Anomalous Gilbert damping and Duffing features of the superconductor-ferromagnet-superconductor φ_0 Josephson junction

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We study the phase dynamics, IV characteristics, and magnetization dynamics of the φ_0 Josephson junction at small values of spin-orbit interaction, ratio of the Josephson junction to magnetic energy, and Gilbert damping. We demonstrate that the coupled Landau-Lifshitz-Gilbert-Josephson dynamics is reduced to a scalar nonlinear Duffing oscillator. The resulting Duffing equation incorporates the Gilbert damping in a special way across the dissipative term and the restoring force. An anomalous shift of the ferromagnetic resonance frequency with decreasing Gilbert damping is found. We demonstrate that there is a critical damping value at which nonlinearity comes into play, and it changes the damping dependence of the ferromagnetic resonance.

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I. INTRODUCTION

The Josephson junctions (JJs) with the current-phase relation $I = I_c \sin(\varphi - \varphi_0)$ where the phase-shift φ_0 is proportional to the magnetic moment of ferromagnetic layer determined by the parameter of spin-orbit interaction, demonstrate a number of unique features important for superconducting spintronics and modern information technology [1–8]. The phase shift allows one to manipulate the internal magnetic moment using the Josephson current and the reverse phenomenon which leads to the appearance of the DC component in the superconducting current [9–11]. In the superconducting circuits, such φ_0 junctions play the role of phase batteries producing spontaneous currents [12–14], important for superconducting electronics and rapid single-flux quantum logics. A remarkable feature of the coupling between the Josephson current and the magnetic moment in the φ_0 junction is the possibility of stimulating a magnetization reversal in the ferromagnetic weak link by the supercurrent. It creates an effective mechanism for the magnetic moment control in the devices of superconducting spintronics [4,6,15–21].

Interactive fields can bring nonlinear phenomena of both classical and quantum natures. A basic example is the magnons as quasiparticles strongly interacting with microwave photons [22]. As a result we could name the Bose-Einstein condensation of the magnons [23,24], and the synchronization of spin torque nano-oscillators as they coherently emit microwave signals in response to the DC current [25]. It is interesting that (semi)classical anharmonic effects in the magnetodynamics described by the Landau-Lifshitz-Gilbert (LLG) model in thin films or heterostructures [26,27], and the quantum anharmonicity in the cavity magnonics [28] can well be modeled by so simple a nonlinear oscillator as Duffing. The Duffing equation contains a cubic term and describes the oscillations of the various nonlinear systems [29].

Despite the fact that nonlinear features of LLG have been studied in different systems, manifestation of the Duffing oscillator in the framework of the LLG model is still not completely studied. Closer to our present paper, the resonant response of the antiferromagnetic bimeron under an alternating current is described by the dynamics of the Duffing equation too. This has applications in weak signal detection [26,30,31]. As another application with the Duffing oscillator at work, we can mention the ultrathin $Co_{20}Fe_{60}B_{20}$ layer and its large-angle magnetization precession under microwave voltage. There are also “ foldover” features, characteristic of the Duffing spring, in the magnetization dynamics of the Co/Ni multilayer excited by a microwave current [27,32,33]. But nonlinear features of φ_0-Josephson junctions have yet to be studied in detail. It is our main purpose here to show properties of a magnetically coupled φ_0 Josephson junction that come as a consequence of nonlinearity.

There are a series of recent experiments demonstrating the modification of Gilbert damping by the superconducting correlations (see Ref. [34] and citations therein). In particular, the pronounced peaks in the temperature dependence of Gilbert damping have been observed for the insulating ferromagnetic/superconductor multilayers [35] which might be explained by the presence of spin-relaxation mechanisms, such as the spin-orbit scattering [34]. Here, we use the noncentrosymmetric ferromagnetic material as a weak link in φ_0 junctions. The suitable candidates may be MnSi or FeGe where the lack of inversion center comes from the crystalline structure [9].

The Gilbert damping determines the magnetization dynamics in ferromagnetic materials, but its origin is not well

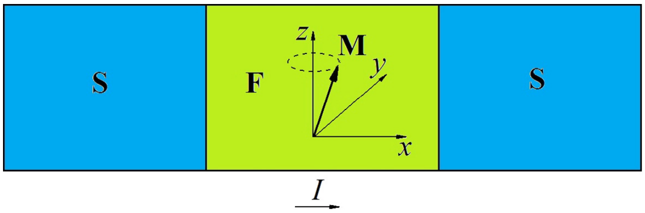

FIG. 1. Schematic of SFS φ_0 Josephson junction. The external current is applied along the x direction, and the ferromagnetic easy axis is along the z direction.

understood yet. Effect of nonlinearity on damping in the system is very important for application of these materials in fast switching spintronics devices. Our paper clarifies such effects. In Ref. [36], the authors discuss the experimental study of temperature-dependent Gilbert damping in permalloy thin films of varying thicknesses by ferromagnetic resonance and provide an important insight into the physical origin of the Gilbert damping in ultrathin magnetic films.

