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Superfield Approach to the Construction of Effective Action in Quantum Field Theory with Extended Supersymmetry

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Abstract—We review the current state of research on the construction of effective actions in supersymmetric quantum field theory. Special attention is paid to gauge models with extended supersymmetry in the superfield approach. The advantages of formulation of such models in harmonic superspace for the calculation of effective action are emphasized. Manifestly supersymmetric and manifestly gauge-invariant methods for constructing the low-energy effective actions and deriving the corrections to them are considered and the possibilities to obtain the exact solutions are discussed. The calculations of one-loop effective actions in $\mathcal{N} = 2$ supersymmetric Yang–Mills theory with hypermultiplets and in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory are analyzed in detail. The relationship between the effective action in supersymmetric quantum field theory and the low-energy limit in superstring theory is discussed.

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[†] Deceased.

1. INTRODUCTION

The method of phenomenological Lagrangians is one of the generally accepted methods of theoretical description of the physical effects in the spectrum of observable particles, such that it takes into account the intermediate states lying outside the energy scales achievable in current experiments. A classic historical example is the Lagrangian of the Fermi theory of weak interactions. As we presently know, a consistent (though far from complete) theory of these interactions is the Weinberg–Salam Standard Model (SM). The Fermi theory then emerges effectively, if we integrate out all massive modes of the vector and Higgs fields and thereby exclude them from consideration.

Another example of the effective Lagrangians is the Euler–Heisenberg Lagrangian [1, 2] in QED, which non-perturbatively describes the quantum interaction of a charged scalar or spinor field with the classical constant electromagnetic field. Such a Lagrangian is invariant under all symmetries of the classical theory. It encodes a complete information about the one-loop amplitude in the form which is most convenient for studying the various low-energy nonlinear QED effects including the photon–photon scattering, the change of the dispersion law and photon splitting, electron–positron pair production in external fields, etc.

At present, the effective action is considered as the fundamental object of the quantum field theory which allows to describe the entire set of quantum effects. By definition, the effective action is a generating functional of connected, one-particle irreducible Green functions. In the framework of effective action the light fiends are used as a tool to probe the structure of vacuum of the complete quantum theory. The problem of constructing the effective action is closely related to solving such fundamental problems of the quantum field theory as finding the structure of vacuum and its low-lying excitations, derivation of quantum corrections to classical equations of motion, the study of phase transitions and dynamic symmetry breaking, as well as of quantum dynamics in strong background fields. The construction of effective action in the quantum field theory was discussed, for example, in Refs. [3–7]. The concept of effective action proved especially useful, while considering various aspects of quantization and renormalization of gauge theories (including aspects of anomalies some of which can be physically relevant; for instance, it is hard to imagine a realistic four-dimensional field theory lacking a conformal anomaly).

The most fruitful approach to constructing the effective action in quantum theory of gauge fields is based on the background field method pioneered by DeWitt [8, 9]. This method is a generalization of the method of generating functionals in the quantum field theory [10–14] to the case of non-vanishing classical background fields and non-Abelian gauge symmetry. The basic object of the background field method is the effective action which is invariant under the classical gauge transformations.

The effective action allows us to describe all aspects of the quantum field theory. It determines the elements of the diagram techniques in perturbation theory (i.e., full propagators and full vertex functions), with all quantum corrections taken into account and, hence, sets up the perturbative S-matrix. On the other hand, the effective action immediately yields the physical amplitudes in external classical fields and so makes it possible to capture all quantum effects in external field (the polarization of vacuum of quantized fields, particle production, etc.) [15].

The effective action functional is an ideal tool to analyze the structure of physical vacuum in various models of quantum field theory with spontaneous symmetry breaking (Higgs vacuum, gluon condensate, superconductivity). The effective action allows one to take into account the back reaction of quantum processes on the classical background, i.e., derive the effective equations of motion for the background fields. However, in this case there arise certain difficulties related to the dependence of the off-hell effective action on the choice of gauge fixing and the parameterization of quantum fields. Nevertheless, in [16, 17] the gauge-invariant renormalizability of Yang–Mills type theories in arbitrary gauges was proven. It was demonstrated that a change of the gauge-fixing condition is equivalent to a certain canonical transformation of both the renormalized action and the generating functional of vertices. This, in turn, implies gauge invariance of the renormalized S-matrix. In other words, the whole dependence of the generating functional of the vertex Green's functions on gauge parameters is entirely accommodated by the argument of the effective action.

