Hot nuclei; thermo field dynamics; quasiparticles and phonons.

PACS Number(s): 21.60.Jz, 24.10.Pa

1. Introduction

The present paper addresses some aspects of theoretical treatment of collective excitations of a hot nucleus within the thermo field dynamics (TFD).

The TFD[1,2] is known as a powerful tool in studying many-body problems at finite temperatures. There exist numerous TFD applications in condensed matter and high-energy physics (see, for example, Refs. 2–5). In spite of several attractive properties, the TFD formalism seems to be less popular in the nuclear theorist community. During the last 20 years only a dozen works dealing with the TFD application to nuclear structure problems were published.[6–19]

An important step in application of the TFD formalism for extending the standard nuclear theory methods (the HFB approximation, RPA, a boson expansion technique) to finite temperatures was made by Hatsuda.[7] In Refs. 6, 11, 14, 16, 17, different approximations going beyond the thermal RPA were formulated. Civitarese and co-workers[9,15] applied the TFD to problems of treating nuclear pairing in the BCS and the random phase approximations as well as quasicontinuum (or resonant) single-particle states at finite temperatures. Moreover, the TFD was used to prove the Bloch–Messiah theorem at finite temperatures[8] and construct number-projection methods for the BCS pairing in hot nuclei.[13,19]

The methods and approximations developed in the papers cited above were not applied to numerical calculations of any nuclear properties. Only simple solvable models were used to demonstrate the new achievements. At the same time, in Refs. 10, 12 the TFD was combined with the quasiparticle–phonon nuclear model (QPM)[20] to formulate an approach allowing one to treat microscopically a coupling of different excitation modes in hot nuclei. This coupling is known to be responsible for the spreading width of giant resonances in cold spherical nuclei, and the TFD-QPM approach was used to analyze a thermal behavior of a giant dipole resonance width.[18]

It appeared that in contrast with earlier studies of the problem[21] (see also Ref. 22) based on the Matsubara Green's function technique and the nuclear field theory,[23] the quasiparticle–phonon interaction at finite temperature in the TFD-QPM approach did not depend on the thermal occupation numbers of phonons. Only the Fermi–Dirac thermal occupation numbers of noninteracting BCS quasiparticles appeared in corresponding formulae of Refs. 10, 18. Some aspects of this difference were discussed in Ref. 18 (see also Ref. 24).

Recently, studying weak transitions of the Gamow–Teller type in hot nuclei[25] within the TFD we found some inconsistencies in the TFD-QPM approach of Refs. 10, 12 related to structures of thermal vibrational phonons. Moreover, new points in the TFD-QPM approach appear if one adopts changes in the TFD formulation proposed by Izumi Ojima,[26,27] which concern the alternative formulation of the double tilde conjugation rule (DTCR).

The reasons given above compel us to reexamine the TFD-QPM approach of Refs. 10, 12, 18 to a quasiparticle–phonon coupling at finite temperatures. This is the main aim of the present paper.

This paper is organized as follows. A brief summary of the TFD formalism is given in Sec. 2. This seems to be necessary because we use here a variant of the theory, which somewhat differs from the standard one.[1,2,7] In Sec. 3, the TFD formalism is applied to the nuclear Hamiltonian consisting of a mean field, the BCS pairing interaction, and separable particle–hole forces, i.e. to the QPM Hamiltonian.[20] In Sec. 3.2, the thermal BCS pairing is considered. This part of the paper is close to that of Refs. 9, 10, 12. A discussion of γ-transitions between a thermal vacuum state and thermal quasiparticle excitations is the only addition. In Sec. 3.3, the equations of the thermal random phase approximation (TRPA) are derived. It is shown why and how the thermal Bose–Einstein occupation factors should appear in the expressions for thermal quasiparticle amplitudes of a thermal phonon wave function. The points missed in Refs. 10, 12, 18 are analyzed and discussed. The new formulae of the quasiparticle–phonon coupling term at finite temperature are evaluated in Sec. 3.4. The summary and conclusions are given in Sec. 4.

2. The TFD Formalism

We consider a system of Fermi particles at finite temperature and treat its statistical properties within the grand canonical ensemble. In the standard statistical mechanics, a heated system is described by a mixed state density matrix ρ, which is the solution to the Liouville–von Neumann equation

$$i\,\frac{\partial\rho}{\partial t} = [H,\rho]\,. \tag{1}$$

Here H is the Hamiltonian of the system under consideration with eigenstates $|n\rangle$ and eigenvalues E_n (the chemical potential is included in H). The thermal average of an arbitrary operator A is given by

$$\langle\langle A\rangle\rangle = \mathrm{Tr}[\rho A]/\mathrm{Tr}[\rho] = \sum_n \mathrm{e}^{-E_n/T}\langle n|A|n\rangle \Bigg/ \sum_n \mathrm{e}^{-E_n/T}\,, \tag{2}$$

where T is the temperature in units of energy. The main idea behind TFD is to define a special state, which is named a thermal vacuum $|0(T)\rangle$, such that the thermal average of A equals the expectation value of A with respect to this state

$$\langle\langle A\rangle\rangle = \langle 0(T)|A|\,0(T)\rangle\,. \tag{3}$$

To construct $|0(T)\rangle$, one should double the Hilbert space of the system by adding the so-called tilde states $|\tilde{n}\rangle$.[1] These tilde states are the eigenstates of the tilde Hamiltonian $\tilde{H}$ with the same eigenvalues as H, i.e. $\tilde{H}|\tilde{n}\rangle = E_n|\tilde{n}\rangle$. Thus, the Hilbert space of a heated system is twice as large as of the corresponding initial (cold) one.

In the enlarged space covered by the states $|n\rangle \otimes |\tilde{m}\rangle$, two types of operators are acting — the ordinary, say, A_i and the tilde ones $\tilde{A}_i$. There is a one-to-one correspondence between these two sets $A_i \leftrightarrow \tilde{A}_i$. However, ordinary operators can change only ordinary states $|n\rangle$ whereas tilde operators change only tilde states $|\tilde{n}\rangle$. There exist the following rules of tilde conjugation operation[1]:

$$\begin{aligned} \widetilde{(A_1A_2)} &= \tilde{A}_1\tilde{A}_2\,,\\ \widetilde{(c_1A_1 + c_2A_2)} &= c_1^*\tilde{A}_1 + c_2^*\tilde{A}_2\,, \end{aligned} \tag{4}$$

where c_1, c_2 are c-numbers. The asterisk denotes the complex conjugation. The tilde operation is supposed to commute with the Hermitian conjugation:

$$(\tilde{A})^\dagger = \widetilde{A^\dagger}\,. \tag{5}$$

Moreover, it is required that ordinary and tilde operators should commute or anticommute with each other

$$[\widetilde{A_1}, A_2]_\mp = 0 \tag{6}$$

depending on their bosonic or fermionic nature.

Introduction of the doubled Hilbert space of a heated system enables one to construct the above thermal vacuum $|0(T)\rangle$ in the following manner:

$$|0(T)\rangle = \sum_n \mathrm{e}^{-E_n/2T}\mathrm{e}^{i\alpha_n}|n\rangle \otimes |\tilde{n}\rangle \Bigg/ \sqrt{\sum_n \mathrm{e}^{-E_n/T}} \quad (\alpha_n \in \mathbb{R})\,. \tag{7}$$

Hereafter, the thermal vacuum (7) will be referred to as the exact thermal vacuum. The thermal vacuum is invariant under the tilde operation.

Till now our discussion follows the line of Refs. 1, 2. The new point appears in the definition of the so-called DTCR. Originally,[1] DTCR was introduced in the form

$$\tilde{\tilde{A}} = \rho_A\, A\,, \tag{8}$$

where $\rho_A = 1$ if A is a bosonic operator and $\rho_A = -1$ if A is a fermionic one. This DTCR form was used in all of the TFD applications to nuclear structure problems, for example, in Refs. 6–19. However, quite long ago the other form of DTCR was proposed by Ojima,[26] namely,

$$\tilde{\tilde{A}} = A\,. \tag{9}$$

That is, the Ojima version of DTCR does not distinguish between bosonic and fermionic operators. This form is based on the equivalence between the algebraic structure of TFD and that of the axiomatic statistical mechanics (c^*-algebra approach) established by Ojima. In the present paper, we will compare consequences of using the two DTCR versions. For that we define DTCR for a fermionic A in the following manner:

$$\tilde{\tilde{A}} = -\sigma^2 A\,, \tag{10}$$

where σ is equal to either 1 ($\tilde{\tilde{A}} = -A$) or i ($\tilde{\tilde{A}} = A$). Thus, the DTCR definition (10) includes both the above DTCR variants.

The Schrödinger equation for a hot system in the doubled Hilbert space reads

$$i\frac{\partial}{\partial t}|\Psi(t,T)\rangle = \mathcal{H}|\Psi(t,T)\rangle\,, \tag{11}$$

since $\mathcal{H} = H - \tilde{H}$ is the time-translation operator and, therefore, the Hamiltonian of a hot system. The operator $\mathcal{H}$ is named the thermal Hamiltonian.[1] The exact thermal vacuum (7) is the eigenstate of $\mathcal{H}$ corresponding to the zero eigenvalue. Equation (11) describes the time evolution of a system at finite temperature and, thus, is the analog of the Liouville–von Neumann equation (1).

The properties of elementary excitations of the system at $T \neq 0$ are determined by $\mathcal{H}$. Excitation energies of various modes at $T \neq 0$ are the eigenvalues of $\mathcal{H}$ but not of the original Hamiltonian H and in general they depend on T. Moreover, any eigenstate of $\mathcal{H}$ with positive energy has its counterpart — the tilde-conjugate eigenstate with negative energy. Following Umezawa, we consider creation of a tilde state with negative energy as annihilation of a thermally excited state. This is a way to treat excitation and de-excitation processes in a heated quantum system within TFD.

Whereas the dynamical development of the system is carried by the thermal Hamiltonian, its thermal behavior is controlled by the thermal vacuum. The grand canonical value of an observable corresponding to operator A should be calculated as $\langle 0(T)|A|0(T)\rangle$, i.e. after diagonalizing $\mathcal{H}$.

Obviously, in most cases one cannot diagonalize $\mathcal{H}$ exactly and, thus, find the exact thermal vacuum and other eigenstates. Usually, one resorts to some approximations and then finds an approximate thermal vacuum state, e.g. the thermal vacua corresponding to the HFB or the random phase approximations. In the case that there appear several solutions in the given approximation, one should find the minimum of thermodynamical potential Ω to see which of them is realized. In TFD, Ω is given as[1,2]

$$\Omega = \langle \Psi_0(T)|H - \hat{K}T|\Psi_0(T)\rangle\,, \tag{12}$$

where $\Psi_0(T)$ is the approximate thermal vacuum and $\hat{K}$ is the entropy operator of the system.

3. The Finite Temperature QPM

3.1. *The thermal QPM Hamiltonian*

In what follows, we will use the microscopic Hamiltonian of the QPM[20]

$$H_{\mathrm{QPM}} = H_{\mathrm{sp}} + H_{\mathrm{pair}} + H_{\mathrm{ph}}\,. \tag{13}$$

The Hamiltonian (13) includes average fields of protons and neutrons

$$H_{\mathrm{sp}} = \sum_{\tau}\sum_{jm}^{\tau}(E_j - \lambda_\tau)a^\dagger_{jm}a_{jm}\,, \tag{14}$$

pairing interactions of the BCS type

$$H_{\mathrm{pair}} = -\frac{1}{4}\sum_{\tau}G_\tau \sum_{\substack{j_1m_1\\ j_2m_2}}^{\tau} a^\dagger_{j_1m_1}a^\dagger_{\overline{j_1m_1}}a_{\overline{j_2m_2}}a_{j_2m_2} \quad (a_{\overline{jm}} = (-1)^{j-m}a_{j-m})\,, \tag{15}$$

and effective multipole–multipole isoscalar and isovector forces

$$H_{\mathrm{ph}} = -\frac{1}{2}\sum_{\lambda\mu}\sum_{\tau\rho=\pm1}(\kappa_0^{(\lambda)} + \rho\kappa_1^{(\lambda)})M^\dagger_{\lambda\mu}(\tau)M_{\lambda\mu}(\rho\tau)\,. \tag{16}$$

The single-particle operator $M^\dagger_{\lambda\mu}(\tau)$ reads

$$M^\dagger_{\lambda\mu}(\tau) = \sum_{\substack{j_1m_1\\ j_2m_2}}^{\tau}\langle j_1m_1|i^\lambda R_\lambda(r)Y_{\lambda\mu}(\theta,\phi)|j_2m_2\rangle a^\dagger_{j_1m_1}a_{j_2m_2}\,. \tag{17}$$

Here, an operator $a^\dagger_{jm}$ (a_{jm}) is the creation (annihilation) operator of nucleon in a single-particle subshell with quantum numbers $nljm \equiv jm$ and energy E_j; index

$\tau = n, p$ is the isotopic one, changing the sign of τ means the interchange $n \leftrightarrow p$, the notation $\sum^{\tau}$ implies a summation over neutron ($\tau = n$) or proton ($\tau = p$) single-particle states only. The parameter G_τ is the proton–proton or neutron–neutron pairing interaction constant; $R_\lambda(r)$ is the radial form factor of the λ-pole separable interaction, $Y_{\lambda\mu}(\theta, \phi)$ is the corresponding spherical harmonic; $\kappa_0^{(\lambda)}$ and $\kappa_1^{(\lambda)}$ are the coupling constants of isoscalar and isovector multipole–multipole interactions of λ multipolarity. The value λ_τ is the neutron or proton chemical potential (the Fermi level).

To find an excitation spectrum of a hot nucleus governed by the QPM Hamiltonian (13), at the beginning we should construct the thermal QPM Hamiltonian $\mathcal{H}_{\mathrm{QPM}}$

$$\mathcal{H}_{\mathrm{QPM}} = H_{\mathrm{QPM}} - \tilde{H}_{\mathrm{QPM}}\,, \tag{18}$$

where $\tilde{H}_{\mathrm{QPM}} = \tilde{H}_{\mathrm{sp}} + \tilde{H}_{\mathrm{pair}} + \tilde{H}_{\mathrm{ph}}$ is the tilde counterpart of H_{QPM} created by the tilde conjugation rules (4). Then we should diagonalize $\mathcal{H}_{\mathrm{QPM}}$. This procedure is quite similar to that used in the standard QPM,[20] i.e. at $T = 0$. The main difference lies in the doubled Hilbert space of a heated nucleus.

3.2. *Thermal quasiparticles*

The first step is diagonalization of part of the full thermal Hamiltonian, namely the sum of the two first terms $\mathcal{H}_{\mathrm{sp}} + \mathcal{H}_{\mathrm{pair}}$, which in the following will be referred to as the thermal BCS Hamiltonian $\mathcal{H}_{\mathrm{BCS}}$. To this aim, we make the Bogoliubov u, v-transformation from nucleon operators $a^\dagger, a$ to quasiparticle operators $\alpha^\dagger, \alpha$:

$$\begin{aligned} \alpha^\dagger_{jm} &= u_j a^\dagger_{jm} - v_j a_{\overline{jm}}\,, \\ \alpha_{jm} &= u_j a_{jm} - v_j a^\dagger_{\overline{jm}} \quad (u_j^2 + v_j^2 = 1)\,. \end{aligned} \tag{19}$$

The same transformation with the same u, v coefficients has to be applied to nucleonic tilde operators $\tilde{a}^\dagger_{jm}$, $\tilde{a}_{jm}$, thus producing the tilde quasiparticle operators $\tilde{\alpha}^\dagger_{jm}$ and $\tilde{\alpha}_{jm}$.

