

Low-energy $6D$, $\mathcal{N} = (1, 1)$ SYM effective action beyond the leading approximation

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Received 11 December 2019; received in revised form 25 February 2020; accepted 15 March 2020

Available online 18 March 2020

Editor: Stephan Stieberger

Abstract

For $6D$, $\mathcal{N} = (1, 1)$ SYM theory formulated in $\mathcal{N} = (1, 0)$ harmonic superspace as a theory of interacting gauge multiplet and hypermultiplet we construct the $\mathcal{N} = (1, 1)$ supersymmetric Heisenberg-Euler-type superfield effective action. The effective action is computed for the slowly varying on-shell background fields and involves, in the bosonic sector, all powers of a constant abelian strength.

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1. Introduction

The low-energy effective action in quantum field theory is a functional of the slowly varying strengths of the vector gauge fields and the matter fields (see e.g., [1]). It provides quantum corrections to the “microscopic” action of given model. The effective action can serve as a bridge between superstring theory and supersymmetric gauge theory. On the one hand, some kind of effective action can be evaluated in string theory; on the other hand, it can be calculated in the framework of the field theory. As a result, there emerges a principal possibility to study the low-

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<https://doi.org/10.1016/j.nucphysb.2020.114995>

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energy effects in string theory by field theory methods (see [2] for review of the effective action in string theory).

The first example of effective action for a quantum field in a constant external electromagnetic field was constructed by Heisenberg and Euler in the pioneer paper [3]. Later, it was reformulated in a covariant way by Schwinger [4]. The Heisenberg-Euler effective action provides the quantum corrections to the Maxwell equations involving all powers of the field strength. The remarkable feature of the Heisenberg-Euler effective action is its non-perturbative dependence on the coupling constant. Later on, the Heisenberg-Euler effective action was computed in one- and two-loop approximations in different field theory models and employed for studying their various properties, such as quantum corrections to the classical equations of motion, particle creation in external fields, finding the low-energy amplitudes, *etc* (see, e.g., [5] for a general review, and [6], [7] for a review of some supersymmetric applications).

In our previous work [8] we have computed the leading low-energy contribution to the one-loop effective action of the six-dimensional $\mathcal{N} = (1, 1)$ SYM theory in the harmonic superspace approach.¹ The contributions to effective action we have found include, in the bosonic sector, the leading terms of the fourth order in the abelian gauge field strength F_{MN} . Such an effective action is in a correspondence with the massless gluon amplitudes in $6D$, $\mathcal{N} = (1, 1)$ SYM theory and is related to the tree-level amplitudes of the massless string modes in the double-scaled little string theory [10], [11] (see [12], [13] for a review of little strings).²

The present paper is a natural continuation of [8]. We calculate the total superfield Heisenberg-Euler-type effective action for $\mathcal{N} = (1, 1)$ SYM theory. This effective action can hopefully be relevant to the little string theory and admit an equivalent formulation within its context.

Like in our previous publications, we deal with the six-dimensional $\mathcal{N} = (1, 1)$ supersymmetric gauge theory formulated in $\mathcal{N} = (1, 0)$ harmonic superspace [17], [18], [19] (for the harmonic superspace approach, see [20], [21]). The theory is quantized in the framework of the harmonic superfield background method which was originally developed for $4D$, $\mathcal{N} = 2$ SYM theories in [22], [23], [24]³ and then generalized to $6D$, $\mathcal{N} = (1, 0)$ gauge theories in [8], [25], [26], [27].

Note that $6D$, $\mathcal{N} = (1, 1)$ SYM theory is non-renormalizable by power counting. However, it was found that this theory is on-shell finite at one and two loops [28], [29], [30], [31], [32], [33], [34], [35]. Recently, the aspects of renormalizability of the theory under consideration were studied by harmonic superspace techniques. It was shown that there is a gauge choice at which the one-loop divergences are completely canceled off shell [25], [26], [27]. Some two-loop harmonic supergraphs are also finite [36].⁴ These results guarantee that the one-loop effective action for the background fields satisfying the classical equations of motions is finite.

The paper is organized as follows. In Section 2 we recall the formulation of the six-dimensional $\mathcal{N} = (1, 1)$ SYM theory in terms of interacting $\mathcal{N} = (1, 0)$ gauge multiplet and hypermultiplet. Assuming that the hypermultiplet is in the adjoint representation of gauge group, we gain an additional implicit $\mathcal{N} = (0, 1)$ supersymmetry and, as a result, obtain the complete $\mathcal{N} = (1, 1)$ supersymmetric gauge theory. Then, in Section 3, we formulate the one-loop effective action of the theory in the framework of the superfield background method. The effective

¹ These results were recently confirmed by the component calculations in [9].

² The relationships of the $6D$, $\mathcal{N} = (1, 1)$ SYM theory with the low-energy dynamics of D5 branes are discussed in [14], [15], [16].

³ Review of various applications of the background harmonic superfields for studying the effective actions of $4D$, $\mathcal{N} = 2, 4$ SYM theories was given in [6], [7].

⁴ Gauge dependence of the one-loop divergences was studied in [37].

action constructed in this way depends on all fields of $\mathcal{N} = (1, 1)$ gauge multiplet. We restrict our consideration to the slowly varying background superfields which are solution of the classical equations of motion, since such an approximation suffices for finding out the low-energy effective action. Section 4 is devoted to deriving the complete one-loop Heisenberg-Euler-type superfield effective action. We follow the $6D, \mathcal{N} = (1, 0)$ harmonic superspace version of the procedure developed in [38], [39], construct the superfield heat kernel for the Green function of the gauge multiplet and find the explicit expression for the Green function in the coincident-points limit. This expression directly yields the low-energy effective action. Section 5 contains a brief summary of our results and a list of possible further studies. In Appendix we give the definition and outline the basic properties of the parallel displacement operator [39] in the harmonic superspace. This operator is of need while constructing the heat kernel for the Green function of gauge multiplet.

2. The model and conventions

We formulate the $6D, \mathcal{N} = (1, 1)$ SYM theory in terms of the gauge multiplet V^{++} and hypermultiplet q_A^+ .⁵ Both these harmonic superfields satisfy the Grassmann analyticity conditions $D_a^+ V^{++} = 0$ and $D_a^+ q_A^+ = 0$, where the spinor derivative in the analytic basis [21] reads $D_a^+ = \frac{\partial}{\partial \theta^{-a}}$. The superfield action of $\mathcal{N} = (1, 1)$ SYM theory is written as a sum of the actions for the gauge multiplet and for the hypermultiplet

$$S_0[V^{++}, q^+] = \frac{1}{f^2} \left\{ \sum_{n=2}^{\infty} \frac{(-i)^n}{n} \text{tr} \int d^{14}z du_1 \dots du_n \frac{V^{++}(z, u_1) \dots V^{++}(z, u_n)}{(u_1^+ u_2^+) \dots (u_n^+ u_1^+)} - \frac{1}{2} \text{tr} \int d\zeta^{(-4)} q^{+A} \nabla^{++} q_A^+ \right\}, \quad (2.1)$$

where f is a dimensionful coupling constant ($[f] = -1$). We include the integration over harmonics into the integration measure over the analytic subspace, $d\zeta^{(-4)} = d^6x_{(\text{an})} du (D^-)^4$. In the action (2.1) the hypermultiplet is minimally coupled with the gauge multiplet by means of the covariant harmonic derivative ∇^{++} ,

$$\nabla^{++} q_A^+ = D^{++} q_A^+ + i[V^{++}, q_A^+]. \quad (2.2)$$

The action (2.1) is invariant under the infinitesimal gauge transformations

$$\delta V^{++} = -\nabla^{++} \Lambda, \quad \delta q_A^+ = i[\Lambda, q_A^+], \quad (2.3)$$

where $\Lambda(\zeta, u) = \tilde{\Lambda}(\zeta, u)$ is a real analytic gauge parameter.