A manifestly reparameterization-invariant functional that does not depend on the gauge-fixing (the so-called Vilkovisky "unified" effective action) was constructed in [18, 19]. The Vilkovisky effective action was analyzed, for example, in [20] in various models of the quantum field theory (including Einstein gravitation) and in quantum gravity with higher derivatives. It was demonstrated in [21] that the unified effective action for scalar QED and Yang–Mills theory to any order of perturbation theory coincides exactly with the gauge-invariant effective action calculated in the Landau–DeWitt gauge. The Vilkovisky effective action was further improved by DeWitt [22]. However, this modification, as a rule, does not alter the results of one-loop calculations and, in fact, creates certain problems basically related to the choice of metric on the configuration space [7].

It follows from the above that the calculation of effective action is of significant interest both in the context of developing the universal formalism of the quantum field theory and for specific applications. The exact calculation of the effective action would mean finding out an exact solution of the corresponding quantum field theory model, which seems infeasible in the general case. Therefore, various approximation approaches (such as the expansions in the number of loops or powers of derivatives of functional arguments) are used. These expansions allow one to describe those physical phenomena in which the role of observables is played by particles and fields with the masses and energies bounded from above by a certain characteristic scale. Only the first nonvanishing terms are kept in the effective action in the leading low-energy approximation. Just these first terms of the low-energy effective action make it possible to explore the vacuum structure of the field-theoretical model and the dynamics of its low-lying excitations.

When finding out the effective action, all fields are split into a background classical part and a quantum perturbation that propagates upon this background. The part of classical action that is quadratic in quantum fields defines the propagators of quantum fields in the background field, and the higher-order terms give the interaction vertices in the perturbation theory. The calculation of effective action requires, first of all, knowing the Green's functions of quantum fields in the background of classical fields of various nature.

Green's functions in the background fields were extensively studied. Fock [23] was the first to propose a method for solving the wave equation in the background of an electromagnetic field via an integral transformation with the proper-time parameter. Later, Schwinger [2] has generalized the proper-time method and applied it to the calculation of one-loop effective action. DeWitt [8] has reformulated the proper-time method in geometric terms and in the presence of a background gravitational field. Note that this development revealed a close relation to the theory of pseudodifferential operators as a tool for the study of partial differential equations in applications to spectral geometry, spectral asymptotics of differential operators, the calculus on manifolds, differential geometry, and other mathematical methods used in quantum theory.[1]

The standard Schwinger–DeWitt techniques were later generalized [32, 33] to the case of arbitrary differential operators satisfying the causality condition. The proper-time method directly yields Green's functions in the vicinity of the light cone. Thus, it is an ideal tool to analyze ultraviolet divergences (i.e., calculate counterterms, β-functions, and anomalies). The primary advantage of the proper-time method consists in the fact that it is manifestly covariant and allows one to introduce different covariant regularizations of divergent integrals (e.g., dimensional regularization, regularization via the generalized ζ-function, etc. (see [34])). Although the authors of the majority of papers in this field limit themselves to the one-loop approximation, the proper-time method can also be applied to higher loops. It was used in [35, 36] to analyze two-loop divergences in diverse models of the quantum field theory (including quantum gravity).

Another important area where the Schwinger–DeWitt proper-time method can be successfuly used is the study of the effect of polarization of vacuum of massive quantum fields by background fields. When the Compton wavelength is much smaller than the characteristic scale on which background fields "live", the proper-time method yields directly the power-series expansion of the effective action over a small parameter $(\lambda/L)^2$. The coefficients of this expansion are proportional to the so-called DeWitt coefficients and are constructed as local invariants of the background fields and their covariant derivatives. The general structure of the Schwinger–DeWitt expansion of effective action for massless fields was discussed in [19, 32, 33]. It was noted that such models require to go beyond the framework of local expansion by summing up the space-time derivatives of the background field. Having introduced certain additional assumptions regarding the convergence of the corresponding series and integrals, the authors of [19] summed the leading derivatives of background fields and obtained nonlocal expressions for one-loop effective action in the case of a massless field.