In this paper, we demonstrate an anomalous dependence of the ferromagnetic resonance frequency with increasing Gilbert damping. We find that the resonance curves demonstrate features of the Duffing oscillator, reflecting the nonlinear nature of Landau-Lifshitz-Gilbert-Josephson (LLGJ) system of equations. The resulting Duffing equation incorporates the Gilbert damping in a special way across the dissipative term and the restoring force. The damped precession of the magnetic moment is dynamically driven by the Josephson supercurrent, and the resonance behavior is given by the dynamics of the Duffing spring. We demonstrate that the resonance frequency ω_F of the magnetic part without dissipation cannot be realized in the LLGJ system. There is a critical damping value at which nonlinearity comes into play, and the damping dependence of ferromagnetic resonance changes. A resonance method for the determination of spin-orbit interaction in noncentrosymmetric materials which play the role of the barrier in φ_0 junctions is proposed.

II. MODEL AND METHODS

In the considered superconductor-ferromegnet-superconductor (SFS) the φ_0 Josephson junction (see Fig. 1), the superconducting phase difference φ, and magnetization $\mathbf{M}$ of the ferromagnetic layer are two coupled dynamical variables. Their dynamics is described by a coupled system of equations we call the LLGJ system of equations. Namely, taking into account LLG equation with effective magnetic-field $\mathbf{H_{eff}}$, resistively capacitively shunted junction model, and Josephson relation, we come to the LLGJ system of equations, which in normalized variables can be written in the form [21,37,38]

$$\begin{aligned}
\frac{d\mathbf{m}}{dt} &= \omega_F \mathbf{h_{eff}} \times \mathbf{m} + \alpha\left(\mathbf{m} \times \frac{d\mathbf{m}}{dt}\right),\\
\mathbf{h_{eff}} &= Gr\,\sin(\varphi - rm_y)\widehat{\mathbf{y}} + m_z\widehat{\mathbf{z}},\\
\frac{dV}{dt} &= \frac{1}{\beta_c}\left[I - V + r\frac{dm_y}{dt} - \sin(\varphi - rm_y)\right],\\
\frac{d\varphi}{dt} &= V. \qquad (1)
\end{aligned}$$

The first term on the right-hand side describes the precession of magnetization vector $\mathbf{m}$ of a ferromagnetic system around the effective field $\mathbf{h_{eff}}$, calculated as the variational derivative of energy with respect to the magnetization. The second term is the damping term, which relaxes the system to an equilibrium state, where $\mathbf{m}$ and $\mathbf{h_{eff}}$ are parallel, and no torque is exerted on the magnetization.

The magnetization vector with components $m_{x,y,z}$ is normalized to the $M_0 = \|\mathbf{M}\|$, and it is satisfied by the constraint $\sum_{i=x,y,z} m_i^2(t) = 1$. The ferromagnetic resonance frequency of the magnetic part of the system $\omega_F = \Omega_F/\omega_c$ with $\Omega_F = \gamma K/\nu$ determines, by gyromagnetic ratio γ, anisotropic constant K and the volume of the ferromagnetic F layer ν is. The vector of effective field $\mathbf{h_{eff}}$ is normalized to the K/M_0 ($\mathbf{h_{eff}} = \mathbf{H_{eff}}M_0/K$), voltage V is normalized to the $V_c = I_cR$ with the I_c—critical current of JJ and R—resistance of JJ. The other parameters are as follows: α is a phenomenological damping constant (Gilbert damping), $G = E_J/(K\nu)$ is the relation of Josephson energy to the magnetic one, r is a parameter of spin-orbit coupling, β_c is the McCumber parameter, I is bias current normalized to the I_c. In this system of equations, time t is normalized to ω_c^{-1}, where $\omega_c = 2eI_cR/\hbar$ is the characteristic frequency. In the chosen normalization, the average voltage corresponds to the Josephson frequency ω_J.

III. FERROMAGNETIC RESONANCE IN THE φ_0 JUNCTION

The ferromagnetic resonance features are demonstrated by the voltage dependence of the maximal amplitude of the m_y component ($m_y^{\max}$), taken at each value of the bias current. We begin with the analytical results for such dependence in the ferromagnetic resonance region. As was discussed in Refs. [9,37,39], in cases $Gr \ll 1$, $m_z \approx 1$, and neglecting quadratic terms m_x and m_y, we get

$$\begin{aligned}
\dot{m}_x &= \xi[-m_y + Gr\,\sin\,\omega_J t - \alpha m_x]\\
\dot{m}_y &= \xi[m_x - \alpha m_y], \qquad (2)
\end{aligned}$$

where $\xi = \omega_F/(1+\alpha^2)$. This system of equations can be written as the second-order differential equation with respect to m_y,

$$\ddot{m}_y = -2\alpha\xi\dot{m}_y - \xi^2(1+\alpha^2)m_y + \xi^2 Gr\,\sin\,\omega_J t. \qquad (3)$$

The corresponding solution for m_y has the form

$$m_y(t) = \frac{\omega_+ - \omega_-}{r}\sin\,\omega_J t - \frac{\alpha_+ + \alpha_-}{r}\cos\,\omega_J t, \qquad (4)$$

where

$$\omega_\pm = \frac{Gr^2\omega_F}{2}\frac{\omega_J \pm \omega_F}{[(\omega_J \pm \omega_F)^2 + (\alpha\omega_J)^2]}, \qquad (5)$$

and

$$\alpha_\pm = \frac{Gr^2\omega_F}{2}\frac{\alpha\omega_J}{[(\omega_J \pm \omega_F)^2 + (\alpha\omega_J)^2]}. \qquad (6)$$