We also introduce the non-analytic superfield V^{--} as a solution of the zero curvature condition [21]

$$D^{++} V^{--} - D^{--} V^{++} + i[V^{++}, V^{--}] = 0, \quad (2.4)$$

and define one more covariant harmonic derivative $\nabla^{--} = D^{--} + iV^{--}$. Using the superfield V^{--} we construct the $\mathcal{N} = (1, 0)$ gauge superfield strength

⁵ The Pauli-Gürsey indices [21] are raised and lowered by ϵ^{AB} : $\widetilde{q}_A^+ \equiv q^{+A} = \epsilon^{AB} q_B$, $A, B = 1, 2$ and $\epsilon^{21} = -\epsilon^{12} = 1$.

$$W^{++} = -\frac{i}{6}\varepsilon^{abcd}D_b^+D_c^+D_d^+V^{--} \quad (2.5)$$

with the useful off-shell properties

$$\nabla^{++}W^{++} = \nabla^{--}W^{--} = 0, \quad W^{--} = \nabla^{--}W^{++}. \quad (2.6)$$

Let us introduce an analytic superfield F^{++} ,

$$F^{++} = (D^+)^4V^{--}, \quad D_a^+F^{++} = \nabla^{++}F^{++} = 0, \quad (2.7)$$

and evaluate the classical equations of motion corresponding to the action (2.1)

$$F^{++} + \frac{1}{2}[q^{+A}, q_A^+] = 0, \quad \nabla^{++}q_A^+ = 0. \quad (2.8)$$

We assume that both V^{++} and q^{+A} superfields take values in the adjoint representation of the gauge group. Hence the action (2.1) possesses the extra implicit $\mathcal{N} = (0, 1)$ supersymmetry,

$$\delta_0 V^{++} = \epsilon^{+A} q_A^+, \quad \delta_0 q^{+A} = -i(D^+)^4(\epsilon_A^- V^{--}), \quad \epsilon_A^\pm = \epsilon_{aA}\theta^{\pm a}, \quad (2.9)$$

which completes the manifest $\mathcal{N} = (1, 0)$ supersymmetry to $\mathcal{N} = (1, 1)$. It is convenient to use the following representation for the variation $\delta_0 q^{+A}$

$$\delta_0 q_A^+ = -\epsilon_{aA}(\theta^{-a}F^{++} - W^{+a}), \quad (2.10)$$

which is expressed through the superfield strengths F^{++} and W_a^+ .

3. One-loop effective action in the background superfield method

Let us apply the background superfield method to the six-dimensional SYM theory.⁶ Following to the method we split the superfields V^{++} , q^+ into the sum of the “background” superfields V^{++} , Q^+ and the “quantum” ones v^{++} , q^+ ,

$$V^{++} \rightarrow \mathbf{V}^{++} + \mathbf{f}v^{++}, \quad q_A^+ \rightarrow \mathbf{Q}_A^+ + \mathbf{f}q_A^+. \quad (3.1)$$

Then we have to expand the action in a power series with respect to the quantum fields. The one-loop quantum correction $\Gamma^{(1)}$ to the classical action for the model (2.1) is given by

$$e^{i\Gamma^{(1)}[\mathbf{V}^{++}, \mathbf{Q}^+]} = \text{Det}^{1/2} \widehat{\square} \int \mathcal{D}v \mathcal{D}q \mathcal{D}\mathbf{b} \mathcal{D}\mathbf{c} \mathcal{D}\varphi \ e^{iS_2[v^{++}, q^+, \mathbf{b}, \mathbf{c}, \varphi, \mathbf{V}^{++}, \mathbf{Q}^+]}, \quad (3.2)$$

where

$$S_2 = S_{gh} - \frac{1}{2}\text{tr} \int d\zeta^{(-4)} v^{++} \widehat{\square} v^{++} - \frac{1}{2}\text{tr} \int d\zeta^{(-4)} q^{+A} \nabla^{++} q_A^+ - \frac{i}{2}\text{tr} \int d\zeta^{(-4)} \left\{ \mathbf{Q}^{+A}[v^{++}, q_A^+] + q^{+A}[v^{++}, \mathbf{Q}_A^+] \right\}, \quad (3.3)$$

$$S_{gh} = \text{tr} \int d\zeta^{(-4)} \mathbf{b}(\nabla^{++})^2 \mathbf{c} + \frac{1}{2}\text{tr} \int d\zeta^{(-4)} \varphi(\nabla^{++})^2 \varphi. \quad (3.4)$$

The action for ghosts superfields S_{gh} (3.4) involves the actions for the Faddeev-Popov ghosts \mathbf{b} and \mathbf{c} and also for the Nielsen-Kallosh ghost φ . The covariantly-analytic d'Alembertian $\widehat{\square}$ is

⁶ The background superfield method for $4D, \mathcal{N} = 2$ gauge theories in harmonic superspace was worked out in [22] and generalized for six-dimensional gauge theory in $\mathcal{N} = (1, 0)$ harmonic superspace in the works [25], [26], [27].

defined as $\widehat{\square} = \frac{1}{2}(D^+)^4(\nabla^{--})^2$, where the harmonic covariant derivative $\nabla^{--} = D^{--} + i\mathbf{V}^{--}$ contains the background superfield \mathbf{V}^{--} . While acting on an analytic superfield, the operator $\widehat{\square}$ is given by

$$\widehat{\square} = \eta^{MN} \nabla_M \nabla_N + \mathbf{W}^{+a} \nabla_a^- + \mathbf{F}^{++} \nabla^{--} - \frac{1}{2}(\nabla^{--} \mathbf{F}^{++}), \quad (3.5)$$

where $\eta^{MN} = \text{diag}(1, -1, -1, -1, -1, -1)$ is the six-dimensional Minkowski metric, $M, N = 0, \dots, 5$, and $\nabla_M = \partial_M + i\mathbf{A}_M$ is the background-dependent vector supercovariant derivative (see [19] for details).

The action S_2 (3.3) contains terms with mixed quantum superfields v^{++} and q^+ . For further use, we diagonalize this quadratic form by means of the special substitution of the quantum hypermultiplet variables in the path integral (3.2), such that it removes the mixed terms,

$$q_A^+(1) = h_A^+(1) - i \int d\zeta_2^{(-4)} G^{(1,1)}(1|2)_A{}^B [v^{++}(2), Q_B^+(2)], \quad (3.6)$$

with h_n^+ being a set of new independent quantum superfields. It is evident that the Jacobian of the variable change (3.6) is equal to one. Here $G^{(1,1)}(\zeta_1, u_1 | \zeta_2, u_2)_A{}^B = i \langle 0 | T q_A^+(\zeta_1, u_1) q^{+B}(\zeta_2, u_2) | 0 \rangle$ is the superfield hypermultiplet Green function in the τ -frame. This Green function is analytic with respect to its both arguments and satisfies the equation

$$\nabla_1^{++} G^{(1,1)}(1|2)_A{}^B = \delta_A{}^B \delta_A^{(3,1)}(1|2). \quad (3.7)$$

In the τ -frame the Green function can be written in the form $G^{(1,1)}(1|2)_A{}^B = \delta_A{}^B G^{(1,1)}(1|2)$, where

$$G^{(1,1)}(1|2) = \frac{(\nabla_1^+)^4 (\nabla_2^+)^4 \delta^{14}(z_1 - z_2)}{\widehat{\square}_1 (u_1^+ u_2^+)^3}. \quad (3.8)$$

Here $\delta_A^{(3,1)}(1|2)$ is the covariantly-analytic delta-function.