Thermal effects appear after the second (or thermal) Bogoliubov transformation, which mixes ordinary and tilde quasiparticle operators and, thus, produces the operators of so-called thermal quasiparticles $\beta^\dagger_{jm}$, β_{jm}, and their tilde counterparts

$$\begin{aligned} \beta^\dagger_{jm} &= x_j \alpha^\dagger_{jm} - \sigma y_j \tilde{\alpha}_{jm}\,, \\ \tilde{\beta}^\dagger_{jm} &= x_j \tilde{\alpha}^\dagger_{jm} + \sigma y_j \alpha_{jm} \quad (x_j^2 + y_j^2 = 1)\,. \end{aligned} \tag{20}$$

Note that in contrast with Refs. 6, 7, 9, 10, 12 and many others, we include the factor σ from the definition of DTCR (10) to the thermal Bogoliubov transformation (20).

To find the coefficients u, v, we express the thermal Hamiltonian in terms of thermal quasiparticle operators (20) and then require that the one-body part of the thermal BCS Hamiltonian has to be diagonal in terms of thermal quasiparticles. This leads to the following equations for the u, v coefficients:

$$u_j^2 = \frac{1}{2}\left(1 + \frac{E_j - \lambda_\tau}{\varepsilon_j}\right), \quad v_j^2 = \frac{1}{2}\left(1 - \frac{E_j - \lambda_\tau}{\varepsilon_j}\right), \tag{21}$$

where $\varepsilon_j = \sqrt{(E_j - \lambda_\tau)^2 + \Delta_\tau^2}$. The gap parameter Δ_τ and the chemical potential λ_τ are the solutions of the equations

$$\Delta_\tau = \frac{G_\tau}{2} \sum_j^\tau (2j+1)(x_j^2 - y_j^2) u_j v_j, \quad N_\tau = \sum_j^\tau (2j+1)(v_j^2 x_j^2 + u_j^2 y_j^2), \tag{22}$$

where N_τ is the number of neutrons or protons in a nucleus. Actually, the equation for N_τ is not a consequence of the procedure described above but rather an additional demand that a number of particles of any kind in the heated system are conserved in average.

With u_j, v_j from (21) the one-body part of the thermal BCS Hamiltonian reads

$$\mathcal{H}_{\text{BCS}} = \mathcal{H}_{\text{sp}} + \mathcal{H}_{\text{pair}} \simeq \sum_\tau \sum_{jm}^\tau \varepsilon_j (\beta_{jm}^\dagger \beta_{jm} - \tilde{\beta}_{jm}^\dagger \tilde{\beta}_{jm}). \tag{23}$$

Thus, $\mathcal{H}_{\text{BCS}}$ describes a system of noninteracting thermal quasiparticles and tilde quasiparticles with energies ε_j and $-\varepsilon_j$, respectively. The vacuum for thermal quasiparticles is given by[1,26]

$$|0(\beta, \tilde{\beta})\rangle = \exp\{-\hat{K}_f/2\} \exp\left\{ \sigma^* \sum_\tau \sum_{jm}^\tau \alpha_{jm}^\dagger \tilde{\alpha}_{jm}^\dagger \right\} |0(\alpha)\rangle |0(\tilde{\alpha})\rangle, \tag{24}$$

where $|0(\alpha)\rangle$ and $|0(\tilde{\alpha})\rangle$ are the vacua for ordinary and tilde Bogoliubov quasiparticles, respectively. The operator $\hat{K}_f$ is the entropy operator. It reads

$$\hat{K}_f = -\sum_\tau \sum_{jm}^\tau \{\alpha_{jm}^\dagger \alpha_{jm} \ln y_j^2 + \alpha_{jm} \alpha_{jm}^\dagger \ln x_j^2\}. \tag{25}$$

We should stress that although the vacuum (24) is the eigenstate of the thermal BCS Hamiltonian (23) with zero eigenvalue it is not yet a thermal vacuum state in the sense of (3). To determine the thermal vacuum state corresponding to $\mathcal{H}_{\text{BCS}}$, we need to fix appropriately the coefficients x_j, y_j. They can be found by minimizing the thermodynamic potential

$$\begin{aligned} \Omega_f &= \langle 0(\beta, \tilde{\beta}) | (H_{\text{sp}} + H_{\text{pair}}) - T\hat{K}_f | 0(\beta, \tilde{\beta}) \rangle \\ &= \sum_\tau \sum_{jm}^\tau \{\varepsilon_j y_j^2 + T(y_j^2 \ln y_j^2 + x_j^2 \ln x_j^2)\}. \end{aligned} \tag{26}$$

Note that Ω_f contains the ordinary operators H_{sp} and H_{pair} but not the thermal ones $\mathcal{H}_{\text{sp}}$ and $\mathcal{H}_{\text{pair}}$. As a result of variational procedure, we obtain

$$y_j = \left[1 + \exp\left(\frac{\varepsilon_j}{T}\right)\right]^{-1/2}, \quad x_j = (1 - y_j^2)^{1/2}. \tag{27}$$

Thus, the coefficients y_j^2 are nothing else than the thermal occupation factors of the Fermi–Dirac statistics.

The thermal quasiparticle vacuum (24) with the coefficients y_j, x_j (27) is the thermal vacuum in the thermal BCS approximation. Hereafter, it will be denoted by $|0(T);\,\mathrm{qp}\rangle_\sigma$.

The average number of thermally excited Bogoliubov quasiparticles with quantum numbers jm in the BCS thermal vacuum $|0(T);\mathrm{qp}\rangle_\sigma$ is

$$_\sigma\langle 0(T);\mathrm{qp}|\alpha^\dagger_{jm}\alpha_{jm}|0(T);\mathrm{qp}\rangle_\sigma = y_j^2\,, \tag{28}$$

whereas the average number of nucleons in the same state is

$$n_j(T) = {}_\sigma\langle 0(T);\mathrm{qp}|a^\dagger_{jm}a_{jm}|0(T);\mathrm{qp}\rangle_\sigma = u_j^2y_j^2 + v_j^2x_j^2\,. \tag{29}$$

The function $n_j(T)$ determines smearing of the Fermi surface due to thermal and pairing effects.

Equations (22) with y_j, x_j (27) are the well-known BCS -equations at finite temperature (see, for example, Refs. 28–30).

Since the thermal vacuum $|0(T);\mathrm{qp}\rangle_\sigma$ contains a certain number of the Bogoliubov quasiparticles, the excited states at finite temperature can be built on the top of $|0(T);\mathrm{qp}\rangle_\sigma$ by either adding or eliminating a Bogoliubov quasiparticle. Due to the relations,

$$\alpha^\dagger_{jm}|0(T);\mathrm{qp}\rangle_\sigma = x_j\beta^\dagger_{jm}|0(T);\mathrm{qp}\rangle_\sigma, \quad \alpha_{\overline{jm}}|0(T);\mathrm{qp}\rangle_\sigma = \sigma^* y_j\tilde\beta^\dagger_{\overline{jm}}|0(T);\mathrm{qp}\rangle_\sigma\,, \tag{30}$$

one can associate the first process with creation of a thermal quasiparticle having a positive energy, whereas the second process can be considered as creation of a tilde thermal quasiparticle having a negative energy.

The simplest excitations on the top of the BCS thermal vacuum in an even–even hot nucleus involve two thermal quasiparticles. Their wave functions and energies are

$$\begin{aligned}
&[\beta^\dagger_{j_1}\beta^\dagger_{j_2}]^\lambda_\mu|0(T);\mathrm{qp}\rangle_\sigma, \quad \omega = \varepsilon_{j_1}+\varepsilon_{j_2} \equiv \varepsilon^{(+)}_{j_1j_2}\,;\\
&[\tilde\beta^\dagger_{\overline{j_1}}\tilde\beta^\dagger_{\overline{j_2}}]^\lambda_\mu|0(T);\mathrm{qp}\rangle_\sigma, \quad \omega = -\varepsilon^{(+)}_{j_1j_2}\,;\\
&[\beta^\dagger_{j_1}\tilde\beta^\dagger_{\overline{j_2}}]^\lambda_\mu|0(T);\mathrm{qp}\rangle_\sigma, \quad \omega = \varepsilon_{j_1}-\varepsilon_{j_2} \equiv \varepsilon^{(-)}_{j_1j_2}\,;\\
&[\tilde\beta^\dagger_{\overline{j_1}}\beta^\dagger_{j_2}]^\lambda_\mu|0(T);\mathrm{qp}\rangle_\sigma, \quad \omega = -\varepsilon^{(-)}_{j_1j_2}\,.
\end{aligned} \tag{31}$$

The square brackets $[\ \ ]^\lambda_\mu$ in (31) mean the coupling of single-particle momenta j_1, j_2 to the total angular momentum λ with the magnetic quantum number μ.

As it should be, any thermal two-quasiparticle state with positive energy ω has a counterpart — a tilde-conjugated state with negative energy $-\omega$.

Quite interesting relations exist between electromagnetic transition probabilities to thermal two-quasiparticle states and their tilde counterparts. Let us write the

$E\lambda$-transition operator $\mathcal{M}(E\lambda\mu)$ in terms of thermal quasiparticles

$$\mathcal{M}(E\lambda\mu) = \frac{1}{\hat{\lambda}} \sum_{\tau} \sum_{j_1 j_2}^{\tau} \Gamma^{(\lambda)}_{j_1 j_2} \{A^\dagger_{\lambda\mu}(j_1 j_2) + A_{\overline{\lambda\mu}}(j_1 j_2) + B_{\lambda\mu}(j_1 j_2)\},$$

$$A^\dagger_{\lambda\mu}(j_1 j_2) = \frac{1}{2} u^{(+)}_{j_1 j_2} (x_{j_1} x_{j_2} [\beta^\dagger_{j_1} \beta^\dagger_{j_2}]^\lambda_\mu - \sigma^2 y_{j_1} y_{j_2} [\tilde{\beta}^\dagger_{\overline{j_1}} \tilde{\beta}^\dagger_{\overline{j_2}}]^\lambda_\mu) - \sigma^* v^{(-)}_{j_1 j_2} x_{j_1} y_{j_2} [\beta^\dagger_{j_1} \tilde{\beta}^\dagger_{\overline{j_2}}]^\lambda_\mu ,$$

$$A_{\overline{\lambda\mu}}(j_1 j_2) = (-1)^{\lambda-\mu} [A^\dagger_{\lambda-\mu}(j_1 j_2)]^\dagger ,$$

$$\begin{aligned} B_{\lambda\mu}(j_1 j_2) = &-v^{(-)}_{j_1 j_2} (x_{j_1} x_{j_2} [\beta^\dagger_{j_1} \beta_{\overline{j_2}}]^\lambda_\mu + y_{j_1} y_{j_2} [\tilde{\beta}^\dagger_{\overline{j_1}} \tilde{\beta}_{j_2}]^\lambda_\mu) \\ &+ u^{(+)}_{j_1 j_2} (\sigma x_{j_1} y_{j_2} [\beta^\dagger_{j_1} \tilde{\beta}_{j_2}]^\lambda_\mu - \sigma^* y_{j_1} x_{j_2} [\tilde{\beta}^\dagger_{\overline{j_1}} \beta_{\overline{j_2}}]^\lambda_\mu) . \end{aligned} \tag{32}$$

In (32), $\Gamma^{(\lambda)}_{j_1 j_2}$ is the reduced single-particle matrix element of the $E\lambda$-transition operator; $u^{(+)}_{j_1 j_2} = u_{j_1} v_{j_2} + u_{j_2} v_{j_1}$, $v^{(-)}_{j_1 j_2} = u_{j_1} u_{j_2} - v_{j_1} v_{j_2}$ and $\hat{\lambda} = \sqrt{2\lambda + 1}$.

Only the terms containing the operators $A^\dagger_{\lambda\mu}(j_1 j_2)$ and $A_{\overline{\lambda\mu}}(j_1 j_2)$ contribute to transitions from the BCS thermal vacuum state to any thermal two-quasiparticle state. Introducing the functions

$$Y(\omega) = \left[\exp\left(\frac{\omega}{T}\right) - 1\right]^{-1/2} ; \quad X(\omega) = [1 + Y^2(\omega)]^{1/2} \tag{33}$$

and taking the advantage of the relations

$$\begin{aligned} y^2_{j_1} y^2_{j_1} &= (1 - y^2_{j_1} - y^2_{j_2}) Y^2(\varepsilon^{(+)}_{j_1 j_2}) , \\ x^2_{j_1} y^2_{j_2} &= (y^2_{j_2} - y^2_{j_1}) Y^2(\varepsilon^{(-)}_{j_1 j_2}) \quad \text{for} \quad \varepsilon_{j_1} > \varepsilon_{j_2} , \end{aligned} \tag{34}$$

we get the squared reduced matrix elements Φ^2_λ of the operator $\mathcal{M}(E\lambda\mu)$ between the BCS thermal vacuum and different thermal two-quasiparticle states (31):

$$\begin{aligned} \Phi^2_\lambda([\beta^\dagger_{j_1} \beta^\dagger_{j_2}]^\lambda_\mu) &= (\Gamma^{(\lambda)}_{j_1 j_2} u^{(+)}_{j_1 j_2})^2 (1 - y^2_{j_1} - y^2_{j_2}) X^2(\varepsilon^{(+)}_{j_1 j_2}), \\ \Phi^2_\lambda([\tilde{\beta}^\dagger_{\overline{j_1}} \tilde{\beta}^\dagger_{\overline{j_2}}]^\lambda_\mu) &= (\Gamma^{(\lambda)}_{j_1 j_2} u^{(+)}_{j_1 j_2})^2 (1 - y^2_{j_1} - y^2_{j_2}) Y^2(\varepsilon^{(+)}_{j_1 j_2}) , \\ \Phi^2_\lambda([\beta^\dagger_{j_1} \tilde{\beta}^\dagger_{\overline{j_2}}]^\lambda_\mu) &= (\Gamma^{(\lambda)}_{j_1 j_2} v^{(-)}_{j_1 j_2})^2 \times \begin{cases} (y^2_{j_2} - y^2_{j_1}) X^2(\varepsilon^{(-)}_{j_1 j_2}), & (\varepsilon_{j_1} > \varepsilon_{j_2}) , \\ (y^2_{j_1} - y^2_{j_2}) Y^2(\varepsilon^{(-)}_{j_2 j_1}), & (\varepsilon_{j_1} < \varepsilon_{j_2}) , \end{cases} \\ \Phi^2_\lambda([\tilde{\beta}^\dagger_{\overline{j_1}} \beta^\dagger_{j_2}]^\lambda_\mu) &= (\Gamma^{(\lambda)}_{j_1 j_2} v^{(-)}_{j_1 j_2})^2 \times \begin{cases} (y^2_{j_2} - y^2_{j_1}) Y^2(\varepsilon^{(-)}_{j_1 j_2}), & (\varepsilon_{j_1} > \varepsilon_{j_2}) , \\ (y^2_{j_1} - y^2_{j_2}) X^2(\varepsilon^{(-)}_{j_2 j_1}), & (\varepsilon_{j_1} < \varepsilon_{j_2}) . \end{cases} \end{aligned} \tag{35}$$

From (35) it follows that transition probabilities to the states tilde-conjugated to each other and, correspondingly, having the energies $\pm\omega$ relate with the factor

$$\Phi^2_\lambda(\omega) = \exp\left(\frac{\omega}{T}\right) \Phi^2_\lambda(-\omega) . \tag{36}$$

Formally, the function $Y(\omega)$ in (35) is the Bose–Einstein distribution function, which determines the average number of bosons with energy ω in a system in the thermal equilibrium at temperature T. This gives an idea to treat two-fermion excitations in a hot nucleus as bosons. The probability to create a boson is proportional to the factor $[1+Y^2(\omega)]$, whereas the probability to annihilate it is proportional to $Y^2(\omega)$.