We see that when Josephson frequency ω_J is approaching the ferromagnetic one ω_F, m_y demonstrates resonance with dissipation. The maximal amplitude $m_y^{\max}$ as a function of voltage (i.e., Josephson frequency ω_J) at different αs, calculated using

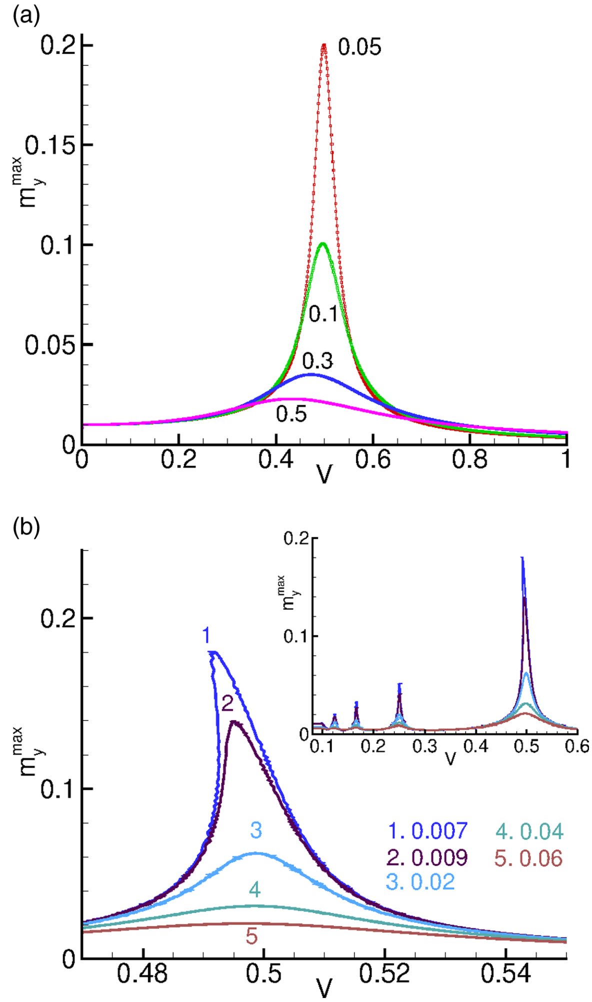

FIG. 2. (a) Analytical results for maximal amplitude $m_y^{\max}$ in the ferromagnetic resonance region for different αs; (b) the same for numerical results based on the system of Eqs. (1). The inset shows the manifestation of the resonance subharmonics. Parameters are as follows: $\beta_c = 25,\ G = 0.05,\ r = 0.05,\ \omega_F = 0.5$.

(4), is presented in Fig. 2(a). It shows the usual characteristic variation of the resonance curve with an increase in dissipation parameter where the maximal amplitude and the position of the resonance peak correspond to the damped resonance. We note that the analytical result in Eq. (4) was obtained for $Gr \ll 1$. Figure 2(b) shows the results of numerical simulations $m_y^{\max}(V)$, based on the system of equation (1) at different values of dissipation parameter. The subharmonics appear at $\omega = 1/2, 1/3, 1/4$ as demonstrated in the inset of Fig. 2(b). The presence of subharmonics already points to nonlinear effects.

The results given in Fig. 2(b) results demonstrate the essential differences with the analytical ones. We stress two important features of the φ_0 junction followed from the presented results. First, the ferromagnetic resonance curves show the foldover effect, i.e., the features of the Duffing oscillator. Different from a linear oscillator, the nonlinear Duffing demonstrates a bistability under external periodic force [40].

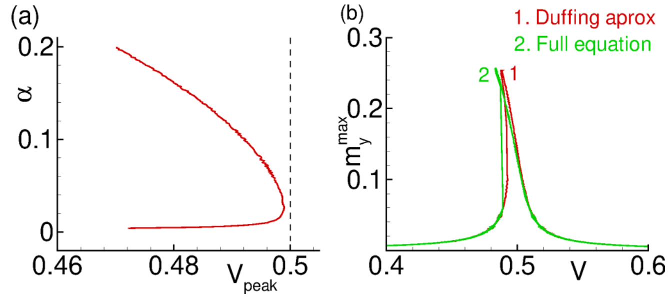

FIG. 3. (a) α effect: Starting from $\alpha = 0.02$, decreasing damping shifts the maximum of the resonance curve $m_y^{\max}(V)$ presented in Fig. 2 away from ω_F. Variation of α is in the interval [0.006–0.2]. The dashed line indicates ferromagnetic resonance ω_F of the magnetic subsystem at $\alpha = 0$; (b) Comparison of the resonance curves $m_y^{\max}(V)$ calculated by full LLGJ equation (1) and the approximate equation (A5).