After performing the shift (3.6), the quadratic part of the action S_2 (3.3) splits into few terms, each being bilinear in quantum superfields:

$$S_2 = S_2^v - \text{tr} \int d\zeta^{(-4)} h^{+A} \nabla^{++} h_A^+ + \text{tr} \int d\zeta^{(-4)} \mathbf{b}(\nabla^{++})^2 \mathbf{c} + \frac{1}{2} \text{tr} \int d\zeta^{(-4)} \varphi (\nabla^{++})^2 \varphi \quad (3.9)$$

$$S_2^v = \frac{1}{2} \text{tr} \int d\zeta_1^{(-4)} d\zeta_2^{(-4)} v_1^{++} \left\{ \widehat{\square} \delta_A^{(3,1)}(1|2) - Q^{+A}(1) G^{(1,1)}(1|2) Q_A^+(2) \right\} v_2^{++}. \quad (3.10)$$

In the action (3.2) the background superfields \mathbf{V}^{++} and \mathbf{Q}^+ are analytic but unconstrained otherwise. The gauge group of the theory (2.1) is assumed to be $SU(N)$. For the further consideration, we will also assume that the background fields \mathbf{V}^{++} and \mathbf{Q}^+ align in a fixed direction in the Cartan subalgebra of $su(N)$

$$\mathbf{V}^{++} = V^{++}(\zeta, u) H, \quad \mathbf{Q}^+ = Q^+(\zeta, u) H, \quad (3.11)$$

where H is a fixed generator in the Cartan subalgebra generating some abelian subgroup $U(1)$.⁷ Our choice of the background corresponds to the spontaneous symmetry breaking $SU(N) \rightarrow SU(N-1) \times U(1)$.

The classical equations of motions (2.8) for the background superfields V^{++} and Ω are free

$$F^{++} = 0, \quad D^{++} Q_A^+ = 0. \quad (3.12)$$

In that follows we assume that the background superfields solve the classical equation of motion (3.12). We will also assume that the background is slowly varying in space-time, *i.e.*,

$$\partial_M W^{+a} \simeq 0, \quad \partial_M Q_A^+ \simeq 0. \quad (3.13)$$

Thus we end up with abelian background analytic superfields V^{++} and Q_A^+ , which satisfy the classical equation of motion (3.12) and the conditions (3.13). Under these assertions the gauge superfield strength W^{+a} is analytic,⁸ $D_a^+ W^{+b} = \delta_a^b F^{++} = 0$. For further analysis it is convenient to use the $\mathcal{N} = (0, 1)$ transformation for gauge superfield strength W^{+a} [19]. In the case of the slowly varying abelian on-shell background superfields the hidden $\mathcal{N} = (0, 1)$ supersymmetry transformations have a simple form,

$$\delta Q_A^+ = \epsilon_{aA} W^{+a} \quad \delta W^{+a} = 0. \quad (3.14)$$

We choose the Cartan-Weyl basis for the $SU(N)$ gauge group generators, so that the quantum superfield v^{++} has the decomposition

$$v^{++} = v_i^{++} H_i + v_\alpha^{++} E_\alpha, \quad i = 1, \dots, N-1, \quad \alpha = 1, \dots, N(N-1), \quad (3.15)$$

where E_α is the generator corresponding to the root α normalized as $\text{tr}(E_\alpha E_{-\beta}) = \delta_{\alpha\beta}$ and H_i are the Cartan subalgebra generators, $[H_i, E_\alpha] = \alpha_{H_i} E_\alpha$. In this case the background covariant d'Alembertian (3.5) under the conditions (3.12) acts on the quantum superfield v^{++} as

$$\begin{aligned} \widehat{\square} v^{++} = & \frac{1}{2} (D^+)^4 \left\{ (D^{--})^2 v^{++} + i \alpha_H D^{--} V^{--} v_\alpha^{++} E_\alpha \right. \\ & \left. + i \alpha_H V^{--} D^{--} v_\alpha^{++} E_\alpha - \alpha_H^2 (V^{--})^2 v_\alpha^{++} E_\alpha \right\} \end{aligned} \quad (3.16)$$

$$= \widehat{\square}_H v_\alpha^{++} E_\alpha + \partial_M \partial^M v_i^{++} H_i, \quad (3.17)$$

where we have introduced the operator

$$\widehat{\square}_H := \nabla^{ab} \nabla_{ab} + \alpha_H W^{+a} D_a^-. \quad (3.18)$$

The one-loop effective action (3.2) with the action S_2 (3.9) for the background superfields V^{++} and Q^+ subjected to the conditions (3.12) and (3.13) thus reads

$$\Gamma^{(1)} = \frac{i}{2} \text{Tr}_{(2,2)} \ln \left(\widehat{\square}_H - \alpha_H^2 Q^{+A} G^{(1,1)} Q_A^+ \right) - \frac{i}{2} \text{Tr}_{(4,0)} \ln \widehat{\square}_H. \quad (3.19)$$

The first term in the expression (3.19) is the contribution from the gauge multiplet (3.9), while the second one comes from $\text{Det}^{1/2} \widehat{\square}$ in (3.2). The contributions from the Faddeev-Popov and Nielsen-Kallosh ghosts are canceled by the contribution from quantum hypermultiplet.

⁷ We denote the H component of V^{++} by the same letter V^{++} as the original non-abelian harmonic connection, with the hope that this will not create a misunderstanding. The same concerns the abelian superfield strength W^{+a} .

⁸ In general this is not true and $F^{++} \neq 0$.

We use the standard definition for the functional trace over harmonic superspace in (3.19)

$$\mathrm{Tr}_{(q,4-q)} \mathcal{O} = \mathrm{tr} \int d\zeta_1^{(-4)} d\zeta_2^{(-4)} \delta_{\mathcal{A}}^{(q,4-q)}(1|2) \mathcal{O}^{(q,4-q)}(1|2).$$

Here $\delta_{\mathcal{A}}^{(q,4-q)}(1|2)$ is an analytic delta-function [21] and $\mathcal{O}^{(q,4-q)}(\zeta_1, u_1 | \zeta_2, u_2)$ is the kernel of an operator acting in the space of analytic superfields with the harmonic U(1) charge q .

As the next step, we rewrite the contribution from $\mathrm{Det}^{1/2} \widehat{\square}$ as the functional integral over a zero-charge analytic superfield σ with the action

$$-\frac{1}{2} \mathrm{tr} \int d\zeta^{(-4)} \sigma (\nabla_H^{++})^2 \widehat{\square}_H \sigma, \quad (3.20)$$

where $\nabla_H^{++} = D^{++} + i\alpha_H V^{++}$. Then we divide the superfield v^{++} into the two orthogonal projections (see the reviews [6], [7])

$$v^{++} = v_T^{++} + \nabla_H^{++} \xi, \quad \nabla_H^{++} v_T^{++} = 0. \quad (3.21)$$

The transversal component v_T^{++} of the superfield v^{++} is defined as

$$v_T^{++}(1) = \int d\zeta_2^{(-4)} \Pi_T^{(2,2)}(1|2) v_2^{++}, \quad (3.22)$$

where $\Pi_T^{(2,2)}(\zeta_1, u_1; \zeta_2, u_2)$ is the projector on the space of covariantly analytic transverse superfields. After substitution of v^{++} (3.21) in the quadratic part of the action for the gauge multiplet (3.10) we obtain an additional contribution from the bosonic superfield ξ ,

$$\frac{1}{2} \mathrm{tr} \int d\zeta^{(-4)} \xi (\nabla_H^{++})^2 \widehat{\square}_H \xi. \quad (3.23)$$

Note that all mixed terms vanish due to the properties $\nabla_H^{++} v_T^{++} = 0$ and $Q^{+A} Q_A^+ = 0$.

The contribution from the superfields ξ and σ cancel each other in the one-loop effective action and finally we obtain

$$\Gamma^{(1)} = \frac{i}{2} \mathrm{Tr}_T \ln \left(\widehat{\square}_H - \alpha_H^2 Q^{+A} G^{(1,1)} Q_A^+ \right), \quad (3.24)$$

where trace is over the space of analytic superfields v_T^{++} constrained by the condition $\nabla_H^{++} v_T^{++} = 0$.