Expressions (35) determine $E\lambda$-strength distribution in a hot nucleus within the independent BCS quasiparticle approximation. An interesting point is that in contrast with $T=0$ case a portion of the $E\lambda$-strength appears in the negative energy region $\omega<0$, i.e. below the thermal vacuum state, at finite temperatures. This strength determines a probability of γ-ray emission by a hot nucleus, whereas the strength at $\omega>0$ determines a photoabsorption cross section. Both the parts of the $E\lambda$-strength contribute to the energy weighted sum rule (EWSR) at $T\neq 0$

$$\begin{aligned}\mathrm{EWSR} &= \sum_\tau \sum_{j_1\geq j_2}^{\tau} \varepsilon^{(+)}_{j_1j_2}[\Phi^2_\lambda([\beta^\dagger_{j_1}\beta^\dagger_{j_2}]^\lambda_\mu) - \Phi^2_\lambda([\tilde\beta^\dagger_{\overline{j_1}}\tilde\beta^\dagger_{\overline{j_2}}]^\lambda_\mu)]\\ &\quad + \sum_\tau \sum_{j_1\geq j_2}^{\tau} \varepsilon^{(-)}_{j_1j_2}[\Phi^2_\lambda([\beta^\dagger_{j_1}\tilde\beta^\dagger_{\overline{j_2}}]^\lambda_\mu) - \Phi^2_\lambda([\tilde\beta^\dagger_{\overline{j_1}}\beta^\dagger_{j_2}]^\lambda_\mu)]\\ &= \sum_\tau \sum_{j_1\geq j_2}^{\tau} (\Gamma^{(\lambda)}_{j_1j_2})^2[\varepsilon^{(+)}_{j_1j_2}(u^{(+)}_{j_1j_2})^2(1-y^2_{j_1}-y^2_{j_2}) - \varepsilon^{(-)}_{j_1j_2}(v^{(-)}_{j_1j_2})^2(y^2_{j_1}-y^2_{j_2})]\,.\end{aligned} \tag{37}$$

3.3. *Thermal phonons*

The second step in the diagonalization of the thermal Hamiltonian is to take into account the long-range particle–hole interaction $H_{\rm ph}$. This interaction is responsible for the existence of different types of collective vibrations in nuclei. In this subsection, the thermal quasiparticle random phase approximation in treating the vibrations of hot nuclei is discussed.

Transformations (19) and (20) with the coefficients determined by Eqs. (21) and (27) should be applied to the rest of the thermal Hamiltonian $H_{\rm ph}-\tilde H_{\rm ph}$ as well. Then the thermal Hamiltonian (18) takes the form

$$\begin{aligned}\mathcal{H}_{\rm QPM} &= \sum_\tau \sum_{jm}^{\tau} \varepsilon_j(\beta^\dagger_{jm}\beta_{jm} - \tilde\beta^\dagger_{jm}\tilde\beta_{jm})\\ &\quad - \frac12 \sum_{\lambda\mu}\sum_{\tau\rho=\pm1}(\kappa^{(\lambda)}_0+\rho\kappa^{(\lambda)}_1)\{M^\dagger_{\lambda\mu}(\tau)M_{\lambda\mu}(\rho\tau) - \tilde M^\dagger_{\lambda\mu}(\tau)\tilde M_{\lambda\mu}(\rho\tau)\}\,.\end{aligned} \tag{38}$$

In terms of thermal quasiparticles, the multipole operator $M^\dagger_{\lambda\mu}(\tau)$ has the same shape as the $E\lambda$-transition operator (32). The only difference is that one should substitute matrix elements $\Gamma^{(\lambda)}_{j_1j_2}$ for matrix elements $f^{(\lambda)}_{j_1j_2} = \langle j_1\| i^\lambda R_\lambda(r) Y_\lambda(\theta,\phi)\| j_2\rangle$.

The thermal Hamiltonian (38) can be approximately reduced to the Hamiltonian of noninteracting bosonic excitations — thermal phonons. This occurs if one omits in (38) the terms containing the operator $B_{\lambda\mu}(j_1j_2)$. Hereafter, the remaining part of the thermal Hamiltonian (38) is denoted $\mathcal{H}_{\mathrm{TRPA}}$.

To diagonalize $\mathcal{H}_{\mathrm{TRPA}}$, it is natural to use a trial wave function, which is a linear superposition of different types of thermal two-quasiparticle operators, namely

$$\begin{aligned} Q^\dagger_{\lambda\mu i} = \frac{1}{2}\sum_\tau \sum^{\tau}_{j_1j_2} (&\psi^{\lambda i}_{j_1j_2}[\beta^\dagger_{j_1}\beta^\dagger_{j_2}]^\lambda_\mu + \tilde{\psi}^{\lambda i}_{j_1j_2}[\tilde{\beta}^\dagger_{\overline{j_1}}\tilde{\beta}^\dagger_{\overline{j_2}}]^\lambda_\mu + 2\sigma\eta^{\lambda i}_{j_1j_2}[\beta^\dagger_{j_1}\tilde{\beta}^\dagger_{\overline{j_2}}]^\lambda_\mu) \\ &+ (-1)^{\lambda-\mu}\left(\phi^{\lambda i}_{j_1j_2}[\beta_{j_1}\beta_{j_2}]^\lambda_{-\mu} + \tilde{\phi}^{\lambda i}_{j_1j_2}[\tilde{\beta}_{\overline{j_1}}\tilde{\beta}_{\overline{j_2}}]^\lambda_{-\mu} + 2\sigma^*\xi^{\lambda i}_{j_1j_2}[\beta_{j_1}\tilde{\beta}_{\overline{j_2}}]^\lambda_{-\mu}\right). \end{aligned} \tag{39}$$

The factors σ and σ^* at the crossover (i.e. tilde–nontilde) terms of (39) are absent in the thermal phonon definition in Refs. 10, 12, 18. They appear due to adoption of the new DTCR (10). The definition (39) coincides with the one in Refs. 10, 12, 18 when $\sigma = 1$.

One more very important assumption has to be accepted. We assume that the thermal biquasiparticle operators contained in (39) commute like bosonic operators

$$\begin{aligned} &[[\beta_{j_1}\beta_{j_2}]^\lambda_\mu, [\beta^\dagger_{j_3}\beta^\dagger_{j_4}]^{\lambda'}_{\mu'}] \approx -\delta_{\lambda\lambda'}\delta_{\mu\mu'}\left(\delta_{j_1j_3}\delta_{j_2j_4} + (-1)^{j_1-j_2+\lambda}\delta_{j_1j_4}\delta_{j_2j_3}\right), \\ &[[\beta_{j_1}\tilde{\beta}_{j_2}]^\lambda_\mu, [\beta^\dagger_{j_3}\tilde{\beta}^\dagger_{j_4}]^{\lambda'}_{\mu'}] \approx -\delta_{\lambda\lambda'}\delta_{\mu\mu'}\delta_{j_1j_3}\delta_{j_2j_4}, \end{aligned} \tag{40}$$

and all other commutators are equal to zero. This assumption is known as the quasiboson approximation.

Moreover, taking into consideration the long-range interaction we should redefine the thermal vacuum state. At first, we define it as the vacuum state $|0(Q,\tilde{Q})\rangle_\sigma$ for thermal phonons

$$Q_{\lambda\mu i}|0(Q,\tilde{Q})\rangle_\sigma = 0, \quad \tilde{Q}_{\lambda\mu i}|0(Q,\tilde{Q})\rangle_\sigma = 0. \tag{41}$$

A thermal one-phonon state is constructed by acting on the thermal phonon vacuum $|0(Q,\tilde{Q})\rangle_\sigma$ by the thermal phonon creation operator

$$Q^\dagger_{\lambda\mu i}|0(Q,\tilde{Q})\rangle_\sigma. \tag{42}$$

A thermal tilde one-phonon state should be define as

$$\tilde{Q}^\dagger_{\overline{\lambda\mu i}}|0(Q,\tilde{Q})\rangle_\sigma \equiv (-1)^{\lambda-\mu}\tilde{Q}^\dagger_{\lambda-\mu i}|0(Q,\tilde{Q})\rangle_\sigma \tag{43}$$

because just the operator $(-1)^{\lambda-\mu}\tilde{Q}^\dagger_{\lambda-\mu i}$ transforms under spatial rotations like a spherical tensor of rank λ.

The set of thermal one-phonon wave functions has to be orthonormalized. This demand together with the quasiboson approximation for commutators of biquasiparticle operators (40) imposes the following constraints on the phonon amplitudes ψ, ϕ, $\tilde{\psi}$, $\tilde{\phi}$, η, ξ:

$$\begin{aligned} &\frac{1}{2}\sum_{\tau}\sum_{j_1j_2}^{\tau} g^{\lambda i}_{j_1j_2} w^{\lambda i'}_{j_1j_2} + \tilde{g}^{\lambda i}_{j_1j_2} \tilde{w}^{\lambda i'}_{j_1j_2} + t^{\lambda i}_{j_1j_2} s^{\lambda i'}_{j_1j_2} + \tilde{t}^{\lambda i}_{j_1j_2} \tilde{s}^{\lambda i'}_{j_1j_2} = \delta_{ii'}\,, \\ &\sum_{\tau}\sum_{j_1j_2}^{\tau} g^{\lambda i}_{j_1j_2} \tilde{w}^{\lambda i'}_{j_1j_2} + \tilde{g}^{\lambda i}_{j_1j_2} w^{\lambda i'}_{j_1j_2} + t^{\lambda i}_{j_1j_2} \tilde{s}^{\lambda i'}_{j_1j_2} + \tilde{t}^{\lambda i}_{j_1j_2} s^{\lambda i'}_{j_1j_2} = 0\,. \end{aligned} \tag{44}$$

In (44), the following notation is introduced for the sums and differences of original phonon amplitudes:

$$\begin{aligned} &\begin{pmatrix} g \\ w \end{pmatrix}^{\lambda i}_{j_1j_2} = \psi^{\lambda i}_{j_1j_2} \pm \phi^{\lambda i}_{j_1j_2}, \quad \begin{pmatrix} \tilde{g} \\ \tilde{w} \end{pmatrix}^{\lambda i}_{j_1j_2} = \tilde{\psi}^{\lambda i}_{j_1j_2} \pm \tilde{\phi}^{\lambda i}_{j_1j_2}\,, \\ &\begin{pmatrix} t \\ s \end{pmatrix}^{\lambda i}_{j_1j_2} = \eta^{\lambda i}_{j_1j_2} \pm \xi^{\lambda i}_{j_1j_2}, \quad \begin{pmatrix} \tilde{t} \\ \tilde{s} \end{pmatrix}^{\lambda i}_{j_1j_2} = \tilde{\eta}^{\lambda i}_{j_1j_2} \pm \tilde{\xi}^{\lambda i}_{j_1j_2}\,. \end{aligned} \tag{45}$$

Moreover, $\tilde{\eta}^{\lambda i}_{j_1j_2} \equiv (-1)^{j_1-j_2+\lambda}\eta^{\lambda i}_{j_2j_1}$, $\tilde{\xi}^{\lambda i}_{j_1j_2} \equiv (-1)^{j_1-j_2+\lambda}\xi^{\lambda i}_{j_2j_1}$.

Constraints (44) imply that thermal phonon operators commute like bosons.

With constraints (44) one can find the inverse transformation to (39)

$$\begin{aligned} [\beta^{\dagger}_{j_1}\beta^{\dagger}_{j_2}]^{\lambda}_{\mu} &= \sum_{i}^{\tau} \psi^{\lambda i}_{j_1j_2} Q^{\dagger}_{\lambda\mu i} + \phi^{\lambda i}_{j_1j_2} Q_{\overline{\lambda\mu} i} + \tilde{\psi}^{\lambda i}_{j_1j_2} \tilde{Q}^{\dagger}_{\overline{\lambda\mu} i} + \tilde{\phi}^{\lambda i}_{j_1j_2} \tilde{Q}_{\lambda\mu i}\,, \\ [\beta^{\dagger}_{j_1}\tilde{\beta}^{\dagger}_{\overline{j_2}}]^{\lambda}_{\mu} &= \sigma^* \sum_{i}^{\tau} \eta^{\lambda i}_{j_1j_2} Q^{\dagger}_{\lambda\mu i} + \xi^{\lambda i}_{j_1j_2} Q_{\overline{\lambda\mu} i} + \tilde{\eta}^{\lambda i}_{j_1j_2} \tilde{Q}^{\dagger}_{\overline{\lambda\mu} i} + \tilde{\xi}^{\lambda i}_{j_1j_2} \tilde{Q}_{\lambda\mu i}\,, \end{aligned} \tag{46}$$

and then evaluate the following expression of the thermal RPA Hamiltonian $\mathcal{H}_{\mathrm{TRPA}}$ in terms of thermal phonon operators:

$$\begin{aligned} \mathcal{H}_{\mathrm{TRPA}} = &\sum_{\tau}\sum_{jm}^{\tau} \varepsilon_j(\beta^{\dagger}_{jm}\beta^{\dagger}_{jm} - \tilde{\beta}^{\dagger}_{jm}\tilde{\beta}^{\dagger}_{jm}) - \frac{1}{8}\sum_{\lambda\mu ii'}\frac{1}{\hat{\lambda}^2}\sum_{\tau\rho=\pm1}(\kappa^{\lambda}_0 + \rho\kappa^{\lambda}_1) \\ &\times \{[D^{\lambda i}_{\tau} D^{\lambda i'}_{\rho\tau} - \tilde{D}^{\lambda i}_{\tau}\tilde{D}^{\lambda i'}_{\rho\tau}](Q^{\dagger}_{\lambda\mu i} + Q_{\overline{\lambda\mu} i})(Q^{\dagger}_{\overline{\lambda\mu} i'} + Q^{\dagger}_{\lambda\mu i'}) \\ &+ [D^{\lambda i}_{\tau}\tilde{D}^{\lambda i'}_{\rho\tau} - \tilde{D}^{\lambda i}_{\tau} D^{\lambda i'}_{\rho\tau}](Q^{\dagger}_{\lambda\mu i} + Q_{\overline{\lambda\mu} i})(\tilde{Q}^{\dagger}_{\lambda\mu i'} + \tilde{Q}_{\overline{\lambda\mu} i'}) - (\mathrm{t.c.})\}\,. \end{aligned} \tag{47}$$

The notation "(t.c.)" in (47) stands for the items, which are tilde conjugated to the displayed ones. The functions $D^{\lambda i}_{\tau}$ and $\tilde{D}^{\lambda i}_{\tau}$ ($\tau = n, p$) are the following combinations of phonon amplitudes:

$$
\begin{aligned}
D_\tau^{\lambda i} &= \sum_{j_1 j_2}^{\tau} f_{j_1 j_2}^{(\lambda)} [u_{j_1 j_2}^{(+)} (x_{j_1} x_{j_2} g_{j_1 j_2}^{\lambda i} - \sigma^2 y_{j_1} y_{j_2} \tilde{g}_{j_1 j_2}^{\lambda i}) - 2\sigma^2 v_{j_1 j_2}^{(-)} x_{j_1} y_{j_2} t_{j_1 j_2}^{\lambda i}], \\
\tilde{D}_\tau^{\lambda i} &= \sum_{j_1 j_2}^{\tau} f_{j_1 j_2}^{(\lambda)} [u_{j_1 j_2}^{(+)} (x_{j_1} x_{j_2} \tilde{g}_{j_1 j_2}^{\lambda i} - \sigma^2 y_{j_1} y_{j_2} g_{j_1 j_2}^{\lambda i}) - 2\sigma^2 v_{j_1 j_2}^{(-)} y_{j_1} x_{j_2} t_{j_1 j_2}^{\lambda i}] .
\end{aligned}
\tag{48}
$$

To find eigenvalues of $\mathcal{H}_{\mathrm{TRPA}}$, we apply the variational principle, i.e. we minimize the expectation value of $\mathcal{H}_{\mathrm{TRPA}}$ over the thermal one-phonon state under constraints (44).