Second, the ferromagnetic resonance curves demonstrate an unusual dependence of the resonance frequency on Gilbert damping α. As shown in Fig. 3(a), an increase in damping leads to a nonuniform change in the resonant frequency, i.e., with an increase in damping the resonance maximum shifts to ω_F at small α, but then moves to the opposite side. So, with an increase in α, the unusual dependence of the resonance voltage transforms to the usual one. For the parameters chosen, the critical value of the damping parameter of this transformation is around $\alpha = 0.02$ to 0.03. We call this unusual damping dependence of $m_y^{\max}$ an "α effect." The essence of the α effect is that the resonance frequency ω_F of the magnetic subsystem without dissipation cannot be realized in the LLGJ system. At critical damping, the nonlinearity comes into play, and the $V_{peak}(\alpha)$ dependence changes.

Both the α effect and the Duffing features in our system appear due to the nonlinear features of the system dynamics at small $G, r, \alpha \ll 1$. To prove it, we have carried out the numerical analysis of each term of the LLGJ full equation [first two equations in (1)] for the set of model parameters $G = 0.05$, $r = 0.05$ $\alpha = 0.005$. After neglecting the terms on the order of 10^{-6}, we have

$$\begin{aligned}
\frac{\dot{m}_x}{\xi} &= -m_y m_z + Grm_z \sin(\varphi - rm_y) - \alpha m_x m_z^2,\\
\frac{\dot{m}_y}{\xi} &= m_x m_z - \alpha m_y m_z^2,\\
\frac{\dot{m}_z}{\xi} &= -Grm_x \sin(\varphi - rm_y) + \alpha m_z (m_x^2 + m_y^2),
\end{aligned} \tag{7}$$

In this approximation, we observe both the α effect and the Duffing oscillator features. Neglecting here the last term $\alpha m_z(m_x^2 + m_y^2)$ in the third equation for $\dot{m}_z$, which is on the order of 10^{-4}, leads to losing the Duffing oscillator features, but still keeps the α effect. We note that Eq. (7) keep the time invariance of the magnetic moment, so that term plays an important role for the manifestation of the Duffing oscillator features by the LLGJ equation.

IV. THE GENERALIZED DUFFING EQUATION FOR THE φ_0 JUNCTION

The LLG is a nonlinear equation, and in the case of simple effective field, it can be transformed to the Duffing equation [26,29]. Such a transformation was used in Ref. [29] to demonstrate the nonlinear dynamics of the magnetic vortex state in a circular nanodisk under a perpendicular alternating magnetic field that excites the radial modes of the magnetic resonance. They showed Duffing-type nonlinear resonance and built a theoretical model corresponding to the Duffing oscillator from the LLG equation to explore the physics of the magnetic vortex core polarity switching for magnetic storage devices.

As we mentioned above, the approximated LLGJ system of Eq. (7) describes both the α effect and the features of the Duffing oscillator. We demonstrate in the Appendix that the generalized Duffing equation for the φ_0 junction,

$$\ddot{m}_y + 2\xi\alpha\dot{m}_y + \xi^2(1+\alpha^2)m_y - \xi^2(1+\alpha^2)m_y^3 = \xi^2 Gr\ \sin\omega_J t \tag{8}$$

can be obtained directly from the LLGJ system of equations. As we see here, for small enough G and r, it is only the dimensionless damping parameter α in (A5) that plays a role in the dynamics of the system. We can think of a harmonic spring with a constant that is hardened or softened by the nonlinear term. For a usual Duffing spring with independent coefficients of the various terms, the resonance peak relative to the harmonic (linear) resonant frequency folds over to the smaller (softening) or larger (hardening) frequencies. In the frequency response, the interplay of the specific dependence of each coefficient on α plays an important role and as Fig. 3(a) shows, there is a particular α that brings the maximum of the resonant curve closest to ferromagnetic resonance.

Simulations of the m_y dynamics in the framework of the Duffing equation can explain the observed foldover effect in the frequency dependence of $m_y^{\max}$. Comparison of the results following from the analytically approximate equation (A5) and the results from the full equation (1) for maximal amplitude $m_y^{\max}$ in the ferromagnetic resonance region is presented in Fig. 3(b). We see a very close qualitatively similar behavior. So, the magnetization dynamics in the SFS φ_0 junction due to the voltage oscillations can effectively be described by a scalar Duffing oscillator, synchronizing the precession of the magnetic moment with the Josephson oscillations.

V. EFFECT OF SPIN-ORBIT INTERACTIONS

The spin-orbit interaction plays an important role in different fields of modern physics (see, for example, Ref. [41] and citations therein). Coupling of superconducting current and magnetization and its manifestation in the magnetization dynamics opens a venue for the resonance methods determination of spin-orbit intensity in noncentrosymmetric materials playing the role of barrier in φ_0 junctions. Here we have suggested a method for its determination in real noncentrosymmetric ferromagnetic materials, such as MnSi or FeGe where the lack of inversion center comes from the crystalline structure [9]. As we see below, based on the obtained results,