Let us consider the quadratic action which produces the effective action (3.24),

$$S^{(2)} = \frac{1}{2} \mathrm{tr} \int d\zeta_1^{(-4)} d\zeta_2^{(-4)} v_T^{++}(1) \left\{ \widehat{\square}_H \delta_A^{(3,1)}(1|2) - \alpha_H^2 Q^{+A}(1) G^{(1,1)}(1|2) Q_A^+(2) \right\} v_T^{++}(2).$$

First of all we study the non-local term $Q^{+A}(1) G^{(1,1)}(1|2) Q_A^+(2)$ in this expression in the coincident harmonic points ($u_2 \rightarrow u_1$) limit. We rewrite the Green function $G^{(1,1)}$ as follows [40]

$$G^{(1,1)}(1|2) = \frac{(D_1^+)^4}{\widehat{\square}} \left\{ (D_1^-)^4 (u_1^+ u_2^+) - \Omega_1^{--} (u_1^- u_2^+) + \widehat{\square} \frac{(u_1^- u_2^+)^2}{(u_1^+ u_2^+)} \right\} \delta^{14}(z_1 - z_2), \quad (3.25)$$

where $\Omega^{--} = i \nabla^{ab} \nabla_a^- \nabla_b^- - W^{-a} \nabla_a^- + \frac{1}{4} (\nabla_a^- W^{-a})$. According to its definition (3.8), the Green function $G^{(1,1)}(1|2)$ is analytic with respect to its both arguments. The representation (3.25) preserves the analyticity in the second argument, though in some implicit way (see, e.g., [40]).

The third term in (3.25) is singular in the $u_2 \rightarrow u_1$ limit. To avoid the singularity, we expand $Q_A^+(2)$ over harmonics using the property $Q_A^+(2) = (u_1^+ u_2^+) Q_A^-(1) - (u_1^- u_2^+) Q_A^+(1)$ [40] and reconstruct the full integration measure by taking off the $(D_1^+)^4$ factor from Green function in (3.25). We obtain the non-singular expression

$$-\frac{\alpha_H^2}{2} \text{tr} \int d^{14} z_1 d\zeta_2^{(-4)} du_1 v_T^{++}(1) v_T^{++}(2) Q^{+A} Q_A^-(1) (u_1^- u_2^+)^2 \delta^{14}(z_1 - z_2) + \dots, \quad (3.26)$$

where dots stand for the rest of terms coming from the expansion of Green function $G^{(1,1)}(1|2)$ in (3.25). These terms are proportional to $(u_1^+ u_2^+)$ and vanish in the effective action for the on-shell background due to property $(u_1^+ u_2^+)|_{2 \rightarrow 1} = 0$ [21].

The combination $Q^{+A} Q_A^-$ is a gauge invariant real superfield. However, the superfield $Q^{+A} Q_A^-$ is not analytic and only the full expression (3.26) preserves the analyticity. For further consideration it will be convenient to replace the background hypermultiplet Q_A^+ by the analytic omega-hypermultiplet Ω , using the correspondence [21]

$$Q_A^+ = u_A^+ \Omega - u_A^- D^{++} \Omega. \quad (3.27)$$

The $\mathcal{N} = (0, 1)$ supersymmetry transformation of Q_A^+ defined in (3.14) implies the following transformation law for the superfield Ω :

$$\delta \Omega = \epsilon_a^- W^{+a}, \quad \delta(D^{++} \Omega) = \epsilon_a^+ W^{+a}, \quad \delta W^{+a} = 0. \quad (3.28)$$

The on-shell condition (3.12) for Q_A^+ and the definition (3.27) give rise to the equation of motion for the Ω hypermultiplet in the form

$$(D^{++})^2 \Omega = 0. \quad (3.29)$$

Now let us discuss the possible structure of effective action (3.24) after passing from the background Q_A^+ hypermultiplet to the Ω hypermultiplet by eq. (3.27). We assume that the background superfields satisfy the classical equations of motion and slowly vary in space-time. As shown in [8], the hidden $\mathcal{N} = (0, 1)$ supersymmetry severely restricts the possible structure of the effective action (3.24). We consider the analytic contributions to the effective action which respect the implicit $\mathcal{N} = (0, 1)$ supersymmetry (3.28) and are local in harmonic superspace. Thus for $\Gamma^{(1)}$ we should have the following general expression:

$$\Gamma^{(1)} = \int d\zeta^{(-4)} (W^+)^4 \mathcal{F}(\Omega, D_a^- W^{+b}), \quad (3.30)$$

where $\mathcal{F}(\Omega, D_a^- W^{+b})$ is a real analytic function with zero harmonic $U(1)$ charge. Here we have to emphasize that within our approximation the function \mathcal{F} can depend only on the background superfield Ω and $D_a^- W^{+b}$. Indeed, including, e.g., the contributions with harmonic derivative D^{++} of superfield Ω will amount to the necessity to compensate the extra harmonic charge $+2$. One can accomplish this, acting on $D^{++} \Omega$ by the spinor derivatives with negative charge, *i.e.* by passing to $D_a^- D_b^- D^{++} \Omega$. Moreover, such contributions are analytic in the constant background approximation which we use. But the covariant d'Alembertian (3.18) includes the operator D_a^- multiplied by the background superfield strength W^{+a} . Thus all contributions of the kind $D_a^- D_b^- D^{++} \Omega$ have to contain W^{+a} and so they immediately vanish due to the presence of the maximal power of $(W^+)^4$ in the integrand of (3.30). Also we exclude from the consideration all contributions containing $D^{--} D^{++} \Omega$. Such terms are not analytical and do not contribute to

the effective action. So in that follows we take into account only the contributions having no harmonic derivatives of the background superfield Ω .

Keeping in mind this discussion, we rewrite the one-loop effective action (3.24), applying the proper-time method

$$\Gamma^{(1)} = -\frac{i}{2} \text{tr} \int d\xi_1^{(-4)} \int_0^\infty \frac{ds}{s} e^{is(\widehat{\square}_1 - \alpha_H^2 \Omega^2)} \Pi_T^{(2,2)}(1|2) \Big|_{2=1}. \quad (3.31)$$

The covariant analytic projector $\Pi_T^{(2,2)}(1|2)$ in the limit $u_2 \rightarrow u_1$ has the simple form [8,40]

$$\Pi_T^{(2,2)}(1|2) \Big|_{u_2=u_1} = -(D_1^+)^4 \delta^{14}(z_1 - z_2). \quad (3.32)$$

Also, in order to avoid the dependence of the effective action on the root α_H , we have to calculate the trace over matrix indices. We will consider the simplest case, when the gauge group of the theory is $SU(2)$. We obtain the following final expression for the one-loop effective action

$$\Gamma^{(1)} = i \int d\xi_1^{(-4)} \int_0^\infty \frac{ds}{s} e^{is(\widehat{\square}_1 - \Omega^2)} (D_1^+)^4 \delta^{14}(z_1 - z_2) \Big|_{2=1}. \quad (3.33)$$

The expression (3.33) is the central object of our further consideration. In the next section we will calculate it under the simplifying assumptions on the background superfields formulated earlier.