It should be stressed that the phonon vacuum $|0(Q,\tilde{Q})\rangle_\sigma$ is not the thermal vacuum in the sense of Eq. (3) and, thus, the expectation value of any physical operator with respect to $|0(Q,\tilde{Q})\rangle_\sigma$ does not correspond to the average over the grand canonical ensemble.

After a variation procedure with respect to functions g, w, $\tilde{g}$, $\tilde{w}$, t, and s one gets a homogeneous system of linear equations. Since we use the separable effective interaction, the system of equations looks simple

$$
\begin{aligned}
\begin{pmatrix} \psi \\ \phi \end{pmatrix}_{\tau j_1 j_2}^{\lambda i} &= \frac{1}{2\hat{\lambda}^2} \frac{f_{j_1 j_2}^{(\lambda)} u_{j_1 j_2}^{(+)}}{\varepsilon_{j_1 j_2}^{(+)} \mp \omega_{\lambda i}} \sum_{\rho=\pm 1} (\kappa_0^{(\lambda)} + \rho \kappa_1^{(\lambda)}) [x_{j_1} x_{j_2} D_{\rho\tau}^{\lambda i} + \sigma^2 y_{j_1} y_{j_2} \tilde{D}_{\rho\tau}^{\lambda i}] , \\
\begin{pmatrix} \tilde{\psi} \\ \tilde{\phi} \end{pmatrix}_{\tau j_1 j_2}^{\lambda i} &= \frac{\sigma^2}{2\hat{\lambda}^2} \frac{f_{j_1 j_2}^{(\lambda)} u_{j_1 j_2}^{(+)}}{\varepsilon_{j_1 j_2}^{(+)} \pm \omega_{\lambda i}} \sum_{\rho=\pm 1} (\kappa_0^{(\lambda)} + \rho \kappa_1^{(\lambda)}) [y_{j_1} y_{j_2} D_{\rho\tau}^{\lambda i} + \sigma^2 x_{j_1} x_{j_2} \tilde{D}_{\rho\tau}^{\lambda i}] , \\
\begin{pmatrix} \eta \\ \xi \end{pmatrix}_{\tau j_1 j_2}^{\lambda i} &= -\frac{\sigma^2}{2\hat{\lambda}^2} \frac{f_{j_1 j_2}^{(\lambda)} v_{j_1 j_2}^{(-)}}{\varepsilon_{j_1 j_2}^{(-)} \mp \omega_{\lambda i}} \sum_{\rho=\pm 1} (\kappa_0^{(\lambda)} + \rho \kappa_1^{(\lambda)}) [x_{j_1} y_{j_2} D_{\rho\tau}^{\lambda i} - y_{j_1} x_{j_2} \tilde{D}_{\rho\tau}^{\lambda i}] .
\end{aligned}
\tag{49}
$$

For a further discussion, it is more convenient to rewrite the system (49) regarding the functions $D_{\rho\tau}^{\lambda i}$ and $\tilde{D}_{\rho\tau}^{\lambda i}$ as unknown variables. Then, we get

$$
\sum_{\rho=\pm 1} (\kappa_0^{(\lambda)} + \rho \kappa_1^{(\lambda)}) \begin{pmatrix} D \\ \tilde{D} \end{pmatrix}_{\rho\tau}^{\lambda i} = [X_\tau^{(\lambda)}(\omega_{\lambda i})]^{-1} \begin{pmatrix} D \\ \tilde{D} \end{pmatrix}_{\tau}^{\lambda i} , \tag{50}
$$

where the function $X_\tau^{\lambda}(\omega)$ reads

$$
X_\tau^{(\lambda)}(\omega) = \frac{1}{\hat{\lambda}^2} \sum_{j_1 j_2}^{\tau} (f_{j_1 j_2}^{(\lambda)})^2 \left[\frac{(u_{j_1 j_2}^{(+)})^2 \varepsilon_{j_1 j_2}^{(+)} (1 - y_{j_1}^2 - y_{j_2}^2)}{(\varepsilon_{j_1 j_2}^{(+)})^2 - \omega^2} - \frac{(v_{j_1 j_2}^{(-)})^2 \varepsilon_{j_1 j_2}^{(-)} (y_{j_1}^2 - y_{j_2}^2)}{(\varepsilon_{j_1 j_2}^{(-)})^2 - \omega^2} \right] . \tag{51}
$$

Demanding the existence of a nontrivial solution to the system of linear equations (50), we derive the secular equation for the energy of the thermal one-phonon state $\omega_{\lambda i}$

$$
(\kappa_0^{(\lambda)} + \kappa_1^{(\lambda)}) [X_p^{(\lambda)}(\omega) + X_n^{(\lambda)}(\omega)] - 4\kappa_0^{(\lambda)} \kappa_1^{(\lambda)} X_p^{(\lambda)}(\omega) X_n^{(\lambda)}(\omega) = 1 . \tag{52}
$$

Equation (52) is strictly the same as was obtained by other methods in Refs. 31–33. It does not differ as well from the previous results of the TFD-QPM approach.[10,12]

One obtains the same secular equation as (52) if the variational procedure is applied to the expectation value of $\mathcal{H}_{\rm TRPA}$ over the tilde thermal one-phonon state $\tilde{Q}^{\dagger}_{\overline{\lambda\mu i}}|0(Q,\tilde{Q})\rangle_{\sigma}$. As it was stated in the previous subsection, both the positive and negative energy excitations of a hot nucleus built on the top of the thermal vacuum have a physical meaning and should be considered on equal footing. Hence, the negative roots of (52) are identified with energies of tilde phonons.

Unfortunately, the values $D^{\lambda i}_{\tau}$ and $\tilde{D}^{\lambda i}_{\tau}$ cannot be determined unambiguously from Eq. (50) and the normalization condition (44). The reason for this indeterminacy is the following. Equations (50) enable one to prove that the TRPA Hamiltonian $\mathcal{H}_{\rm TRPA}$ is diagonal in terms of the thermal phonon operators, i.e.

$$\mathcal{H}_{\rm TRPA} = \sum_{\lambda\mu i} \omega_{\lambda i}(Q^{\dagger}_{\lambda\mu i}Q_{\lambda\mu i} - \tilde{Q}^{\dagger}_{\lambda\mu i}\tilde{Q}_{\lambda\mu i})\,. \tag{53}$$

It is seen that the Hamiltonian (53) is invariant under the unitary transformation, which mixes nontilde and tilde thermal phonons but preserves the secular equation (52). Just due to this invariance the systems of Eqs. (49) or (50) cannot be solved unambiguously. To overcome this problem, one needs to involve additional considerations. To this aim, minimization of the thermodynamic potential was used while considering the pairing correlations at finite temperature (see Sec. 3.3). However, in the case of a thermal phonon system this procedure is not so straightforward.

Let us turn back to Eqs. (49). The following interrelations between $D^{\lambda i}_{\tau}$ and $\tilde{D}^{\lambda i}_{\tau}$ can be easily found from the normalization condition (44):

$$(D^{\lambda i}_{\tau})^2 - (\tilde{D}^{\lambda i}_{\tau})^2 = 4\hat{\lambda}^4 \frac{X^{(\lambda)}_{\tau}(\omega_{\lambda i})}{\mathcal{N}^{\lambda i}_{\tau}}\,, \tag{54}$$

where $\mathcal{N}^{\lambda i}_{\tau}$ is given by

$$\mathcal{N}^{\lambda i}_{\tau} = \hat{\lambda}^2 \left[\frac{\partial}{\partial\omega} X^{(\lambda)}_{\tau}(\omega)|_{\omega=\omega_{\lambda i}} \right. \\ \left. + \left(\frac{1 - X^{(\lambda)}_{\tau}(\omega_{\lambda i})(\kappa^{(\lambda)}_0 + \kappa^{(\lambda)}_1)}{X^{(\lambda)}_{-\tau}(\omega_{\lambda i})(\kappa^{(\lambda)}_0 - \kappa^{(\lambda)}_1)} \right)^2 \frac{\partial}{\partial\omega} X^{(\lambda)}_{-\tau}(\omega)|_{\omega=\omega_{\lambda i}} \right]. \tag{55}$$

Instead of the functions $D^{\lambda i}_{\tau}$ and $\tilde{D}^{\lambda i}_{\tau}$, it is convenient to introduce the new variables

$$X_{\lambda i} = \frac{D^{\lambda i}_{\tau}\sqrt{\mathcal{N}^{\lambda i}_{\tau}}}{2\hat{\lambda}^2 X^{(\lambda)}_{\tau}(\omega_{\lambda i})}\,, \quad Y_{\lambda i} = \frac{\tilde{D}^{\lambda i}_{\tau}\sqrt{\mathcal{N}^{\lambda i}_{\tau}}}{2\hat{\lambda}^2 X^{(\lambda)}_{\tau}(\omega_{\lambda i})}\,, \tag{56}$$

which obey the following condition: $X^2_{\lambda i} - Y^2_{\lambda i} = 1$.

Then we substitute $X_{\lambda i}$ and $Y_{\lambda i}$ for $D_\tau^{\lambda i}$ and $\tilde{D}_\tau^{\lambda i}$ in Eqs. (49) and get

$$\begin{aligned}
\begin{pmatrix}\psi\\ \phi\end{pmatrix}^{\lambda i}_{j_1j_2} &= \frac{1}{\sqrt{\mathcal{N}_\tau^{\lambda i}}}\frac{f^{(\lambda)}_{j_1j_2}u^{(+)}_{j_1j_2}}{\varepsilon^{(+)}_{j_1j_2}\mp\omega_{\lambda i}}(x_{j_1}x_{j_2}X_{\lambda i}+\sigma^2 y_{j_1}y_{j_2}Y_{\lambda i})\,,\\
\begin{pmatrix}\tilde{\psi}\\ \tilde{\phi}\end{pmatrix}^{\lambda i}_{j_1j_2} &= \frac{\sigma^2}{\sqrt{\mathcal{N}_\tau^{\lambda i}}}\frac{f^{(\lambda)}_{j_1j_2}u^{(+)}_{j_1j_2}}{\varepsilon^{(+)}_{j_1j_2}\pm\omega_{\lambda i}}(y_{j_1}y_{j_2}X_{\lambda i}+\sigma^2 x_{j_1}x_{j_2}Y_{\lambda i})\,,\\
\begin{pmatrix}\eta\\ \xi\end{pmatrix}^{\lambda i}_{j_1j_2} &= -\frac{\sigma^2}{\sqrt{\mathcal{N}_\tau^{\lambda i}}}\frac{f^{(\lambda)}_{j_1j_2}v^{(-)}_{j_1j_2}}{\varepsilon^{(-)}_{j_1j_2}\mp\omega_{\lambda i}}(x_{j_1}y_{j_2}X_{\lambda i}-y_{j_1}x_{j_2}Y_{\lambda i})\,.
\end{aligned} \tag{57}$$

Some conclusions concerning variables $X_{\lambda i}$ and $Y_{\lambda i}$ can be achieved if one analyzes their behavior at the limit $T\to 0$. Obviously, the phonon wave function (39) at $T=0$ should consist of two components only $\alpha^\dagger_{j_1}\alpha^\dagger_{j_2}$ and $\alpha_{j_2}\alpha_{j_1}$. Since at $T\to 0$ the thermal occupation numbers of quasiparticles y_j tend to zero ($x_j\to 1$), the demand can be fulfilled only if $Y_{\lambda i}\to 0$ (and synchronously $X_{\lambda i}\to 1$). In such a case the tilde amplitudes $\tilde{\psi}^{\lambda i}_{j_1j_2}$ and $\tilde{\phi}^{\lambda i}_{j_1j_2}$ tend to zero whereas $\psi^{\lambda i}_{j_1j_2}$ and $\phi^{\lambda i}_{j_1j_2}$ survive.