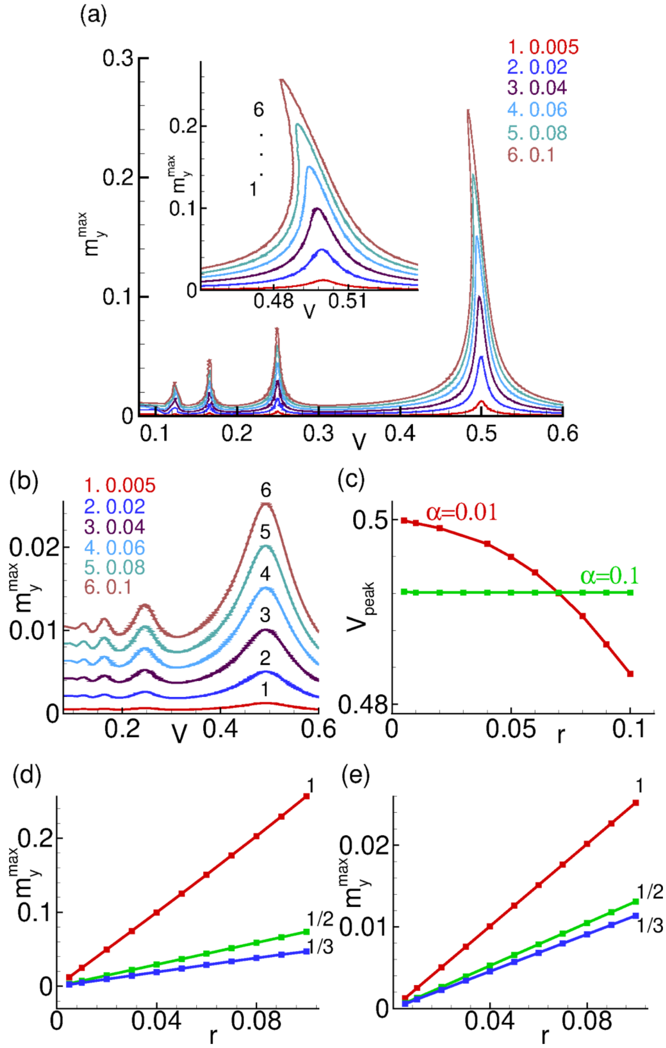

FIG. 4. (a) Voltage dependence of $m_y^{\max}$ in the ferromagnetic resonance region at different values of spin-orbit interaction based on (1) at $G = 0.05$, $\alpha = 0.01$. The inset enlarges the main harmonic; (b) The same for $\alpha = 0.1$; (c) The shift of $m_y^{\max}$ peak as a function of spin-orbit interaction at two values of Gilbert damping; (d) r dependence of the main harmonic and subharmonic peaks in case (a); (e) the same for case (b).

presented in Fig. 4, we may propose three different versions of the resonance method for the determination of spin-orbit interaction in these materials. Particularly, in Fig. 4(a), we present the simulation results of maximal amplitude $m_y^{\max}$ based on LLGJ system (1) at $G = 0.05$, $\alpha = 0.01$ at different values of spin-orbit parameter r in the ferromagnetic resonance region. This case corresponds to the nonlinear approximation leading to Duffing equation (A5). The same characteristics calculated by Eq. (1) for larger value $\alpha = 0.1$, i.e., corresponding to the linear approximation (3) are presented in Fig. 4(b). We see there one order difference in the $m_y^{\max}$ value in these two cases. As was expected, in the case $\alpha = 0.01$, the foldover effect is more distinct.

The simulated r dependence of the resonance peak position for these two values of damping parameter ($\alpha = 0.01$ and $\alpha = 0.1$) are shown in Fig. 4(c). The calculations were performed for the same set of model and simulation parameters. We see clearly a manifestation of nonlinearity of φ_0 junction at small damping, leading to the Duffing's shift of the $m_y^{\max}$ peak of main harmonic with r.

Despite the noted differences between results for $\alpha = 0.01$ and $\alpha = 0.1$, in Fig. 4(c), we found a monotonic linear increase in $m_y^{\max}$ peaks of main harmonic and subharmonics with r in both cases, demonstrated in Figs. 4(d) and 4(e). Such linear dependence can be noted from Eq. (6) of Ref. [26], but the authors did not discuss it. Such dependence of $m_y^{\max}$ peaks of the main harmonic and subharmonics with r might serve as a calibrated curve for spin-orbit interaction intensity, thus, creating the harmonic and subharmonic resonance methods for r determination.

VI. CONCLUSIONS

Based on the reported features of the φ_0 Josephson junction at small values of spin-orbit interaction, the ratio of Josephson to magnetic energy and Gilbert damping, we have demonstrated that the coupling of the superconducting current and the magnetic moments in the φ_0 junction results in the current phase relation intertwining with the ferromagnetic LLGJ dynamics. The ferromagnetic resonance clearly shows this interplay, in particular, an anomalous shift of the ferromagnetic resonance frequency with a decrease in Gilbert damping. The ferromagnetic resonance curves demonstrate features of the Duffing oscillator, reflecting the nonlinear nature of the LLGJ equation. We have shown that due to the nonlinearity as modeled by the generalized Duffing equation, the parameters of the system can compensate each other resulting in unusual response. We have demonstrated that the resonance frequency ω_F of the magnetic subsystem without dissipation cannot be achieved in the LLGJ system. There is a critical damping at which nonlinearity comes into play and the (α) dependence of ferromagnetic resonance changes. There are also foldover effects that were explained by the nonlinearity of the proposed model. A linear dependence of main harmonic and subharmonic maxima on the spin-orbit parameter might be used for its determination in noncentrosymmetric materials. A detailed discussion of this problem will be performed somewhere else.