4. Complete contribution to one-loop effective action

To find the complete low-energy effective Lagrangian we should calculate (3.33). We use the covariantly constant on-shell gauge and omega-hypermultiplet background superfields subject to the constraints (3.12) and (3.29). We also introduce the notation

$$D_a^- W^{+b} = -D_a^+ W^{-b} = N_a^b, \quad (4.1)$$

where the superfield N_a^b is related to the gauge field strength $F_a^b = i(\sigma^{MN})_a^b F_{MN}$ as

$$F_a^b = D_a^- W^{+b} - D_a^+ W^{-b} = 2N_a^b. \quad (4.2)$$

We use the following definition for the generator of spinor representation $(\sigma^{MN})_a^b$

$$(\sigma_{MN})^a_b = \frac{1}{2} (\tilde{\gamma}^M \gamma^N - \tilde{\gamma}^N \gamma^M)^a_b, \quad (4.3)$$

where the antisymmetric six-dimensional $(\gamma^M)_{ab}$ and $(\tilde{\gamma}^M)^{ab}$ matrices are related as

$$(\tilde{\gamma}_M)^{ab} = \frac{1}{2} \varepsilon^{abcd} (\gamma_M)_{cd}, \quad (4.4)$$

and ε^{abcd} is the totally skew-symmetric 4th rank tensor. The matrices γ_M and $\tilde{\gamma}_M$ are subject to the basic relations for Weyl matrices

$$(\gamma_M)_{ac} (\tilde{\gamma}_N)^{cb} + (\gamma_N)_{ac} (\tilde{\gamma}_M)^{cb} = -2\delta_a^b \eta_{MN}, \quad (\gamma^M)_{ac} (\gamma_M)_{cb} = 2\varepsilon_{abcd}. \quad (4.5)$$

As before, we choose the Minkowski metric η_{MN} , $M, N = 0, \dots, 5$, with the mostly negative signature (see its definition after eq. (3.5)).

Then, as in the $4D, \mathcal{N} = 2$ case [40], we introduce the operator Δ ,

$$\Delta = \widehat{\square} - W^{-\alpha} D_{\alpha}^{+}, \quad (4.6)$$

which coincides with $\widehat{\square} = \nabla^{ab} \nabla_{ab} + W^{+a} D_a^{-}$ on the space of covariantly analytic superfields.⁹ Thus the expression (3.33) takes the form

$$\Gamma^{(1)} = i \int d\zeta_1^{(-4)} \int_0^{\infty} \frac{ds}{s} e^{is(\Delta_1 - \Omega^2)} (D_1^{+})^4 \delta^{14}(z_1 - z_2) \Big|_{2=1}. \quad (4.7)$$

Note that the spinor derivative D_a^{-} can act on the superfield W^{-a} in the operator Δ . However, the operator $\Delta - \Omega^2$ standing in the exponential does not commute with $(D^{+})^4$ even in the case of constant on-shell background. Thus, pulling the exponential with the argument $\Delta - \Omega^2$ through $(D^{+})^4$, we obtain

$$\Gamma^{(1)} = i \int d\zeta_1^{(-4)} \int_0^{\infty} \frac{ds}{s} (e^{-isN} D_1^{+})^4 e^{is(\Delta_1 - \Omega^2)} \delta^{14}(z_1 - z_2) \Big|_{2=1}. \quad (4.8)$$

Let us introduce the heat kernel for the operator $\Delta - \Omega^2$,

$$K(z_1, z_2|s) = e^{is(\Delta_1 - \Omega^2)} \delta^{14}(z_1 - z_2), \quad (4.9)$$

as a formal solution of the equation

$$\left(i \frac{d}{ds} + \Delta_1 - \Omega^2\right) K(z_1, z_2|s) = \delta^{14}(z_1 - z_2). \quad (4.10)$$

In terms of the kernel $K(z_1, z_2|s)$ the one-loop effective action (4.8) can be rewritten as

$$\Gamma^{(1)} = i \int d\zeta_1^{(-4)} \int_0^{\infty} \frac{ds}{s} (e^{-isN} D_1^{+})^4 K(z_1, z_2|s) \Big|_{2=1}. \quad (4.11)$$

We denote by Υ the first-order operator appearing in Δ , *i.e.* write the latter as follows

$$\Delta = \nabla^{ab} \nabla_{ab} + \Upsilon, \quad \Upsilon := W^{+a} D_a^{-} - W^{-a} D_a^{+}. \quad (4.12)$$

We provide the calculation in the case of a covariantly constant vector multiplet (3.13). The vector covariant derivative ∇_{ab} turns out to commute with the operator Υ , as well as with the additional term Ω^2 . This allows us to represent $e^{is(\Delta - \Omega^2)}$ in the factorized form $e^{is(\Upsilon - \Omega^2)} e^{is \nabla^{ab} \nabla_{ab}}$ and to calculate the heat kernel $K(z_1, z_2|s)$

$$K(z_1, z_2|s) = e^{is(\Upsilon - \Omega^2)} e^{is \nabla^{ab} \nabla_{ab}} \delta^{14}(z_1 - z_2) = e^{is(\Upsilon - \Omega^2)} \tilde{K}(z_1, z_2|s). \quad (4.13)$$

The further steps in calculation of (4.11) are similar to those performed in [41]. We use the momentum representation of the delta function, $\delta^{14}(z_1 - z_2) = \delta^6(x_1 - x_2) \delta^4(\theta_1^{+} - \theta_2^{+}) \delta^4(\theta_1^{-} - \theta_2^{-})$,

⁹ Note that in $6D, \mathcal{N} = (1, 0)$ hypermultiplet theory, the operator (4.6) differs from the analogical operator in $4D, \mathcal{N} = 2$ hypermultiplet theory [40].

$$\mathbf{1}\delta^{(14)}(z_1 - z_2) = \int \frac{d^6 p}{(2\pi)^6} e^{i\rho^M p_M} \zeta^{+4} \zeta^{-4} I(z_1, z_2), \quad (4.14)$$

where $I(z_1, z_2)$ is a parallel displacement operator in superspace [38,39] (see details in Appendix) and

$$\rho^M = (x_1 - x_2)^M - 2i\zeta^{+a}(\gamma^M)_{ab}\theta_1^{-b}, \quad \zeta^{\pm a} = (\theta_1^{\pm} - \theta_2^{\pm})^a. \quad (4.15)$$

The reduced heat kernel $\tilde{K}(z_1, z_2|s)$ can now be evaluated in the same way by generalizing the Schwinger construction [38],

$$\tilde{K}(z_1, z_2|s) = \frac{i}{(4\pi is)^3} \det^{\frac{1}{2}} \left(\frac{sF}{\sinh sF} \right) e^{\frac{i}{4}\rho^M (F \coth sF)_{MN} \rho^N} \zeta^{+4} \zeta^{-4} I(z_1, z_2), \quad (4.16)$$

where the determinant is taken with respect to Lorentz indices. To compute the kernel $K(z_1, z_2|s)$ we need to evaluate the action of $e^{is\Upsilon}$ on $\tilde{K}(z_1, z_2|s)$. However, the operator Υ does not commute with Ω^2 even on shell. To separate its contribution in $\exp(is(\Upsilon - \Omega^2))$, we use the Baker-Campbell-Hausdorff formula

$$e^{is(\Upsilon - \Omega^2)} = e^{(-is\Omega^2 + \frac{(is)^2}{2}[\Upsilon, \Omega^2] - \frac{(is)^3}{3!}[\Upsilon, [\Upsilon, \Omega^2]] + \dots)} e^{is\Upsilon}. \quad (4.17)$$

Using the explicit expression for the commutator $[\Upsilon, \Omega^2] = W^{+a}(D_a^- \Omega^2)$, one can show that the series in eq. (4.17) can be summed up to the concise expression

$$e^{is(\Upsilon - \Omega^2)} = e^{\left(\frac{\exp(-is \frac{W^+ D^-}{W^+ D^-}) - 1}{W^+ D^-} \right) \Omega^2} e^{is\Upsilon}. \quad (4.18)$$

The complete structure of the last expression is rather complicated but this does not matter. It is crucial for us that it has the form

$$e^{\left(\frac{\exp(-is \frac{W^+ D^-}{W^+ D^-}) - 1}{W^+ D^-} \right) \Omega^2} = e^{-is\Omega^2 + W^{+a} f_a(W^+, N, \Omega^2, s)}, \quad (4.19)$$

where the function $f_a(W^+, N, \Omega^2, s)$ encodes the whole information about the series (4.18).