Let us define the thermal phonons with the amplitudes corresponding to $Y_{\lambda i}=0$ and $X_{\lambda i}=1$ as $q^\dagger_{\lambda\mu i}, q_{\lambda\mu i}$ and name them "reference phonons" (r-phonons). The vacuum for r-phonons is denoted by $|0(q,\tilde{q})\rangle_\sigma$. From (57) one can conclude that the thermal phonons corresponding to other values of $X_{\lambda i}, Y_{\lambda i}$ can be produced by applying the unitary $\{X_{\lambda i},\, Y_{\lambda i}\}$ transformation to the r-phonons

$$\begin{aligned}
Q^\dagger_{\lambda\mu i} &= X_{\lambda i}q^\dagger_{\lambda\mu i} - Y_{\lambda i}\tilde{q}_{\lambda\mu i}\,,\\
\tilde{Q}^\dagger_{\lambda\mu i} &= X_{\lambda i}\tilde{q}^\dagger_{\lambda\mu i} - Y_{\lambda i}q_{\lambda\mu i}\,.
\end{aligned} \tag{58}$$

The vacuum for the phonons $Q^\dagger_{\lambda\mu i}$, $\tilde{Q}^\dagger_{\lambda\mu i}$ has the form[1]

$$|0(Q,\tilde{Q})\rangle_\sigma = \exp\{-\hat{K}_b/2\}\exp\left\{\sum_{\lambda\mu i} q^\dagger_{\lambda\mu i}\tilde{q}^\dagger_{\lambda\mu i}\right\}|0(q,\tilde{q})\rangle_\sigma\,, \tag{59}$$

where $\hat{K}_b$ is the entropy operator for noninteracting bosons

$$\hat{K}_b = -\sum_{\lambda\mu i}\{q^\dagger_{\lambda\mu i}q_{\lambda\mu i}\ln Y^2_{\lambda i} - q_{\lambda\mu i}q^\dagger_{\lambda\mu i}\ln X^2_{\lambda i}\}\,. \tag{60}$$

Now we are ready to determine the values $X_{\lambda i}$, $Y_{\lambda i}$. To this aim, we minimize the thermodynamic potential Ω_b for the r-phonon system, which reads

$$\begin{aligned}
\Omega_b &= {}_\sigma\langle 0(Q,\tilde{Q})|\sum_{\lambda\mu i}\omega_{\lambda i}q^\dagger_{\lambda\mu i}q_{\lambda\mu i} - T\hat{K}_b|0(Q,\tilde{Q})\rangle_\sigma\\
&= \sum_{\lambda\mu i}\left\{\omega_{\lambda i}Y^2_{\lambda i} + T(Y^2_{\lambda i}\ln Y^2_{\lambda i} + X^2_{\lambda i}\ln X^2_{\lambda i})\right\}\,.
\end{aligned} \tag{61}$$

Varying (61) with respect to $Y_{\lambda i}$ and equating the result to zero we get

$$Y_{\lambda i} = \left[\exp\left(\frac{\omega_{\lambda i}}{T}\right) - 1\right]^{-1/2}, \quad X_{\lambda i} = [1 + Y_{\lambda i}^2]^{1/2}. \tag{62}$$

The coefficients $Y_{\lambda i}^2$ are the thermal occupation factors of the Bose–Einstein statistics. They determine the average number of r-phonons in the thermal phonon vacuum

$$_\sigma\langle 0(Q,\tilde{Q})|\, q^\dagger_{\lambda\mu i} q_{\lambda\mu i} |0(Q,\tilde{Q})\rangle_\sigma = Y_{\lambda i}^2 . \tag{63}$$

It is worthwhile to note that the problems of correct construction of the thermal RPA phonon operator and the thermal RPA vacuum state were already discussed in Ref. 9 using the BCS Hamiltonian as an example. The authors of Ref. 9 also confronted with an ambiguity of RPA solutions at finite temperature. To get it out, they introduced the "nonrotated" thermal pairing phonons and then made a thermal rotation, which minimized the thermodynamic potential. At the same time, these circumstances, i.e. the invariance of the $\mathcal{H}_{\text{TRPA}}$ under the thermal rotation (58), were overlooked in Refs. 10, 12. In these papers, it was assumed that $Y_{\lambda i} = 0$ at any temperature and the vacuum for "r-phonons" was identified as the "true" thermal vacuum state. That is why thermal bosonic occupation numbers did not appear in the corresponding formulae in Refs. 10, 12, 18.

Now we can fix the factor σ. To this aim, we consider the behavior of thermal phonon amplitudes (57) when the coupling constants of a separable multipole interaction $\kappa_{0,1}^{(\lambda)}$ tend to zero. When $\kappa_{0,1}^{(\lambda)} \to 0$ the thermal phonon energy $\omega_{\lambda i} \to \varepsilon^{(\pm)}_{j_1 j_2}$ and only one amplitude has to survive in the corresponding phonon wave function. In particular, all the amplitudes of backward going components vanish when $\kappa_{0,1}^{(\lambda)} \to 0$. Taking advantage of (34) it can be shown that if $\omega_{\lambda i} \to \varepsilon^{(-)}_{j_1 j_2}$ $\xi^{\lambda i}_{j_1 j_2} \to 0$ whereas $(\eta^{\lambda i}_{j_1 j_2})^2 \to 1$ at any σ value.

When $\omega_{\lambda i} \to \varepsilon^{(+)}_{j_1 j_2}$ the amplitudes ϕ and $\tilde{\psi}$ also vanish at any value of σ. However, limiting values of ψ and $\tilde{\phi}$ appear to be dependent on σ, namely,

$$\lim_{\omega_{\lambda i}\to\varepsilon^{(+)}_{j_1 j_2}} \psi^{\lambda i}_{j_1 j_2} = \frac{x_{j_1} x_{j_2} X(\varepsilon^{(+)}_{j_1 j_2}) + \sigma^2 y_{j_1} y_{j_2} Y(\varepsilon^{(+)}_{j_1 j_2})}{(1 - y^2_{j_1} - y^2_{j_2})^{1/2}} = X^2(\varepsilon^{(+)}_{j_1 j_2}) + \sigma^2 Y^2(\varepsilon^{(+)}_{j_1 j_2}),$$

$$\lim_{\omega_{\lambda i}\to\varepsilon^{(+)}_{j_1 j_2}} \tilde{\phi}^{\lambda i}_{j_1 j_2} = \frac{\sigma^2 y_{j_1} y_{j_2} X(\varepsilon^{(+)}_{j_1 j_2}) + x_{j_1} x_{j_2} Y(\varepsilon^{(+)}_{j_1 j_2})}{(1 - y^2_{j_1} - y^2_{j_2})^{1/2}} = X(\varepsilon^{(+)}_{j_1 j_2}) Y(\varepsilon^{(+)}_{j_1 j_2})(\sigma^2 + 1). \tag{64}$$

Since from a physical point of view $\tilde{\phi}^{\lambda i}_{j_1 j_2}$ should vanish, we choose $\sigma = i$. Then not only $\tilde{\phi}^{\lambda i}_{j_1 j_2} = 0$ but also $\psi^{\lambda i}_{j_1 j_2} = 1$. The present result on DTCR agrees with the conclusion in Ref. 27. In Ref. 27, the appropriate choice of DTCR was specified by the fermion number conservation in the system. If the number of fermions in the system is not conserved, DTCR should have the form (9), i.e. $\tilde{\tilde{A}} = A$. This seems to be just our case since the number of quasiparticles in a nucleus is not conserved.

Now we complete constructing the thermal phonon operator and the "true" thermal vacuum state in the random phase approximation. They should be calculated with formulae (39) and (57), where $X_{\lambda i}$ and $Y_{\lambda i}$ are given by (62) and $\sigma = i$. Hereafter, the thermal phonon vacuum is denoted by $|0(T); \mathrm{TRPA}\rangle$. We would like to stress once again that the thermal RPA vacuum $|0(T); \mathrm{TRPA}\rangle$ reduces to the thermal BCS vacuum when the particle–hole interaction vanishes only if the coefficients $X_{\lambda i}$, $Y_{\lambda i}$ are the phonon thermal occupation numbers given by (62) and $\sigma = i$.

At the end of this subsection, we calculate a matrix element of the $E\lambda$-transition between the RPA thermal vacuum and a thermal one-phonon state. To this aim, one has to write the operator $\mathcal{M}(E\lambda\mu)$ (32) in terms of thermal phonon operators. Taking into account only the term of (32) with the operators $A^\dagger_{\lambda\mu}(j_1 j_2)$ and $A_{\overline{\lambda\mu}}(j_1 j_2)$, one gets

$$
\begin{aligned}
\mathcal{M}(E\lambda\mu) &= \frac{1}{2\hat{\lambda}} \sum_i \sum_\tau \sum_{j_1 j_2}^{\tau} \Gamma^{(\lambda)}_{j_1 j_2} \\
&\quad \times \{[u^{(+)}_{j_1 j_2}(x_{j_1} x_{j_2} g^{\lambda i}_{j_1 j_2} + y_{j_1} y_{j_2} \tilde{g}^{\lambda i}_{j_1 j_2}) + 2 v^{(-)}_{j_1 j_2} x_{j_1} y_{j_2} t^{\lambda i}_{j_1 j_2}](Q^\dagger_{\lambda\mu i} + Q_{\overline{\lambda\mu i}}) \\
&\quad + [u^{(+)}_{j_1 j_2}(x_{j_1} x_{j_2} \tilde{g}^{\lambda i}_{j_1 j_2} + y_{j_1} y_{j_2} g^{\lambda i}_{j_1 j_2}) + 2 v^{(-)}_{j_1 j_2} y_{j_1} x_{j_2} t^{\lambda i}_{j_1 j_2}](\tilde{Q}^\dagger_{\overline{\lambda\mu i}} + \tilde{Q}_{\lambda\mu i})\} \\
&= \frac{1}{\hat{\lambda}} \sum_i \sum_\tau \Gamma^{\lambda i}_\tau \{X_{\lambda i}(Q^\dagger_{\lambda\mu i} + Q_{\overline{\lambda\mu i}}) + Y_{\lambda i}(\tilde{Q}^\dagger_{\overline{\lambda\mu i}} + \tilde{Q}_{\lambda\mu i})\}\,,
\end{aligned} \tag{65}
$$

where

$$
\Gamma^{\lambda i}_\tau = \sum_{j_1 j_2}^{\tau} \frac{\Gamma^{(\lambda)}_{j_1 j_2} f^{(\lambda)}_{j_1 j_2}}{\sqrt{\mathcal{N}^{\lambda i}_\tau}} \left[\frac{(u^{(+)}_{j_1 j_2})^2 \varepsilon^{(+)}_{j_1 j_2}(1 - y^2_{j_1} - y^2_{j_2})}{(\varepsilon^{(+)}_{j_1 j_2})^2 - \omega^2_{\lambda i}} - \frac{(v^{(-)}_{j_1 j_2})^2 \varepsilon^{(-)}_{j_1 j_2}(y^2_{j_1} - y^2_{j_2})}{(\varepsilon^{(-)}_{j_1 j_2})^2 - \omega^2_{\lambda i}} \right]. \tag{66}
$$

Expression (65) differs significantly from that in Refs. 12, 18. Specifically, there appear terms with tilde-phonon operators in (65). This is a consequence of $\{X, Y\}$ rotation of thermal phonons. The item proportional to the factor $Y_{\lambda i}$ is responsible for transitions to tilde phonon states lying below the thermal vacuum state, i.e. for decay of the thermal vacuum.

Thus, there are two types of matrix elements corresponding to excitation and de-excitation processes of the thermal vacuum

$$
\begin{aligned}
\Phi_{\lambda i} &= \langle 0(T); \mathrm{TRPA} \| \mathcal{M}(E\lambda\mu) Q^\dagger_{\lambda\mu i} \| 0(T); \mathrm{TRPA}\rangle = X_{\lambda i} \sum_\tau \Gamma^{\lambda i}_\tau\,, \\
\tilde{\Phi}_{\lambda i} &= \langle 0(T); \mathrm{TRPA} \| \mathcal{M}(E\lambda\mu) \tilde{Q}^\dagger_{\overline{\lambda\mu i}} \| 0(T); \mathrm{TRPA}\rangle = Y_{\lambda i} \sum_\tau \Gamma^{\lambda i}_\tau\,.
\end{aligned} \tag{67}
$$

The factors $X_{\lambda i}$ and $Y_{\lambda i}$ in (67) were missed in Refs. 12, 18 since the r-phonons $q^\dagger_{\lambda\mu i}, \tilde{q}^\dagger_{\lambda\mu i}$ were explored rather than the "rotated" $Q^\dagger_{\lambda\mu i}$, $\tilde{Q}^\dagger_{\lambda\mu i}$.

The factor $X^2_{\lambda i}$ in the photoabsorption cross section for hot nuclei as well as its role was discussed in detail in Ref. 34. This factor occurred also in the response function in Ref. 32 but was missed in the $B(E\lambda)\downarrow$ expression in Ref. 33. Note that in both the papers[32,33] the thermal RPA equations were derived with the equation of motion method for bifermion operators $\alpha^\dagger_{j_1}\alpha^\dagger_{j_2}$ and $\alpha^\dagger_{j_1}\alpha_{j_2}$.

As in the case of transitions to thermal two-quasiparticle states (see Sec. 3.2), there exists the following relation between transition probabilities from the nontilde and tilde one-phonon states

$$\Phi^2_{\lambda i} = \exp(\omega_{\lambda i}/T)\tilde{\Phi}^2_{\lambda i}\,. \tag{68}$$

This relation is equivalent to the principle of detailed balancing connecting the probabilities for the probe to transfer energy ω to the heated system and to absorb energy ω from the heated system (see, for example, Ref. 32).

The model energy-weighed sum rule at the thermal RP approximation is given by

$$\mathrm{EWSR} = \sum_{\omega_{\lambda i}>0} \omega_{\lambda i}(\Phi^2_{\lambda i} - \tilde{\Phi}^2_{\lambda i}) = \sum_{\omega_{\lambda i}>0} \omega_{\lambda i}\left(\Gamma^{\lambda i}_p + \Gamma^{\lambda i}_n\right)^2\,. \tag{69}$$

It is worthwhile to note that the EWSR (69) appears to be independent on the thermal phonon occupation factors. The numerical value of (69) should coincide with that of (37).

Nominally, this expression coincides with that in Ref. 33. However, the essential difference between the present result and Ref. 33 is the contribution of TRPA states with negative energies $\omega_{\lambda i}$ (i.e. tilde states). In Ref. 33, the total strength of $E\lambda$-transitions is located in the positive energy region whereas in the present approach some fraction of strength is carried by the states with $\omega_{\lambda i} < 0$. As a result, part of the EWSR pertaining to positive excitation energies is greater than the total one that is obvious from (69).

3.4. *Interaction of thermal phonons*

In this subsection, we deal with a coupling of thermal phonons, i.e. go beyond the thermal RPA. Physical effects that can be treated in this order of approximation relate to fragmentation of basic nuclear excitations like quasiparticles and phonons, their spreading widths and/or more consistent description of transition strength distributions over a nuclear spectrum. The problem of temperature dependence of the giant resonance width, which was intensively discussed not long ago (see, e.g., reviews Refs. 35–37) also belongs to this set.