The experimental testing of our results would involve SFS structures with ferromagnetic material having small enough Gilbert damping, in particular, ferromagnetic metals or insulators with damping parameter $\alpha \sim 10^{-3}$–10^{-4}. In Ref. [42], the authors report on a binary alloy of cobalt and iron that exhibits a damping parameter approaching 10^{-4}, which is comparable to values reported only for ferrimagnetic insulators [43,44]. Using a superconductor-ferromagnetic insulator superconductor on a three-dimensional topological insulator might be a way to have strong spin-orbit coupling needed for the realization of φ_0 JJ and small Gilbert dissipation to observe the α effect [7]. We note in this connection that the yttrium iron garnet is especially interesting because of its small Gilbert damping $\alpha \sim 10^{-5}$.

The interaction between the Josephson current and the magnetization is also determined by the ratio of the Josephson to the magnetic anisotropy energy $G = E_J/(K\nu)$. The value of the Rashba-type parameter r in a permalloy doped with Pt and in the ferromagnets without inversion symmetry, such as MnSi or FeGe, is usually estimated within the range 0.1–1 [45]. The value of the product Gr in the material with weak magnetic anisotropy of $K \sim 4 \times 10^{-5}$ K A^{-3} [46], and in junction with a relatively high critical current density of $3 \times 10^5 - 5 \times 10^6$ A/cm^2 [47] is in the range 1–100. This gives the set of ferromagnetic layer and junction parameters that make it possible to reach the values used in our numerical calculations for the possible experimental observation of the predicted effects. The giant oscillatory Gilbert damping in the superconducting niobium/nickel iron/niobium junctions with respect to the nickel iron thickness was experimentally observed in Ref. [48]. This observation could be important for further exploring the exotic physical properties of ferromagnet/superconductor heterostructures, including the α effect. The search for technological routes to anomalous Josephson junctions fabrication and the development of information-processing methods using circuits based on such junctions are urgent tasks in this area of research [38] and important also for potential applications in quantum computing.

TABLE I. Numerical analysis of Eq. (A4) terms.

a_1	$\frac{\alpha}{\xi}$	$a_1\dot{m}_y^3$	$\sim 1.76 \times 10^{-5}$
a_2	α^2	$a_2 m_y \dot{m}_y^2$	$\sim 3.4 \times 10^{-8}$
a_3	$\xi\alpha^3$	$a_3 m_y^4 \dot{m}_y$	$\sim 7.7 \times 10^{-12}$
a_4	$\xi(3\alpha - \alpha^3)$	$a_4 m_y^2 \dot{m}_y$	$\sim 2 \times 10^{-5}$
a_5	$2\xi\alpha$	$a_5 \dot{m}_y$	$\sim 6 \times 10^{-4}$
a_6	$\xi^2(\alpha^2 + 2\alpha^4)$	$a_6 m_y^5$	$\sim 5.56 \times 10^{-9}$
a_7	$\xi^2(1 + \alpha^2 - \alpha^4)$	$a_7 m_y^3$	$\sim 3.7 \times 10^{-3}$
a_8	$\xi^2(1 + \alpha^2)$	$a_8 m_y$	$\sim 6.1 \times 10^{-2}$
c_1	Gr	$c_1 \dot{m}_y^2 \sin\varphi$	$\sim 3.6 \times 10^{-5}$
c_2	$2\xi^2\alpha^2 Gr$	$c_2 m_y^4 \sin\varphi$	$\sim 5.3 \times 10^{-11}$
c_3	$\xi^2 Gr(\alpha^2 - 2)$	$c_3 m_y^2 \sin\varphi$	$\sim 4.5 \times 10^{-5}$
A	$\xi^2 Gr$	$A \sin \omega_J t$	$\sim 6.25 \times 10^{-4}$

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APPENDIX: GENERALIZED DUFFING EQUATION

Here, we demonstrate by numerical methods that a generalized Duffing equation can be obtained directly from the LLG system of equations for small system parameters of the SFS junction.

Both the α effect and the Duffing features obtained by the LLG system of equations appear due to the nonlinear features of its dynamics at small $G, r, \alpha \ll 1$. To prove it, we have carried out the numerical analysis of each term of the LLG full equation [first two equations in Eq. (1) of the main text] for the set of model parameters $G = 0.05$, $r = 0.05$, and $\alpha = 0.005$. After neglecting the terms on the order of 10^{-6}, we have

$$\frac{\dot{m}_x}{\xi} = -m_y m_z + Grm_z \sin(\varphi - rm_y) - \alpha m_x m_z^2,$$

$$\frac{\dot{m}_y}{\xi} = m_x m_z - \alpha m_y m_z^2,$$

$$\frac{\dot{m}_z}{\xi} = -Grm_x \sin(\varphi - rm_y) + \alpha m_z\left(m_x^2 + m_y^2\right), \quad \text{(A1)}$$