As the next step, we act by the operator $e^{is\Upsilon}$ on the kernel $\tilde{K}(z_1, z_2|s)$. The formal result reads

$$K(z_1, z_2|s) = \frac{i}{(4\pi is)^3} \det^{\frac{1}{2}} \left(\frac{sF}{\sinh sF} \right) e^{\frac{i}{4}\rho^M(s)(F \coth sF)_{MN} \rho^N(s)} \zeta^{+4}(s) \zeta^{-4}(s) I(z_1, z_2|s), \quad (4.20)$$

where we denoted,

$$\zeta^A(s) = e^{is\Upsilon} \zeta^A e^{-is\Upsilon}, \quad I(z_1, z_2|s) = e^{is\Upsilon} I(z_1, z_2), \quad (4.21)$$

and $\zeta^A = (\rho^a, \zeta^{\pm a})$. Using the formula $e^A B e^{-A} = B + [A, B] + \dots$ and our constraints on the background (4.1), we obtain¹⁰

$$\zeta^{+a}(s) = \zeta^{+a} - W^{+b} \mathcal{N}_b^a, \quad \zeta^{-a}(s) = \zeta^{-a} - W^{-b} \mathcal{N}_b^a, \quad (4.22)$$

$$\rho^M(s) = \rho^M - 2 \int_0^s dt W^{-a}(t) (\gamma^M)_{ab} \zeta^{+b}(t), \quad W^{-a}(s) = W^{-b} (e^{isN})_b^a. \quad (4.23)$$

¹⁰ Here we use $D_a^+ \zeta^{b-} = \delta_a^b$ and $D_a^- \zeta^{b+} = -\delta_a^b$.

Here we made use of the definition (4.1) and $\mathcal{N}_a^b := \left(\frac{e^{isN}-1}{N}\right)_a^b$. We do not need the explicit expression for $I(z_1, z_2|s)$. However, it is easy to check, by differentiating with respect to the proper time s , that the following identity holds

$$I(z_1, z_2|s) = \exp \left[\int_0^s dt \Sigma(z_1, z_2|t) \right] I(z_1, z_2), \quad (4.24)$$

$$\Sigma(z_1, z_2|t) = e^{it\Upsilon} \Sigma(z_1, z_2) e^{-it\Upsilon}, \quad (4.25)$$

where $\Sigma(z_1, z_2)$ is defined by the relation

$$(W^{+\alpha} D_a^- - W^{-\alpha} D_a^+) I(z_1, z_2) = \Sigma(z_1, z_2) I(z_1, z_2). \quad (4.26)$$

For what follows it is important that $\Sigma(z_1, z_2|s) = W^{+a} \rho_{ab} W^{-b} + \dots$ (see (A.7) in the Appendix).

Now we can come back to the calculation of the effective action (4.11). We need to calculate the coincident-points limit for $(e^{-isN} D_1^+)^4 K^{z_1, z_2|s}$. The operator $(e^{-isN} D_1^+)^4$ acts on the two-point function $\zeta^{-4}(s)$ and in the coincident-points limit gives the unity

$$(e^{-isN} D_1^+)^4 \zeta^{-4}(s) \Big|_{2=1} = 1. \quad (4.27)$$

For $\zeta^{+4}(s)$ we have

$$\zeta^{+4}(s) \Big|_{2=1} = (W^+)^4 \det \left(\frac{e^{isN} - 1}{N} \right). \quad (4.28)$$

We observe that in the coincident-points limit all terms with $\rho^M(s)$ and $I(z_1, z_2|s)$ have the formal structure $\exp(W^{+a} + \dots)$. Due to the presence of the maximal power of the gauge superfield strength $(W^+)^4$ in (4.28) we can replace the exponential in such terms just by unity.

As the result, we obtain

$$\Gamma^{(1)} = \frac{1}{(4\pi)^3} \int d\zeta^{(-4)} (W^+)^4 \xi(F, N, \Omega^2), \quad (4.29)$$

$$\xi(F, N, \Omega^2) = \int_0^\infty \frac{ds}{s^4} e^{-s\Omega^2} \det \left(\frac{e^{sN} - 1}{N} \right) \det^{\frac{1}{2}} \left(\frac{sF}{\sin sF} \right). \quad (4.30)$$

This is the final expression for the complete low-energy effective action in the theory under consideration. The effective action (4.29), (4.30) is manifestly gauge invariant and manifestly $\mathcal{N} = (1, 0)$ supersymmetric by construction. The action (4.29) is also invariant under the implicit $\mathcal{N} = (0, 1)$ supersymmetry (3.28). Indeed, according to (3.28), the transformation of Ω^2 is proportional to the superfield strength W^{+a} , $\delta\Omega \sim W^{+a}$. Consequently, all such terms vanish due to the presence of the maximal power of the spinorial superfield $(W^+)^4$ in the integrand of (4.29).

In our previous work [8] we calculated the leading low-energy contribution to the one-loop effective action. It has the form

$$\Gamma_{\text{lead}}^{(1)} = \frac{1}{(4\pi)^3} \int d\zeta^{(-4)} \frac{(W^+)^4}{\Omega^2}. \quad (4.31)$$

The expression (4.31) was obtained under the assumption of the simplest background, $D_a^- W^{+b} = N_a^b = 0$. We see that the leading contribution (4.31) immediately follows from (4.29) when $N = F = 0$. In this case, $\xi(0, 0, \Omega^2) = \frac{1}{\Omega^2}$.

5. Conclusions

In this paper we considered the quantum aspects of the six-dimensional $\mathcal{N} = (1, 1)$ SYM theory. We used the $\mathcal{N} = (1, 0)$ harmonic superspace formulation of the theory in terms of $\mathcal{N} = (1, 0)$ analytic vector gauge multiplet and hypermultiplet. We assumed that both gauge and matter $\mathcal{N} = (1, 0)$ supermultiplets are in the adjoint representation of gauge group. By construction, the theory is invariant under the manifest $\mathcal{N} = (1, 0)$ supersymmetry and the second implicit $\mathcal{N} = (0, 1)$ one.

We calculated the complete one-loop effective action for the considered theory in the framework of the background superfield method in $\mathcal{N} = (1, 0)$ harmonic superspace. We restricted our attention to the special case of the slowly varying background superfields satisfying the free classical equations of motion. We also assumed that background superfields align in the Cartan subalgebra of $su(2)$. The obtained result (4.29) for the effective action is the complete one-loop effective action for the six-dimensional $\mathcal{N} = (1, 1)$ SYM theory in the constant background approximation.

A few comments on the calculation procedure are needed. In six dimensions the gauge superfield strength is the spinor superfield W^{+a} . The general analysis of the structure of the leading low-energy effective action [8] implies that the effective Lagrangian as a function of W^{+a} and the Ω hypermultiplet has to be an analytic superfield of the $U(1)$ harmonic charge $+4$. Namely, $\mathcal{L}^{(+4)} = (W^+)^4 \xi(F, N, \Omega^2)$, where the function ξ was defined in (4.30). It is analytic and contains the whole information about one-loop quantum corrections. We have also to recall that, initially, we formulated the theory in terms of the gauge $\mathcal{N} = (1, 0)$ multiplet and the charge $+1$ q_A^+ -hypermultiplet. But during the calculation we were forced to pass from the background q_A^+ -hypermultiplet to the zero-charge Ω hypermultiplet. It is known that the matter sector of the supersymmetric gauge theories can be equivalently described either by a complex q_A^+ -hypermultiplet or by a real Ω hypermultiplet [21]. The reason for making use of Ω is that it provides a possibility to define the uncharged analytic superfield combination playing the role of the background UV cutoff term in the function ξ (4.30). One can see that the use of the q_A^+ -hypermultiplet does not ensure the analyticity required, because the uncharged combination $q^{+A} q_A^-$ is not analytic.