The part of $\mathcal{H}$ (18), which is responsible for these effects, is the so-called quasiparticle–phonon interaction (or the cubic anharmonic term) $\mathcal{H}_{\rm qph}$

$$\mathcal{H}_{\rm qph} = -\frac{1}{2}\sum_{\lambda\mu i}\sum_{\tau}\sum^{\tau}_{j_1j_2} \frac{f^{(\lambda)}_{j_1j_2}}{\sqrt{\mathcal{N}^{\lambda i}_{\tau}}}\{(Q^\dagger_{\overline{\lambda\mu} i} + Q_{\lambda\mu i})B_{\lambda\mu i}(j_1j_2) + (\mathrm{h.c.}) - (\mathrm{t.c.})\},$$

$$B_{\lambda\mu i}(j_1j_2) = iu^{(+)}_{j_1j_2}(\mathcal{Z}^{\lambda i}_{j_1j_2}[\beta^\dagger_{j_1}\tilde{\beta}_{j_2}]^\lambda_\mu + \mathcal{Z}^{\lambda i}_{j_2j_1}[\tilde{\beta}^\dagger_{\overline{j_1}}\beta_{\overline{j_2}}]^\lambda_\mu) - v^{(-)}_{j_1j_2}(\mathcal{X}^{\lambda i}_{j_1j_2}[\beta^\dagger_{j_1}\beta_{\overline{j_2}}]^\lambda_\mu + \mathcal{Y}^{\lambda i}_{j_1j_2}[\tilde{\beta}^\dagger_{\overline{j_1}}\tilde{\beta}_{j_2}]^\lambda_\mu)\,. \tag{70}$$

The coefficients $\mathcal{X}^{\lambda i}_{j_1j_2}$, $\mathcal{Y}^{\lambda i}_{j_1j_2}$, and $\mathcal{Z}^{\lambda i}_{j_1j_2}$ are given by

$$\begin{pmatrix}\mathcal{X}\\ \mathcal{Y}\end{pmatrix}^{\lambda i}_{j_1j_2} = x_{j_1}x_{j_2}\begin{pmatrix}X\\ Y\end{pmatrix}_{\lambda i} + y_{j_1}y_{j_2}\begin{pmatrix}Y\\ X\end{pmatrix}_{\lambda i},\quad \mathcal{Z}^{\lambda i}_{j_1j_2} = x_{j_1}y_{j_2}X_{\lambda i} + y_{j_1}x_{j_2}Y_{\lambda i}\,. \tag{71}$$

The interaction $\mathcal{H}_{\rm qph}$ couples the multiphonon states whose structures differ by one phonon, i.e. one-phonon states with two-phonon states, two-phonon states with three-phonon states, etc. This term can be named the "cubic anharmonic" one because in the lowest order of boson expansion the fermionic operator $B_{\lambda\mu i}(j_1j_2)$ is substituted by the product of two bosonic operators (or phonons). The remaining part of the thermal Hamiltonian consists of the items $\sim B^\dagger_{\lambda-\mu}(j_1j_2)B_{\lambda\mu}(j_3j_4)$, which are equivalent to the sum of products of four bosonic operators. Its contribution will not be considered here.

To take into account the term $\mathcal{H}_{\rm qph}$, we again apply the variational principle. A new trial wave function is assumed to be of the form

$$|\Psi_\nu(JM)\rangle = \Bigg\{\sum_i R_i(J\nu)Q^\dagger_{JMi} + \sum_{\substack{\lambda_1 i_1\\ \lambda_2 i_2}} P^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)[Q^\dagger_{\lambda_1 i_1}Q^\dagger_{\lambda_2 i_2}]^J_M + \sum_{\substack{\lambda_1 i_1\\ \lambda_2 i_2}} S^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)[Q^\dagger_{\lambda_1 i_1}\tilde{Q}^\dagger_{\overline{\lambda_2 i_2}}]^J_M + \sum_{\substack{\lambda_1 i_1\\ \lambda_2 i_2}} T^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)[\tilde{Q}^\dagger_{\overline{\lambda_1 i_1}}\tilde{Q}^\dagger_{\overline{\lambda_2 i_2}}]^J_M\Bigg\} \times |0(T);\mathrm{RPA}\rangle\,, \tag{72}$$

where R, P, S, T are the variational parameters, which should be determined. As one can see in (72), the thermal vacuum is kept the same as in the TRPA. At $T=0$ the latter approximation is valid if the quasiparticle–phonon interaction is relatively weak.

The trial wave function (72) has to be normalized and, therefore, the amplitudes R, P, S, T should satisfy the following constraints:

$$\sum_i [R_i(J\nu)]^2 + \sum_{\substack{\lambda_1 i_1\\ \lambda_2 i_2}}\{2[P^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)]^2 + [S^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)]^2 + 2[T^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)]^2\} = 1\,. \tag{73}$$

Since the trial wave function contains three different types of components, there are three types of interaction matrix elements, which couple a thermal one-phonon state with two-phonon ones

$$\begin{aligned}
U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= \langle \mathrm{RPA}; 0(T)|Q_{JMi}\mathcal{H}_{\rm qph}[Q^\dagger_{\lambda_1 i_1}Q^\dagger_{\lambda_2 i_2}]^J_M|0(T);\mathrm{RPA}\rangle\,,\\
V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= \langle \mathrm{RPA}; 0(T)|Q_{JMi}\mathcal{H}_{\rm qph}[Q^\dagger_{\lambda_1 i_1}\tilde{Q}^\dagger_{\overline{\lambda_2 i_2}}]^J_M|0(T);\mathrm{RPA}\rangle\,,\\
W^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= \langle \mathrm{RPA}; 0(T)|Q_{JMi}\mathcal{H}_{\rm qph}[\tilde{Q}^\dagger_{\overline{\lambda_1 i_1}}\tilde{Q}^\dagger_{\overline{\lambda_2 i_2}}]^J_M|0(T);\mathrm{RPA}\rangle\,.
\end{aligned} \tag{74}$$

The matrix element $U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)$ corresponds to a transition from one to two phonons, whereas the matrix elements $V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)$ and $W^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)$ describe a thermal phonon scattering and a thermal phonon absorption, respectively. The explicit forms of U, V, and W are given in Appendix A.

Thus, we should minimize the expectation value of the thermal Hamiltonian $\mathcal{H}_{\rm RPA} + \mathcal{H}_{\rm qph}$ over $|\Psi_\nu(JM)\rangle$ at constraint (73). The expectation value is given by

$$\begin{aligned}
\langle\Psi_\nu(JM)|\mathcal{H}_{\rm RPA} + \mathcal{H}_{\rm qph}|\Psi_\nu(JM)\rangle &= \sum_i \omega_{Ji}\big[R_i(J\nu)\big]^2 \\
&+ 2\sum_{\substack{\lambda_1 i_1\\ \lambda_2 i_2}}(\omega_{\lambda_1 i_1} + \omega_{\lambda_2 i_2})[P^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)]^2 + \sum_{\substack{\lambda_1 i_1\\ \lambda_2 i_2}}(\omega_{\lambda_1 i_1} - \omega_{\lambda_2 i_2})[S^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)]^2 \\
&- 2\sum_{\substack{\lambda_1 i_1\\ \lambda_2 i_2}}(\omega_{\lambda_1 i_1} + \omega_{\lambda_2 i_2})[T^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)]^2 + 2\sum_i\sum_{\substack{\lambda_1 i_1\\ \lambda_2 i_2}} R_i(J\nu)\{P^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) \\
&+ S^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)W^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) + T^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)\}\,.
\end{aligned} \tag{75}$$

After the standard variation procedure, one gets a homogeneous system of linear equations (η_ν is the energy of the state $|\Psi_\nu(JM)\rangle$)

$$\begin{gathered}
R_i(J\nu)(\omega_{\lambda i} - \eta_\nu) + \sum_{\substack{\lambda_1 i_1\\ \lambda_2 i_2}}\{P^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) \\
+ S^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) + T^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)W^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)\} = 0\,, \\
P^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)(\omega_{\lambda_1 i_1} + \omega_{\lambda_2 i_2} - \eta_\nu) + \frac{1}{2}\sum_i R_i(J\nu)U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) = 0\,, \\
S^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)(\omega_{\lambda_1 i_1} - \omega_{\lambda_2 i_2} - \eta_\nu) + \sum_i R_i(J\nu)V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) = 0\,, \\
T^{\lambda_1 i_1}_{\lambda_2 i_2}(J\nu)(\omega_{\lambda_1 i_1} + \omega_{\lambda_2 i_2} + \eta_\nu) - \frac{1}{2}\sum_i R_i(J\nu)W^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) = 0\,.
\end{gathered} \tag{76}$$

The system has a solution if η_ν is the root of the following secular equation:

$$\begin{aligned}
\det\Bigg|(\omega_{Ji} - \eta_\nu)\delta_{ii'} - \frac{1}{2}\sum_{\substack{\lambda_1 i_1\\ \lambda_2 i_2}}\Bigg\{&\frac{U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji')}{\omega_{\lambda_1 i_1} + \omega_{\lambda_2 i_2} - \eta_\nu} \\
&+ 2\frac{V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji')}{\omega_{\lambda_1 i_1} - \omega_{\lambda_2 i_2} - \eta_\nu} - \frac{W^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)W^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji')}{\omega_{\lambda_1 i_1} + \omega_{\lambda_2 i_2} + \eta_\nu}\Bigg\}\Bigg| = 0\,.
\end{aligned} \tag{77}$$

Minimizing the expectation value of the thermal Hamiltonian $\mathcal{H}_{\rm RPA} + \mathcal{H}_{\rm qph}$ over the wave function $|\tilde{\Psi}_\nu(JM)\rangle$ we get the corresponding equations for tilde-conjugated states. They also can be obtained from (76) and (77) by changing the sign of η_ν.

That is, if η_ν is the energy of a given state $|\Psi_\nu(JM)\rangle$, then $-\eta_\nu$ is the energy of the tilde-conjugated state $|\tilde{\Psi}_\nu(JM)\rangle$.

Certainly, the above results deviate from those of Ref. 10. Owing to the changes of thermal RPA-phonon amplitudes and, in particular, their dependence on the thermal phonon occupation numbers, the quasiparticle–phonon interaction (70) couples one-phonon states with three types of thermal two-phonon states whereas in Ref. 10 only one type was taken into account (namely, the first term in (72)).

The terms $Q^\dagger\tilde{Q}^\dagger$ and $\tilde{Q}^\dagger\tilde{Q}^\dagger$ in the trial wave function describe the processes, which were not considered in Ref. 10 and later in Ref. 18. The processes of thermal phonon scattering and absorption became possible due to the presence of thermal r-phonons in the thermal phonon vacuum. Since in Ref. 10 the thermal quasiparticle–phonon interaction was treated on the basis of thermal vacuum for r-phonons, the above effects were missed.

The inclusion of the terms $Q^\dagger\tilde{Q}^\dagger$ and $\tilde{Q}^\dagger\tilde{Q}^\dagger$ in the trial wave function produces the new poles in the secular equation (77) in comparison with the previous study,[10] namely, $(\omega_{\lambda_1 i_1} - \omega_{\lambda_2 i_2})$ and $-(\omega_{\lambda_1 i_1} + \omega_{\lambda_2 i_2})$.

The new equations of the TFD-QPM approach (76) and (77) are in qualitative agreement with those in Ref. 22 and, in some respects, in Ref. 21.

As it was mentioned in Sec. 1, the consideration in Refs. 21, 22 was based on the Matsubara Green's function technique and nuclear field theory. The approach of Ref. 22 is especially close to our approach since it treats a hot nucleus as a system of interacting TRPA phonons. Although the equations of both the approaches seem to be hardly compared "term-to-term" because of quite different formalisms explored, in both the cases the negative TRPA roots are in the game, the poles in the equations are of the same types, phonon interaction matrix elements are similar, etc. That is why we can establish their "qualitative" agreement.

The formulae for the matrix elements of $E\lambda$ transitions from the thermal vacuum state to the state (72) and its tilde-counterpart are quite obvious. In the leading approximation the operator $\mathcal{M}(E\lambda)$ induces a transition from the thermal vacuum to one-phonon components of the thermal state $|\Psi_\nu(JM)\rangle$ (or $|\tilde{\Psi}_\nu(JM)\rangle$) only. Matrix elements of direct transitions from $|0(T);\mathrm{RPA}\rangle$ to two-phonon components are very weak although do not vanish. Thus, one gets

$$
\begin{aligned}
\Phi(J\nu) &= \langle \Psi_\nu(JM) \| \mathcal{M}(E\lambda) \| 0(T);\mathrm{RPA}\rangle = \sum_i R_i(J\nu)\Phi_{Ji}\,,\\
\tilde{\Phi}(J\nu) &= \langle \tilde{\Psi}_\nu(JM) \| \mathcal{M}(E\lambda) \| 0(T);\mathrm{RPA}\rangle = \sum_i R_i(J\nu)\tilde{\Phi}_{Ji}\,,
\end{aligned}
\tag{78}
$$

where Φ_{Ji} and $\tilde{\Phi}_{Ji}$ are given by (67).

4. Summary and Conclusions

The present study was motivated by the necessity to reexamine the TFD-QPM approach[10,12,18] in theory of hot nuclei. In this respect, a couple of new facets of the TRPA formulated within the TFD were found and their influence on the coupling of thermal phonons was established.

We showed that for the Hamiltonian consisting of a mean field, the BCS pairing interaction and separable particle–hole effective interactions, amplitudes of a thermal phonon wave function (39) were not determined unambiguously by diagonalization of the RPA-part of the thermal Hamiltonian. To fix the coefficients of a linear transformation from the set of thermal two-quasiparticle operators to phonon operators, one should impose an additional demand — to minimize the thermodynamic potential of the system of free thermal phonons. This was achieved by using one more unitary transformation — a thermal rotation of "reference" phonons (r-phonons). As a consequence, the Bose–Einstein thermal occupation numbers (thermal occupation numbers of phonons) come to play. It should be stressed that the new ingredient does not affect the main TRPA equation, i.e. the equation for thermal phonon energies.

As far as we know, the thermal rotation of phonons in the TRPA framework was discussed only in Ref. 9 when treating the pairing BCS Hamiltonian. The thermal rotation of phonons was missed not only in the preceding TFD-QPM studies[10,12,18] but it was not also mentioned in Refs. 6, 7. Moreover, the Bose–Einstein thermal occupation numbers do not appear when the TRPA equations are derived by exploring the phonon operator with scattering terms.[33,38,39]

Thus, according to the present results amplitudes in the thermal phonon wave function depend not only on thermal occupation numbers of the Bogoliubov quasiparticles making up the phonon but also on the thermal occupation numbers of the phonon. Moreover, presently the corresponding thermal phonon vacuum contains some amount of r-phonons with probabilities determined by their energies in accordance with the Bose statistics. During to this, the processes of excitation and de-excitation of a hot nucleus can be regarded on equal footing.

The above results have weighty consequences. The first one is the appearance of the factor $1/(1-\exp(-\omega/T))$ in the $E\lambda$ transition strength. Its role was discussed in Ref. 34 where it was derived using the Green function technique (see also Ref. 32). This factor somewhat enhances the low energy part of the $E\lambda$ transition strength. It is also essential when ω can take negative values, i.e. when considering a decay of a hot nucleus.

The second consequence concerns the thermal quasiparticle–phonon interaction $\mathcal{H}_{\rm qph}$. This consequence is two-fold: (a) renormalization of phonon–phonon interaction vertices (cf. matrix element $U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji)$ (A.1) with that in Ref. 10); (b) essential extension of the thermal two-phonon configuration space and the corresponding complication of trial wave function (72).

With the above new ingredients the TFD-QPM approach well conform (at least qualitatively) with a more traditional one exploring the Matsubara Green's function method.[21,22] Now in both the approaches one has two kinds of thermal occupation numbers (fermionic and bosonic) and the identical sets of processes giving rise to the phonon–phonon coupling.

In view of the above discussion, we conclude that the results of calculations in Ref. 18 should be revised. At the same time, it should be stressed that the thermal rotation of phonons cannot be regarded as a mandatory ingredient of a microscopic treatment of boson-like excitations in many fermion systems. Its necessity seems to be intimately related with a two-body interaction character. For example, in Refs. 14, 16, 17, 24, 25, some approximations going beyond the TRPA were constructed by using the TFD formalism and their validity was examined by the example of the Lipkin model. In those studies, the transformation from thermal bifermion operators to thermal phonon operators was unambiguously determined by diagonalization of the thermal model Hamiltonian. No additional demands or assumptions were needed to be involved.