After expanding m_z^n in a series with the degree of $(m_z - 1)$, we can find

$$m_z^n = nm_z - (n-1). \tag{A2}$$

From expression $m_x^2 + m_y^2 + m_z^2 = 1$ and (A2), we obtain

$$m_z = \frac{2 - m_y^2}{2}. \tag{A3}$$

Using approximation $\sin(\varphi - rm_y) = \sin(\omega_J t)$ in (A1), differentiating the second equation of the system (A1) and substituting $\dot{m}_x$, m_x, and $\dot{m}_z$ from the first second and third equations of system (A1), respectively, and using the expressions (A2), (A3), and assuming $m_z = 1$ only in denominators, we come to a second-order differential equation with respect to m_y,

$$\begin{aligned}\ddot{m}_y &= a_1\dot{m}_y^3 + a_2 m_y \dot{m}_y^2 + a_3 m_y^4 \dot{m}_y + a_4 m_y^2 \dot{m}_y + a_5 \dot{m}_y \\ &\quad + a_6 m_y^5 + a_7 m_y^3 + a_8 m_y - c_1 \dot{m}_y^2 \sin\omega_J t \\ &\quad + c_2 m_y^4 \sin\omega_J t + c_3 m_y^2 \sin\omega_J t \\ &\quad + A\sin\omega_J t. \end{aligned}\tag{A4}$$

The numerical calculation for the used set of model parameters allows us to estimate each of the terms in the equation as presented in Table I.

Now, if we neglect those terms smaller than 10^{-4}, Eq. (A4) takes on the form of the Duffing equation,

$$\begin{aligned}&\ddot{m}_y + 2\xi\alpha\dot{m}_y + \xi^2(1+\alpha^2)m_y - \xi^2(1+\alpha^2)m_y^3 \\ &\quad = \xi^2 Gr\,\sin\omega_J t. \end{aligned}\tag{A5}$$

with damping dependent coefficients, i.e., we have a generalized Duffing equation. In the main text, this equation is used to demonstrate the foldover and α effects in φ_0 Josephson junctions.

[1] A. Buzdin, Phys. Rev. Lett. **101**, 107005 (2008).
[2] A. A. Reynoso, G. Usaj, C. A. Balseiro, D. Feinberg, and M. Avignon, Phys. Rev. Lett. **101**, 107001 (2008).
[3] A. Zazunov, R. Egger, T. Jonckheere, and T. Martin, Phys. Rev. Lett. **103**, 147004 (2009).
[4] J. Linder and W. A. Jason Robinson, Nat. Phys. **11**, 307 (2015).
[5] S. V. Mironov, A. S. Mel'nikov, and A. I. Buzdin, Phys. Rev. Lett. **114**, 227001 (2015).
[6] I. V. Bobkova, A. M. Bobkov, and M. A. Silaev, Phys. Rev. B **98**, 014521 (2018).
[7] I. V. Bobkova, A. M. Bobkov, I. R. Rahmonov, A. A. Mazanik, K. Sengupta, and Y. M. Shukrinov, Phys. Rev. B **102**, 134505 (2020).
[8] D. B. Szombati, S. Nadj-Perge, D. Car, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoveb, Nat. Phys. **12**, 568 (2016).
[9] F. Konschelle and A. Buzdin, Phys. Rev. Lett. **102**, 017001 (2009).
[10] Y. M. Shukrinov, I. R. Rahmonov, and K. Sengupta, Phys. Rev. B **99**, 224513 (2019).
[11] Y. M. Shukrinov, Phys.-Usp. (to be published).
[12] A. K. Feofanov, V. A. Oboznov, V. V. Bol'ginov, J. Lisenfeld, S. Poletto, V. V. Ryazanov, A. N. Rossolenko, M. Khabipov, D. Balashov, A. B. Zorin, P. N. Dmitriev, V. P. Koshelets, and A. V. Ustinov, Nat. Phys. **6**, 593 (2010).
[13] T. Ortlepp, Ariando, O. Mielke, C. J. M. Verwijs, K. F. K. Foo, H. Rogalla, F. H. Uhlmann, and H. Hilgenkamp, Science **312**, 1495 (2006).
[14] E. Strambini, A. Iorio, O. Durante, R. Citro, C. Sanz-Fernández, C. Guarcello, I. V. Tokatly, A. Braggio, M. Rocci, N. Ligato, V. Zannier, L. Sorba, F. S. Bergeret, and F. Giazotto, Nat. Nanotechnol. **15**, 656 (2020).
[15] A. A. Mazanik, I. R. Rahmonov, A. E. Botha, and Y. M. Shukrinov, Phys. Rev. Appl. **14**, 014003 (2020).
[16] L. Cai and E. M. Chudnovsky, Phys. Rev. B **82**, 104429 (2010).
[17] L. Cai, D. A. Garanin, and E. M. Chudnovsky, Phys. Rev. B **87**, 024418 (2013).
[18] J. Linder and T. Yokoyama, Phys. Rev. B **83**, 012501 (2011).
[19] S. Hoffman, Y. M. Blanter, and Y. Tserkovnyak, Phys. Rev. B **86**, 054427 (2012).
[20] M. Eschrig, Rep. Prog. Phys. **78**, 104501 (2015).
[21] Y. M. Shukrinov, I. R. Rahmonov, K. Sengupta, and A. Buzdin, Appl. Phys. Lett. **110**, 182407 (2017).
[22] X. Zhang, C.-L. Zou, L. Jiang, and H. X. Tang, Phys. Rev. Lett. **113**, 156401 (2014).
[23] A. A. Serga, V. S. Tiberkevich, C. W. Sandweg, V. I. Vasyuchka, D. A. Bozhko, A. V. Chumak, T. Neumann, B. Obry, G. A. Melkov, A. N. Slavin, and B. Hillebrands, Nat. Commun. **5**, 3452 (2014).
[24] V. E. Demidov, O. Dzyapko, S. O. Demokritov, G. A. Melkov, and A. N. Slavin, Phys. Rev. Lett. **100**, 047205 (2008).
[25] S. Kaka, M. R. Pufall, Wi. H. Rippard, T. J. Silva, S. E. Russek, and J. A. Katine, Nature (London) **437**, 389 (2005).
[26] L. Shen, J. Xia, X. Zhang, M. Ezawa, O. A. Tretiakov, X. Liu, G. Zhao, Y. Zhou, and Y. Zhou, Phys. Rev. Lett. **124**, 037202 (2020).
[27] A. I. Nikitchenko and N. A. Pertsev, Phys. Rev. Appl. **14**, 034022 (2020).
[28] M. Elyasi, Y. M. Blanter, and G. E. W. Bauer, Phys. Rev. B **101**, 054402 (2020).
[29] K. W. Moon, B. S. Chun, W. Kim, Z. Q. Qiu, and C. Hwang, Sci. Rep. **4**, 6170 (2015).
[30] G. Wang, D. Chen, J. Lin, and X. Chen, IEEE Trans. Indust. Electron. **46**, 440 (1999).
[31] R. Almog, S. Zaitsev, O. Shtempluck, and E. Buks, Phys. Rev. Lett. **98**, 078103 (2007).
[32] A. H. Nayfeh and D. T. Mook, *Nonlinear Oscillations* (Wiley, New York, 1979).
[33] W. Chen, G. de Loubens, J.-M. L. Beaujour, J. Z. Sun, and A. D. Kent, Appl. Phys. Lett. **95**, 172513 (2009).
[34] M. A. Silaev, Phys. Rev. B **102**, 144521 (2020).
[35] Y. Yao, Q. Song, Y. Takamura, J. P. Cascales, W. Yuan, Y. Ma, Y. Yun, X. C. Xie, J. S. Moodera, and W. Han, Phys. Rev. B **97**, 224414 (2018).
[36] Y. Zhao, Q. Song, S.-H. Yang, T. Su, W. Yuan, S. S. P. Parkin, J. Shi, and W. Han, Sci. Rep. **6**, 22890 (2016).