As the final remark, we emphasize that there are two interesting further directions of applying the background field method used here. One such direction amounts to studying the structure of the effective action in six-dimensional $\mathcal{N} = (1, 0)$ SYM theory with higher derivatives [42], [43], [44], another one concerns deriving the Born-Infeld-type effective action associated with D5-brane. The latter problem will require carrying out the superspace multi-loop calculations (see the relevant discussion in [45] on the Born-Infeld-type action related to D3-brane in the framework of $4D, \mathcal{N} = 4$ SYM theory). Some aspects of the superfield two-loop calculations of the effective action in $4D, \mathcal{N} = 4$ SYM theory have been considered in [38], [45], [46], [47], [48].

Declaration of competing interest

We declare that we have no known competing financial interests or personal relationships. We declare that there are no financial interests/personal relationships which may be considered as potential competing interests.

Acknowledgements

This research was supported in part by RFBR grant, project No. 18-02-01046. I.L.B. and B.S.M. are grateful to RFBR grant, project No. 18-02-00153, for partial support. The work of B.S.M. was supported in part by the Russian Federation President grant, the project MK-1649.2019.2.

Appendix A. Parallel displacement operator

Let us briefly discuss the basic properties of the parallel displacement operator $I(z, z')$. By definition, it is defined as a two-point superspace function depending on the gauge superfields with the following properties [38,39]:

- (i) Under the gauge transformations it transforms as

$$I(z, z') = e^{i\tau(z)} I(z, z') e^{-i\tau(z')}; \quad (\text{A.1})$$

- (ii) It obeys the equation

$$\zeta^A \nabla_A I(z, z') = 0, \quad (\text{A.2})$$

where $\zeta^A = (\rho^M, \rho^{a\pm})$ was defined in (4.15);

- (iii) For the coincident superspace points $z = z'$ it reduces to the identity operator in the gauge group,

$$I(z, z) = 1. \quad (\text{A.3})$$

The general form of the superalgebra of covariant derivatives is as follows

$$[\nabla_A, \nabla_B] = \mathbf{T}_{AB}{}^C \nabla_C + i \mathbf{F}_{AB}, \quad (\text{A.4})$$

where $\mathbf{T}_{AB}{}^C$ is a supertorsion and \mathbf{F}_{AB} is a supercurvature for gauge superfield connections. In [38] it was proved that, owing to (A.2), the action of the derivative ∇_B on $I(z, z')$ can be expressed in terms $\mathbf{T}_{AB}{}^C$, \mathbf{F}_{AB} and their covariant derivatives,

$$\begin{aligned} \nabla_B I(z, z') = i \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} \left[-\zeta^{A_n} \dots \zeta^{A_1} \nabla_{A_1} \dots \nabla_{A_{n-1}} \mathbf{F}_{A_n B}(z) \right. \\ \left. + \frac{(n-1)}{2} \zeta^{A_n} \mathbf{T}_{A_n B}{}^C \zeta^{A_{n-1}} \dots \zeta^{A_1} \nabla_{A_1} \dots \nabla_{A_{n-2}} \mathbf{F}_{A_{n-1} C}(z) \right] I(z, z'). \end{aligned} \quad (\text{A.5})$$

In our case we do not need the detailed analysis of (A.5), and we addressed only the simplest background, $N_a^b = 0$. We have

$$\begin{aligned} D_a^\pm I(z, z') = \left[\frac{1}{2} \rho_{ab} W^{b\pm} - \frac{i}{6} (\gamma^M)_{ab} \zeta^{\pm b} \left(\zeta^{+c} (\gamma_M)_{cd} W^{-d} \right. \right. \\ \left. \left. + \zeta^{-c} (\gamma_M)_{cd} W^{+d} - i \rho^N F_{NM} \right) \right] I(z, z'). \end{aligned} \quad (\text{A.6})$$

Then the superfield $\Sigma(z, z')$ introduced in (4.26) has the form

$$\begin{aligned} \Sigma(z, z') = & W^{+a} \rho_{ab} W^{-b} - \frac{i}{6} (W^{+a} (\gamma^M)_{ab} \zeta^{-b} - W^{-a} (\gamma^M)_{ab} \zeta^{+b}) \\ & \times (\zeta^{+c} (\gamma_M)_{cd} W^{-d} + \zeta^{-c} (\gamma_M)_{cd} W^{+d} - i \rho^N F_{NM}). \end{aligned} \quad (\text{A.7})$$

Thus the decomposition of the superfield $\Sigma(z, z')$ begins with the gauge superfield strength W^{+a} . This is one of the crucial properties used in the computation of the coincident-points limit of the kernel $K(z_1, z_2|s)$.