One more new feature of the present study is exploring the double tilde-conjugation rule in the form proposed by Ojima.[26] More precisely, we found that just the Ojima formulation of DTCR guarantees the correct behavior of the thermal phonon wave function in the limit of vanishing particle–hole interaction. Furthermore, the Ojima form of DTCR implies the complex thermal rotation for fermions. An interesting point is that the effect of the DTCR choice was revealed only at the TRPA stage while constructing a thermal phonon operator, whereas at the pure fermionic stage when, e.g., pairing correlations were considered, the role of DTCR did not show up in and both the versions gave the same results.

Acknowledgments

The valuable discussions with Dr A. Storozhenko and Prof. D. Kosov are gratefully acknowledged.

Appendix A

Here we display the matrix elements of the quasiparticle–phonon interaction $\mathcal{H}_{\rm qph}$ between thermal one-phonon and different two-phonon configurations. The phonon amplitudes are shown in (57) and include both quasiparticle and phonon occupation factors. The symbol $\{\cdots\}$ is the standard $6j$-symbol.

$$
\begin{aligned}
U^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) = \hat{\lambda}_1\hat{\lambda}_2 \sum_{\tau} \sum_{j_1 j_2 j_3}^{\tau} \Bigg[& \frac{f^{(J)}_{j_1 j_2}}{\sqrt{\mathcal{N}^{Ji}_{\tau}}} \begin{Bmatrix} J & \lambda_1 & \lambda_2 \\ j_3 & j_1 & j_2 \end{Bmatrix} \mathcal{G}^{Ji}_{\lambda_1 i_1 \lambda_2 i_2}(j_1 j_2 j_3) \\
& + \frac{f^{(\lambda_1)}_{j_1 j_2}}{\sqrt{\mathcal{N}^{\lambda_1 i_1}_{\tau}}} \begin{Bmatrix} \lambda_1 & \lambda_2 & J \\ j_3 & j_1 & j_2 \end{Bmatrix} \mathcal{F}^{\lambda_1 i_1}_{\lambda_2 i_2 Ji}(j_1 j_2 j_3) \\
& + (-1)^{\lambda_1+\lambda_2+J} \frac{f^{(\lambda_2)}_{j_1 j_2}}{\sqrt{\mathcal{N}^{\lambda_2 i_2}_{\tau}}} \begin{Bmatrix} \lambda_2 & \lambda_1 & J \\ j_3 & j_1 & j_2 \end{Bmatrix} \mathcal{F}^{\lambda_2 i_2}_{\lambda_1 i_1 Ji}(j_1 j_2 j_3) \Bigg] ,
\end{aligned}
\tag{A.1}
$$

where

$$\begin{aligned}
\mathcal{G}^{Ji}_{\lambda_1 i_1 \lambda_2 i_2}(j_1 j_2 j_3) &= \mathcal{X}^{Ji}_{j_1 j_2} v^{(-)}_{j_1 j_2} (\psi^{\lambda_1 i_1}_{j_2 j_3} \phi^{\lambda_2 i_2}_{j_3 j_1} + \phi^{\lambda_1 i_1}_{j_2 j_3} \psi^{\lambda_2 i_2}_{j_3 j_1} + \eta^{\lambda_1 i_1}_{j_2 j_3} \tilde{\xi}^{\lambda_2 i_2}_{j_3 j_1} + \xi^{\lambda_1 i_1}_{j_2 j_3} \tilde{\eta}^{\lambda_2 i_2}_{j_3 j_1}) \\
&\quad - \mathcal{Y}^{Ji}_{j_1 j_2} v^{(-)}_{j_1 j_2} (\tilde{\psi}^{\lambda_1 i_1}_{j_2 j_3} \tilde{\phi}^{\lambda_2 i_2}_{j_3 j_1} + \tilde{\phi}^{\lambda_1 i_1}_{j_2 j_3} \tilde{\psi}^{\lambda_2 i_2}_{j_3 j_1} + \tilde{\eta}^{\lambda_1 i_1}_{j_2 j_3} \xi^{\lambda_2 i_2}_{j_3 j_1} + \tilde{\xi}^{\lambda_1 i_1}_{j_2 j_3} \eta^{\lambda_2 i_2}_{j_3 j_1}) \\
&\quad + \mathcal{Z}^{Ji}_{j_1 j_2} u^{(+)}_{j_1 j_2} (\tilde{\psi}^{\lambda_1 i_1}_{j_2 j_3} \tilde{\xi}^{\lambda_2 i_2}_{j_3 j_1} + \tilde{\phi}^{\lambda_1 i_1}_{j_2 j_3} \tilde{\eta}^{\lambda_2 i_2}_{j_3 j_1} - \tilde{\eta}^{\lambda_1 i_1}_{j_2 j_3} \phi^{\lambda_2 i_2}_{j_3 j_1} - \tilde{\xi}^{\lambda_1 i_1}_{j_2 j_3} \psi^{\lambda_2 i_2}_{j_3 j_1}) \\
&\quad - \mathcal{Z}^{Ji}_{j_2 j_1} u^{(+)}_{j_1 j_2} (\psi^{\lambda_1 i_1}_{j_2 j_3} \xi^{\lambda_2 i_2}_{j_3 j_1} + \phi^{\lambda_1 i_1}_{j_2 j_3} \eta^{\lambda_2 i_2}_{j_3 j_1} - \eta^{\lambda_1 i_1}_{j_2 j_3} \tilde{\phi}^{\lambda_2 i_2}_{j_3 j_1} - \xi^{\lambda_1 i_1}_{j_2 j_3} \tilde{\psi}^{\lambda_2 i_2}_{j_3 j_1})\,, \\
\mathcal{F}^{\lambda_1 i_1}_{\lambda_2 i_2 Ji}(j_1 j_2 j_3) &= \mathcal{X}^{\lambda_1 i_1}_{j_1 j_2} v^{(-)}_{j_1 j_2} (\psi^{\lambda_2 i_2}_{j_2 j_3} \psi^{Ji}_{j_3 j_1} + \phi^{\lambda_2 i_2}_{j_2 j_3} \phi^{Ji}_{j_3 j_1} + \eta^{\lambda_2 i_2}_{j_2 j_3} \tilde{\eta}^{Ji}_{j_3 j_1} + \xi^{\lambda_2 i_2}_{j_2 j_3} \tilde{\xi}^{Ji}_{j_3 j_1}) \\
&\quad - \mathcal{Y}^{\lambda_1 i_1}_{j_1 j_2} v^{(-)}_{j_1 j_2} (\tilde{\psi}^{\lambda_2 i_2}_{j_2 j_3} \tilde{\psi}^{Ji}_{j_3 j_1} + \tilde{\phi}^{\lambda_2 i_2}_{j_2 j_3} \tilde{\phi}^{Ji}_{j_3 j_1} + \tilde{\eta}^{\lambda_2 i_2}_{j_2 j_3} \eta^{Ji}_{j_3 j_1} + \tilde{\xi}^{\lambda_2 i_2}_{j_2 j_3} \xi^{Ji}_{j_3 j_1}) \\
&\quad + \mathcal{Z}^{\lambda_1 i_1}_{j_1 j_2} u^{(+)}_{j_1 j_2} (\tilde{\psi}^{\lambda_2 i_2}_{j_2 j_3} \tilde{\eta}^{Ji}_{j_3 j_1} + \tilde{\phi}^{\lambda_2 i_2}_{j_2 j_3} \tilde{\xi}^{Ji}_{j_3 j_1} - \tilde{\eta}^{\lambda_2 i_2}_{j_2 j_3} \psi^{Ji}_{j_3 j_1} - \tilde{\xi}^{\lambda_2 i_2}_{j_2 j_3} \phi^{Ji}_{j_3 j_1}) \\
&\quad - \mathcal{Z}^{\lambda_1 i_1}_{j_2 j_1} u^{(+)}_{j_1 j_2} (\psi^{\lambda_2 i_2}_{j_2 j_3} \eta^{Ji}_{j_3 j_1} + \phi^{\lambda_2 i_2}_{j_2 j_3} \xi^{Ji}_{j_3 j_1} - \eta^{\lambda_2 i_2}_{j_2 j_3} \tilde{\psi}^{Ji}_{j_3 j_1} - \xi^{\lambda_2 i_2}_{j_2 j_3} \tilde{\phi}^{Ji}_{j_3 j_1})\,.
\end{aligned}$$

$$\begin{aligned}
V^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) &= \hat{\lambda}_1 \hat{\lambda}_2 \sum_{\tau} \sum_{j_1 j_2 j_3}^{\tau} \Bigg[\frac{f^{(J)}_{j_1 j_2}}{\sqrt{\mathcal{N}^{Ji}_{\tau}}} \begin{Bmatrix} J & \lambda_1 & \lambda_2 \\ j_3 & j_1 & j_2 \end{Bmatrix} \mathcal{S}^{Ji}_{\lambda_1 i_1 \lambda_2 i_2}(j_1 j_2 j_3) \\
&\quad + \frac{f^{(\lambda_1)}_{j_1 j_2}}{\sqrt{\mathcal{N}^{\lambda_1 i_1}_{\tau}}} \begin{Bmatrix} \lambda_1 & \lambda_2 & J \\ j_3 & j_1 & j_2 \end{Bmatrix} \tilde{\mathcal{R}}^{\lambda_1 i_1}_{\lambda_2 i_2 Ji}(j_1 j_2 j_3) \\
&\quad - (-1)^{\lambda_1 + \lambda_2 + J} \frac{f^{(\lambda_2)}_{j_1 j_2}}{\sqrt{\mathcal{N}^{\lambda_2 i_2}_{\tau}}} \begin{Bmatrix} \lambda_2 & \lambda_1 & J \\ j_3 & j_1 & j_2 \end{Bmatrix} \tilde{\mathcal{F}}^{\lambda_2 i_2}_{\lambda_1 i_1 Ji}(j_1 j_2 j_3) \Bigg]\,, \qquad \text{(A.2)}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{S}^{Ji}_{\lambda_1 i_1 \lambda_2 i_2}(j_1 j_2 j_3) &= \mathcal{X}^{Ji}_{j_1 j_2} v^{(-)}_{j_1 j_2} (\psi^{\lambda_1 i_1}_{j_2 j_3} \tilde{\phi}^{\lambda_2 i_2}_{j_3 j_1} + \phi^{\lambda_1 i_1}_{j_2 j_3} \tilde{\psi}^{\lambda_2 i_2}_{j_3 j_1} + \eta^{\lambda_1 i_1}_{j_2 j_3} \xi^{\lambda_2 i_2}_{j_3 j_1} + \xi^{\lambda_1 i_1}_{j_2 j_3} \eta^{\lambda_2 i_2}_{j_3 j_1}) \\
&\quad - \mathcal{Y}^{Ji}_{j_1 j_2} v^{(-)}_{j_1 j_2} (\tilde{\psi}^{\lambda_1 i_1}_{j_2 j_3} \phi^{\lambda_2 i_2}_{j_3 j_1} + \tilde{\phi}^{\lambda_1 i_1}_{j_2 j_3} \psi^{\lambda_2 i_2}_{j_3 j_1} + \tilde{\eta}^{\lambda_1 i_1}_{j_2 j_3} \tilde{\xi}^{\lambda_2 i_2}_{j_3 j_1} + \tilde{\xi}^{\lambda_1 i_1}_{j_2 j_3} \tilde{\eta}^{\lambda_2 i_2}_{j_3 j_1}) \\
&\quad + \mathcal{Z}^{Ji}_{j_1 j_2} u^{(+)}_{j_1 j_2} (\tilde{\psi}^{\lambda_1 i_1}_{j_2 j_3} \xi^{\lambda_2 i_2}_{j_3 j_1} + \tilde{\phi}^{\lambda_1 i_1}_{j_2 j_3} \eta^{\lambda_2 i_2}_{j_3 j_1} - \tilde{\eta}^{\lambda_1 i_1}_{j_2 j_3} \tilde{\phi}^{\lambda_2 i_2}_{j_3 j_1} - \tilde{\xi}^{\lambda_1 i_1}_{j_2 j_3} \tilde{\psi}^{\lambda_2 i_2}_{j_3 j_1}) \\
&\quad - \mathcal{Z}^{Ji}_{j_2 j_1} u^{(+)}_{j_1 j_2} (\psi^{\lambda_1 i_1}_{j_2 j_3} \tilde{\xi}^{\lambda_2 i_2}_{j_3 j_1} + \phi^{\lambda_1 i_1}_{j_2 j_3} \tilde{\eta}^{\lambda_2 i_2}_{j_3 j_1} - \eta^{\lambda_1 i_1}_{j_2 j_3} \phi^{\lambda_2 i_2}_{j_3 j_1} - \xi^{\lambda_1 i_1}_{j_2 j_3} \psi^{\lambda_2 i_2}_{j_3 j_1})\,, \\
\tilde{\mathcal{R}}^{\lambda_1 i_1}_{\lambda_2 i_2 Ji}(j_1 j_2 j_3) &= \mathcal{X}^{\lambda_1 i_1}_{j_1 j_2} v^{(-)}_{j_1 j_2} (\tilde{\psi}^{\lambda_2 i_2}_{j_2 j_3} \psi^{Ji}_{j_3 j_1} + \tilde{\phi}^{\lambda_2 i_2}_{j_2 j_3} \phi^{Ji}_{j_3 j_1} + \tilde{\eta}^{\lambda_2 i_2}_{j_2 j_3} \tilde{\eta}^{Ji}_{j_3 j_1} + \tilde{\xi}^{\lambda_2 i_2}_{j_2 j_3} \tilde{\xi}^{Ji}_{j_3 j_1}) \\
&\quad - \mathcal{Y}^{\lambda_1 i_1}_{j_1 j_2} v^{(-)}_{j_1 j_2} (\psi^{\lambda_2 i_2}_{j_2 j_3} \tilde{\psi}^{Ji}_{j_3 j_1} + \phi^{\lambda_2 i_2}_{j_2 j_3} \tilde{\phi}^{Ji}_{j_3 j_1} + \eta^{\lambda_2 i_2}_{j_2 j_3} \eta^{Ji}_{j_3 j_1} + \xi^{\lambda_2 i_2}_{j_2 j_3} \xi^{Ji}_{j_3 j_1}) \\
&\quad + \mathcal{Z}^{\lambda_1 i_1}_{j_1 j_2} u^{(+)}_{j_1 j_2} (\psi^{\lambda_2 i_2}_{j_2 j_3} \tilde{\eta}^{Ji}_{j_3 j_1} + \phi^{\lambda_2 i_2}_{j_2 j_3} \tilde{\xi}^{Ji}_{j_3 j_1} - \eta^{\lambda_2 i_2}_{j_2 j_3} \psi^{Ji}_{j_3 j_1} - \xi^{\lambda_2 i_2}_{j_2 j_3} \phi^{Ji}_{j_3 j_1}) \\
&\quad - \mathcal{Z}^{\lambda_1 i_1}_{j_2 j_1} u^{(+)}_{j_1 j_2} (\tilde{\psi}^{\lambda_2 i_2}_{j_2 j_3} \eta^{Ji}_{j_3 j_1} + \tilde{\phi}^{\lambda_2 i_2}_{j_2 j_3} \xi^{Ji}_{j_3 j_1} - \tilde{\eta}^{\lambda_2 i_2}_{j_2 j_3} \tilde{\psi}^{Ji}_{j_3 j_1} - \tilde{\xi}^{\lambda_2 i_2}_{j_2 j_3} \tilde{\phi}^{Ji}_{j_3 j_1})\,, \\
\tilde{\mathcal{F}}^{\lambda_2 i_2}_{\lambda_1 i_1 Ji}(j_1 j_2 j_3) &= \mathcal{X}^{\lambda_2 i_2}_{j_1 j_2} v^{(-)}_{j_1 j_2} (\tilde{\psi}^{\lambda_1 i_1}_{j_2 j_3} \tilde{\psi}^{Ji}_{j_3 j_1} + \tilde{\phi}^{\lambda_1 i_1}_{j_2 j_3} \tilde{\phi}^{Ji}_{j_3 j_1} + \tilde{\eta}^{\lambda_1 i_1}_{j_2 j_3} \eta^{Ji}_{j_3 j_1} + \tilde{\xi}^{\lambda_1 i_1}_{j_2 j_3} \xi^{Ji}_{j_3 j_1}) \\
&\quad - \mathcal{Y}^{\lambda_2 i_2}_{j_1 j_2} v^{(-)}_{j_1 j_2} (\psi^{\lambda_1 i_1}_{j_2 j_3} \psi^{Ji}_{j_3 j_1} + \phi^{\lambda_1 i_1}_{j_2 j_3} \phi^{Ji}_{j_3 j_1} + \eta^{\lambda_2 i_3}_{j_3 j_1} \tilde{\eta}^{Ji}_{j_3 j_1} + \xi^{\lambda_2 i_3}_{j_3 j_1} \tilde{\xi}^{Ji}_{j_3 j_1})
\end{aligned}$$