[37] Y. M. Shukrinov and I. R. Rahmonov, Phys. Part. Nuclei **51**, 816 (2020).

[38] C. Guarcello and F. S. Bergeret, Phys. Rev. Appl. **13**, 034012 (2020).

[39] Y. M. Shukrinov, I. R. Rahmonov, and A. E. Botha., Low Temp. Phys. **46**, 932 (2020).

[40] I. Kovacic and J. B. Michael, *The Duffing Equation : Nonlinear Oscillators and their Behaviour* (John Wiley & Sons, Hoboken, NJ, 2011).

[41] A. Manchon, H. C. Koo, J. Nitta, S. M. Frolov, and R. A. Duine, Nat. Mater. **4**, 871 (2015).

[42] M. A. W. Schoen, D. Thonig, M. L. Schneider, T. J. Silva, H. T. Nembach, O. Eriksson, O. Karis, and J. M. Shaw, Nat. Phys. **12**, 842 (2016).

[43] O. A. Kelly, A. Anane, R. Bernard, J. B. Youssef, C. Hahn, A. H. Molpeceres, C. Carrétéro, E. Jacquet, C. Deranlot, P. Bortolotti, R. Lebourgeois, J.-C. Mage, G. de Loubens, O. Klein, V. Cros, and A. Fert, Appl. Phys. Lett. **103**, 082408 (2013).

[44] M. C. Onbasli, A. Kehlberger, D. H. Kim, G. Jakob, M. Kläui, A. V. Chumak, B. Hillebrands, and C. A. Ross, APL Mater. **2**, 106102 (2014).

[45] A. Hrabec, F. J. T. Goncalves, C. S. Spencer, E. Arenholz, A. T. N'Diaye, R. L. Stamps, and C. H. Marrows, Phys. Rev. B **93**, 014432 (2016).

[46] A. Y. Rusanov, M. Hesselberth, J. Aarts, and A. I. Buzdin, Phys. Rev. Lett. **93**, 057002 (2004).

[47] J. W. A. Robinson, F. Chiodi, M. Egilmez, G. B. Halász, and M. G. Blamire, Sci. Rep. **2**, 699 (2012).

[48] Y. Yao, R. Cai, T. Yu, Y. Ma, W. Xing, Y. Ji, X. C. Xie, S.-H. Yang, and W. Han, Sci. Adv. **7**, eabh3686 (2021).