References

- [1] S. Weinberg, *The Quantum Theory of Fields, Volume II: Modern Applications*, Cambridge Univ. Press, 1996.
- [2] A.A. Tseytlin, Born-Infeld action, supersymmetry and string theory, arXiv:hep-th/9908105.
- [3] W. Heisenberg, H. Euler, Consequences of Dirac's theory of positrons, *Z. Phys.* 98 (1936) 714, arXiv:physics/0605038.
- [4] J. Schwinger, On gauge invariance and vacuum polarization, *Phys. Rev.* 82 (1951) 664.
- [5] G.V. Dunne, The Heisenberg-Euler effective action: 75 years on, *Int. J. Mod. Phys. A* 27 (2012) 1260004, arXiv:1202.1557 [hep-th].
- [6] I.L. Buchbinder, E.A. Ivanov, N.G. Pletnev, Superfield approach to the construction of effective action in quantum field theory with extended supersymmetry, *Phys. Part. Nucl.* 47 (2016) 291.
- [7] I.L. Buchbinder, E.A. Ivanov, I.B. Samsonov, The low-energy $N = 4$ SYM effective action in diverse harmonic superspaces, *Phys. Part. Nucl.* 48 (2017) 333, arXiv:1603.02768 [hep-th].
- [8] I.L. Buchbinder, E.A. Ivanov, B.S. Merzlikin, Leading low-energy effective action in $6D$, $\mathcal{N} = (1, 1)$ SYM theory, *J. High Energy Phys.* 1809 (2018) 039, arXiv:1711.03302 [hep-th].
- [9] I.L. Buchbinder, A.S. Budekhina, B.S. Merzlikin, On the component structure of one-loop effective actions in $6D$, $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ supersymmetric gauge theories, arXiv:1909.10789 [hep-th].
- [10] C.M. Chang, Y.H. Lin, S.H. Shao, Y. Wang, X. Yin, Little string amplitudes (and the unreasonable effectiveness of $6D$ SYM), *J. High Energy Phys.* 1412 (2014) 176, arXiv:1407.7511 [hep-th].
- [11] S.H. Shao, Y.H. Lin, Y. Wang, X. Yin, Interpolating the Coulomb phase of little string theory, *J. High Energy Phys.* 1512 (2015) 022, arXiv:1502.01751 [hep-th].
- [12] O. Aharony, A brief review of "little string theories", *Class. Quantum Gravity* 17 (2000) 929, arXiv:hep-th/9911147.
- [13] D. Kutasov, Introduction to little string theory, *ICTP Lect. Notes Ser.* 7 (2002) 165.
- [14] N. Lambert, M-theory and maximally supersymmetric gauge theories, *Annu. Rev. Nucl. Part. Sci.* 62 (2012) 285, arXiv:1203.4244 [hep-th].
- [15] J. Bagger, N. Lambert, S. Mukhi, C. Papageorgakis, Multiple membranes in M-theory, *Phys. Rep.* 527 (2013) 1, arXiv:1203.3546 [hep-th].
- [16] A. Giveon, D. Kutasov, Brane dynamics and gauge theory, *Rev. Mod. Phys.* 71 (1999) 983, arXiv:hep-th/9802067.
- [17] P.S. Howe, K.S. Stelle, P.C. West, $N = 1$ $d = 6$ harmonic superspace, *Class. Quantum Gravity* 2 (1985) 815.
- [18] B.M. Zupnik, Six-dimensional supergauge theories in the harmonic superspace, *Sov. J. Nucl. Phys.* 44 (1986) 512.
- [19] G. Bossard, E. Ivanov, A. Smilga, Ultraviolet behaviour of $6D$ supersymmetric Yang-Mills theories and harmonic superspace, *J. High Energy Phys.* 1512 (2015) 085, arXiv:1509.08027 [hep-th].
- [20] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky, E. Sokatchev, Unconstrained $N = 2$ matter, Yang-Mills and supergravity theories in harmonic superspace, *Class. Quantum Gravity* 1 (1984) 469.
- [21] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky, E.S. Sokatchev, *Harmonic Superspace*, Cambridge University Press, Cambridge, 2001, p. 306.
- [22] I.L. Buchbinder, E.I. Buchbinder, S.M. Kuzenko, B.A. Ovrut, The background field method for $N = 2$ super Yang-Mills theories in harmonic superspace, *Phys. Lett. B* 417 (1998) 61, arXiv:hep-th/9704214.
- [23] I.L. Buchbinder, S.M. Kuzenko, Comments on the background field method in harmonic superspace: non-holomorphic corrections in $\mathcal{N} = 4$ SYM, *Mod. Phys. Lett. A* 13 (1998) 1623, arXiv:hep-th/9804168.
- [24] E.I. Buchbinder, I.L. Buchbinder, S.M. Kuzenko, Non-holomorphic effective potential in $\mathcal{N} = 4SU(n)$ SYM, *Phys. Lett. B* 446 (1999) 216, arXiv:hep-th/9810239.
- [25] I.L. Buchbinder, E.A. Ivanov, M.B. Merzlikin, K.V. Stepanyantz, One-loop divergences in the $6D$, $\mathcal{N} = (1, 0)$ Abelian gauge theory, *Phys. Lett. B* 763 (2016) 375, arXiv:1609.00975 [hep-th].
- [26] I.L. Buchbinder, E.A. Ivanov, B.S. Merzlikin, K.V. Stepanyantz, One-loop divergences in $6D$, $N = (1, 0)$ SYM theory, *J. High Energy Phys.* 1701 (2017) 128, arXiv:1612.03190 [hep-th].
- [27] I.L. Buchbinder, E.A. Ivanov, B.S. Merzlikin, K.V. Stepanyantz, Supergraph analysis of the one-loop divergences in $6D$, $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ gauge theories, *Nucl. Phys. B* 921 (2017) 127, arXiv:1704.02530 [hep-th].

- [28] E.S. Fradkin, A.A. Tseytlin, Quantum properties of higher dimensional and dimensionally reduced supersymmetric theories, Nucl. Phys. B 227 (1983) 252.
- [29] N. Markus, A. Sagnotti, A test of finiteness predictions for supersymmetric theories, Phys. Lett. B 135 (1984) 85.
- [30] N. Markus, A. Sagnotti, The ultraviolet behavior of $\mathcal{N} = 4$ Yang-Mills and power counting of extended superspace, Nucl. Phys. B 256 (1985) 77.
- [31] P.S. Howe, K.S. Stelle, Ultraviolet divergences in higher dimensional supersymmetric Yang-Mills theories, Phys. Lett. B 137 (1984) 175.
- [32] P.S. Howe, K.S. Stelle, Supersymmetry counterterms revisited, Phys. Lett. B 554 (2003) 190, arXiv:hep-th/0211279.
- [33] G. Bossard, P.S. Howe, K.S. Stelle, The ultra-violet question in maximally supersymmetric theories, Gen. Relativ. Gravit. 41 (2009) 919, arXiv:0901.4661 [hep-th].
- [34] G. Bossard, P.S. Howe, K.S. Stelle, A note on the UV behaviour of maximally supersymmetric Yang-Mills theories, Phys. Lett. B 682 (2009) 137, arXiv:0908.3883 [hep-th].
- [35] L.V. Bork, D.I. Kazakov, M.V. Kompaniets, D.M. Tolkachev, D.E. Vlasenko, Divergences in maximal supersymmetric Yang-Mills theories in diverse dimensions, J. High Energy Phys. 1511 (2015) 059, arXiv:1508.05570 [hep-th].
- [36] I.L. Buchbinder, E.A. Ivanov, B.S. Merzlikin, K.V. Stepanyantz, On the two-loop divergences of the 2-point hypermultiplet supergraphs for $6D, \mathcal{N} = (1, 1)$ SYM theory, Phys. Lett. B 778 (2018) 252, arXiv:1711.11514 [hep-th].
- [37] I.L. Buchbinder, E.A. Ivanov, B.S. Merzlikin, K.V. Stepanyantz, On gauge dependence of the one-loop divergences in $6D, \mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1)$ SYM theories, Phys. Lett. B 798 (2019) 134957, arXiv:1907.12302 [hep-th].
- [38] S.M. Kuzenko, I.N. McArthur, On the background field method beyond one loop: a manifestly covariant derivative expansion in super Yang-Mills theories, J. High Energy Phys. 0305 (2003) 015, arXiv:hep-th/0302205.
- [39] S.M. Kuzenko, Exact propagators in harmonic superspace, Phys. Lett. B 600 (2004) 163, arXiv:hep-th/0407242.
- [40] S.M. Kuzenko, I.N. McArthur, Hypermultiplet effective action: $\mathcal{N} = 2$ superspace approach, Phys. Lett. B 513 (2001) 213, arXiv:hep-th/0105121.
- [41] I.L. Buchbinder, B.S. Merzlikin, N.G. Pletnev, Induced low-energy effective action in the $6D, \mathcal{N} = (1, 0)$ hypermultiplet theory on the vector multiplet background, Phys. Lett. B 759 (2016) 626, arXiv:1604.06186 [hep-th].
- [42] E.A. Ivanov, A.V. Smilga, B.M. Zupnik, Renormalizable supersymmetric gauge theory in six dimensions, Nucl. Phys. B 726 (2005) 131, arXiv:hep-th/0505082.
- [43] E.A. Ivanov, A.V. Smilga, Conformal properties of hypermultiplet actions in six dimensions, Phys. Lett. B 637 (2006) 374, arXiv:hep-th/0510273.
- [44] L. Casarin, A.A. Tseytlin, One-loop β -functions in 4-derivative gauge theory in 6 dimensions, J. High Energy Phys. 1908 (2019) 159, arXiv:1907.02501 [hep-th].
- [45] I.L. Buchbinder, A.Yu. Petrov, A.A. Tseytlin, Two-loop $\mathcal{N} = 4$ Super Yang Mills effective action and interaction between $D3$ -branes, Nucl. Phys. B 621 (2002) 179, arXiv:hep-th/0110173.
- [46] S.M. Kuzenko, I.N. McArthur, Low-energy dynamics in $\mathcal{N} = 2$ super QED: two loop approximation, J. High Energy Phys. 0310 (2003) 029, arXiv:hep-th/0308136.
- [47] S.M. Kuzenko, I.N. McArthur, On the two loop four derivative quantum corrections in $4D, \mathcal{N} = 2$ superconformal field theories, Nucl. Phys. B 683 (2004) 3, arXiv:hep-th/0310025.
- [48] S.M. Kuzenko, Self-dual effective action of $\mathcal{N} = 4$ SYM revisited, J. High Energy Phys. 0503 (2005) 008, arXiv:hep-th/0410128.