$$+ \mathcal{Z}^{\lambda_2 i_2}_{j_1 j_2} u^{(+)}_{j_1 j_2} (\psi^{\lambda_1 i_1}_{j_2 j_3} \eta^{Ji}_{j_3 j_1} + \phi^{\lambda_1 i_1}_{j_2 j_3} \xi^{Ji}_{j_3 j_1} - \eta^{\lambda_1 i_1}_{j_2 j_3} \tilde{\psi}^{Ji}_{j_3 j_1} - \xi^{\lambda_1 i_1}_{j_2 j_3} \tilde{\phi}^{Ji}_{j_3 j_1})$$
$$- \mathcal{Z}^{\lambda_2 i_2}_{j_2 j_1} u^{(+)}_{j_1 j_2} (\tilde{\psi}^{\lambda_1 i_1}_{j_2 j_3} \tilde{\eta}^{Ji}_{j_3 j_1} + \tilde{\phi}^{\lambda_1 i_1}_{j_2 j_3} \tilde{\xi}^{Ji}_{j_3 j_1} - \tilde{\eta}^{\lambda_1 i_1}_{j_2 j_3} \psi^{Ji}_{j_3 j_1} - \tilde{\xi}^{\lambda_1 i_1}_{j_2 j_3} \phi^{Ji}_{j_3 j_1}) .$$

$$\begin{aligned} W^{\lambda_1 i_1}_{\lambda_2 i_2}(Ji) = \hat{\lambda}_1 \hat{\lambda}_2 \sum_{\tau} \sum_{j_1 j_2 j_3}^{\tau} \Bigg[& \frac{f^{(J)}_{j_1 j_2}}{\sqrt{\mathcal{N}^{Ji}_{\tau}}} \begin{Bmatrix} J & \lambda_1 & \lambda_2 \\ j_3 & j_1 & j_2 \end{Bmatrix} \tilde{\mathcal{G}}^{Ji}_{\lambda_1 i_1 \lambda_2 i_2}(j_1 j_2 j_3) \\ & - \frac{f^{(\lambda_1)}_{j_1 j_2}}{\sqrt{\mathcal{N}^{\lambda_1 i_1}_{\tau}}} \begin{Bmatrix} \lambda_1 & \lambda_2 & J \\ j_3 & j_1 & j_2 \end{Bmatrix} \mathcal{R}^{\lambda_1 i_1}_{\lambda_2 i_2 Ji}(j_1 j_2 j_3) \\ & - (-1)^{\lambda_1 + \lambda_2 + J} \frac{f^{(\lambda_2)}_{j_1 j_2}}{\sqrt{\mathcal{N}^{\lambda_2 i_2}_{\tau}}} \begin{Bmatrix} \lambda_2 & \lambda_1 & J \\ j_3 & j_1 & j_2 \end{Bmatrix} \mathcal{R}^{\lambda_2 i_2}_{\lambda_1 i_1 Ji}(j_1 j_2 j_3) \Bigg] , \end{aligned} \qquad (A.3)$$

where

$$\begin{aligned} \tilde{\mathcal{G}}^{Ji}_{\lambda_1 i_1 \lambda_2 i_2}(j_1 j_2 j_3) = & \mathcal{X}^{Ji}_{j_1 j_2} v^{(-)}_{j_1 j_2} (\tilde{\psi}^{\lambda_1 i_1}_{j_2 j_3} \tilde{\phi}^{\lambda_2 i_2}_{j_3 j_1} + \tilde{\phi}^{\lambda_1 i_1}_{j_2 j_3} \tilde{\psi}^{\lambda_2 i_2}_{j_3 j_1} + \tilde{\eta}^{\lambda_1 i_1}_{j_2 j_3} \xi^{\lambda_2 i_2}_{j_3 j_1} + \tilde{\xi}^{\lambda_1 i_1}_{j_2 j_3} \eta^{\lambda_2 i_2}_{j_3 j_1}) \\ & - \mathcal{Y}^{Ji}_{j_1 j_2} v^{(-)}_{j_1 j_2} (\psi^{\lambda_1 i_1}_{j_2 j_3} \phi^{\lambda_2 i_2}_{j_3 j_1} + \phi^{\lambda_1 i_1}_{j_2 j_3} \psi^{\lambda_2 i_2}_{j_3 j_1} + \eta^{\lambda_1 i_1}_{j_2 j_3} \tilde{\xi}^{\lambda_2 i_2}_{j_3 j_1} + \xi^{\lambda_1 i_1}_{j_2 j_3} \tilde{\eta}^{\lambda_2 i_2}_{j_3 j_1}) \\ & + \mathcal{Z}^{Ji}_{j_1 j_2} u^{(+)}_{j_1 j_2} (\psi^{\lambda_1 i_1}_{j_2 j_3} \xi^{\lambda_2 i_2}_{j_3 j_1} + \phi^{\lambda_1 i_1}_{j_2 j_3} \eta^{\lambda_2 i_2}_{j_3 j_1} - \eta^{\lambda_1 i_1}_{j_2 j_3} \tilde{\phi}^{\lambda_2 i_2}_{j_3 j_1} - \xi^{\lambda_1 i_1}_{j_2 j_3} \tilde{\psi}^{\lambda_2 i_2}_{j_3 j_1}) \\ & - \mathcal{Z}^{Ji}_{j_1 j_2} u^{(+)}_{j_1 j_2} (\tilde{\psi}^{\lambda_1 i_1}_{j_2 j_3} \tilde{\xi}^{\lambda_2 i_2}_{j_3 j_1} + \tilde{\phi}^{\lambda_1 i_1}_{j_2 j_3} \tilde{\eta}^{\lambda_2 i_2}_{j_3 j_1} - \tilde{\eta}^{\lambda_1 i_1}_{j_2 j_3} \phi^{\lambda_2 i_2}_{j_3 j_1} - \tilde{\xi}^{\lambda_1 i_1}_{j_2 j_3} \psi^{\lambda_2 i_2}_{j_3 j_1}) , \end{aligned}$$

$$\begin{aligned} \mathcal{R}^{\lambda_1 i_1}_{\lambda_2 i_2 Ji}(j_1 j_2 j_3) = & \mathcal{X}^{\lambda_1 i_1}_{j_1 j_2} v^{(-)}_{j_1 j_2} (\psi^{\lambda_2 i_2}_{j_2 j_3} \tilde{\psi}^{Ji}_{j_3 j_1} + \phi^{\lambda_2 i_2}_{j_2 j_3} \tilde{\phi}^{Ji}_{j_3 j_1} + \eta^{\lambda_2 i_2}_{j_2 j_3} \eta^{Ji}_{j_3 j_1} + \xi^{\lambda_2 i_2}_{j_2 j_3} \xi^{Ji}_{j_3 j_1}) \\ & - \mathcal{Y}^{\lambda_1 i_1}_{j_1 j_2} v^{(-)}_{j_1 j_2} (\tilde{\psi}^{\lambda_2 i_2}_{j_2 j_3} \psi^{Ji}_{j_3 j_1} + \tilde{\phi}^{\lambda_2 i_2}_{j_2 j_3} \phi^{Ji}_{j_3 j_1} + \tilde{\eta}^{\lambda_2 i_2}_{j_2 j_3} \tilde{\eta}^{Ji}_{j_3 j_1} + \tilde{\xi}^{\lambda_2 i_2}_{j_2 j_3} \tilde{\xi}^{Ji}_{j_3 j_1}) \\ & + \mathcal{Z}^{\lambda_1 i_1}_{j_1 j_2} u^{(+)}_{j_1 j_2} (\tilde{\psi}^{\lambda_2 i_2}_{j_2 j_3} \eta^{Ji}_{j_3 j_1} + \tilde{\phi}^{\lambda_2 i_2}_{j_2 j_3} \xi^{Ji}_{j_3 j_1} - \tilde{\eta}^{\lambda_2 i_2}_{j_2 j_3} \tilde{\psi}^{Ji}_{j_3 j_1} - \tilde{\xi}^{\lambda_2 i_2}_{j_2 j_3} \tilde{\phi}^{Ji}_{j_3 j_1}) \\ & - \mathcal{Z}^{\lambda_1 i_1}_{j_2 j_1} u^{(+)}_{j_1 j_2} (\psi^{\lambda_2 i_2}_{j_2 j_3} \tilde{\eta}^{Ji}_{j_3 j_1} + \phi^{\lambda_2 i_2}_{j_2 j_3} \tilde{\xi}^{Ji}_{j_3 j_1} - \eta^{\lambda_2 i_2}_{j_2 j_3} \psi^{Ji}_{j_3 j_1} - \xi^{\lambda_2 i_2}_{j_2 j_3} \phi^{Ji}_{j_3 j_1}) . \end{aligned}$$

References

1. Y. Takahashi and H. Umezawa, *Collect. Phenom.* **2** (1975) 55.
2. H. Umezawa, H. Matsumoto and M. Tachiki, *Thermo Field Dynamics and Condensed States* (North-Holland, Amsterdam, New York, Oxford, 1982).
3. T. Arimitsu, in *Thermal Field Theories*, eds. H. Ezawa, T. Arimitsu and Y. Hashimoto (North-Holland, 1991), p. 207.
4. P. Henning, *Phys. Rep.* **253** (1995) 235.
5. T. Arimitsu, in *Stochastic Processes and their Applications*, eds. A. Vijayakumar and M. Sreenivasan (Narosa Publishing House, Madras, 1999), p. 279.
6. K. Tanabe, *Phys. Rev. C* **37** (1988) 2802.
7. T. Hatsuda, *Nucl. Phys. A* **492** (1989) 187.
8. K. Tanabe and K. Sugawara-Tanabe, *Phys. Lett. B* **247** (1990) 202; *ibid. Z. Phys. A: Hadrons and Nuclei* **339** (1991) 91.
9. O. Civitarese and A. L. DePaoli, *Z. Phys. A* **344** (1993) 243.
10. D. S. Kosov and A. I. Vdovin, *Mod. Phys. Lett. A* **9** (1994) 1735.

11. K. Tanabe, *Nucl. Phys. A* **569** (1994) 27.
12. A. I. Vdovin and D. S. Kosov, *Phys. Atom. Nucl.* **58** (1995) 776.
13. D. S. Kosov and A. I. Vdovin, *Z. Phys. A* **355** (1996) 17.
14. A. V. Avdeenkov, D. S. Kosov and A. I. Vdovin, *Mod. Phys. Lett. A* **11** (1996) 853.
15. O. Civitarese, A. L. DePaoli and M. C. Rocca, *Nucl. Phys. A* **642** (1998) 541.
16. A. N. Storozhenko and A. I. Vdovin, *Eur. Phys. J. A* **5** (1999) 263.
17. A. I. Vdovin and A. N. Storozhenko, *Phys. Atom. Nucl.* **64** (2001) 1069.
18. A. N. Storozhenko and A. I. Vdovin *et al.*, *Phys. Rev. C* **69** (2004) 064320.
19. K. Tanabe and H. Nakada, *Phys. Rev. C* **71** (2005) 024314.
20. V. G. Soloviev, *Theory of Atomic Nuclei: Quasiparticles and Phonons* (Institute of Physics, Bristol, Philadelphia, 1992).
21. P. F. Bortignon, R. A. Broglia, G. F. Bertch and J. Pacheco, *Nucl. Phys. A* **460** (1986) 149.
22. E. C. Seva and H. M. Sofia, *Phys. Rev. C* **56** (1997) 3107.
23. P.-F. Bortignon, R. A. Broglia, D. R. Bes and R. J. Liotta, *Phys. Rep. C* **30** (1977) 305.
24. Z. Aouissat, A. Storozhenko, A. Vdovin and J. Wambach, *Phys. Rev. C* **64** (2001) 015201.
25. A. Dzhioev, A. Vdovin, V. Yu. Ponomarev and J. Wambach, *Bull. Russ. Acad. Sci. Phys.* **72** (2008) 269.
26. I. Ojima, *Ann. Phys.* **137** (1981) 1.
27. H. Matsumoto, Y. Nakani and H. Umezawa, *Phys. Rev. D* **31** (1985) 429.
28. A. L. Goodman, *Nucl. Phys. A* **352** (1981) 30.
29. O. Civitarese, G. G. Dussel and R. P. J. Perazzo, *Nucl. Phys. A* **404** (1983) 15.
30. A. V. Ignatyuk, *Statistical Properties of Excited Atomic Nuclei* (Moscow, Energoatomizdat, 1983, in Russian).
31. A. V. Ignatyuk, *Yad. Phys.* **21** (1975) 20.
32. H. M. Sommermann, *Ann. Phys. (NY)* **151** (1983) 163.
33. F. Alasia and O. Civitarese, *Phys. Rev C* **42** (1990) 1335.
34. Ph. Chomaz, D. Vautherin and N. Vinh Mau, *Phys. Lett. B* **242** (1990) 313.
35. J. L. Egido and P. Ring, *J. Phys G: Nucl. Part. Phys.* **19** (1993) 1.
36. M. Di Toro, V. Baran and M. Cabibbo *et al.*, *Phys. Part. Nucl.* **31** (2000) 876.
37. S. Shlomo and V. M. Kolomietz, *Rep. Progr. Phys* **68** (2005) 1.
38. G. G. Dussel, H. Fortunato and H. M. Sofia, *Phys. Rev. C* **42** (1990) 2093.
39. O. Civitarese and M. Reboiro, *Rhys. Rev. C* **63** (2001) 034323.