No exact procedure for the calculation of effective action exists in the general case, since such a calculation would require the knowledge of propagators in arbitrary external fields, while it is impossible to construct such propagators in the exact closed form. All

[1] See, for example, [24–31].

the calculations known to date were performed either in the local approximation in the framework of derivative expansion of the effective action or assuming a certain specific configuration of the background fields (constant fields, homogeneous spaces, etc.), where an exact solution of the corresponding quantum-mechanical problem may be obtained. Since exact calculations can be carried out only in certain specific cases, the development of the perturbative methods of covariant calculation of effective action is a research area of the actual interest. It is especially important to develop such universal methods for quantum theory of gauge fields and gravity [33], as well as for supersymmetric extensions of these theories.

The background field method, being successfully used for studies of the structure of effective action in Yang–Mills theory and gravity, may be generalized to supersymmetric gauge theories formulated in superspace. Such generalizations were constructed for $\mathcal{N} = 1$ supersymmetric Yang–Mills theory and supergravity [37–40], as well as for $\mathcal{N} = 2$ supersymmetric Yang– Mills theory [41–45] (see also [46] for an early attempt at developing the background field method for certain $\mathcal{N} = 2$ supersymmetric Yang–Mills models including $\mathcal{N} = 4$ supersymmetric Yang–Mills theory).

The analysis of phenomenological and formal aspects of supersymmetric field theory models is an essential part of the modern theoretical high-energy physics. An interest in supersymmetry in the field theory as a symmetry between bosons and fermions $\delta\phi \sim \epsilon^{\alpha}\psi_{\alpha}, \delta\psi_{\alpha} \sim \sigma^{m}_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\partial_{m}\phi + \ldots$, is caused by a number of reasons. Let us distinguish some of them.

(1) Supersymmetry [47–50] provides a natural mechanism of unifying bosons and fermions and, consequently, should be regarded as an integral part of any theory pretending to be the unified theory of fundamental interactions (formulations of supersymmetric theories are given, for example, in [37, 39, 50–52]).

(2) Supersymmetry imposes strong restrictions on the structure and coupling constants of the interaction between bosons and fermions, thus weakening the possible divergences. Certain problems of the grand unification theory (such as the hierarchy problem, the problem of exact convergence of three running gauge coupling constants, the proton lifetime problem, etc.) are also solved.

(3) Supersymmetry reveals an intimate relationship between physically motivated supersymmetric field theory nonlinear sigma models and such mathematical concepts as the geometry of Kählerian, hyper-Kählerian, and quaternion manifolds.

(4) It is currently believed that superstring theory is a candidate unified theory of all fundamental interactions (including the gravitational one) [53, 54]. Supersymmetry, which guarantees absence of tachyons in the string spectrum, is a key concept of this theory.

The characteristic energy scale of superstring theory is set by the Planck mass. At energies much lower than the Planck mass, all quantum-field phenomena should be characterized by an effective (low-energy, from the standpoint of superstring theory) supersymmetric field theory. This is the reason why the study of various (super)field limits of superstring theories attracts so much interest (see, for example, [53–55]). These studies allow one to probe string effects using the methods of the field theory (and, vice versa, field-theoretical effects using the methods of superstring theory). They also offer the opportunity to construct new (super)field models with intriguing features, including intrinsic mechanisms of supersymmetry breaking. Supersymmetry is "hidden" at the currently accessible energies; this implies that it is broken at a certain scale. The determination of this energy scale remains an intriguing problem. Theoretical analysis of the low-energy quantum aspects of supersymmetric models is a key to the study of probable phenomenological manifestations of supersymmetry at energy scales achievable in current experiments.

To date, it has been established that a special low-energy effective action may be determined exactly in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ extended supersymmetric Yang–Mills theories. However, it is to the point here to mention that certain problems arise, while constructing quantum theories with extended supersymmetry. They are related to the fact that, in the general case, the algebra of extended supersymmetry closes only on the equations of motion. In superfield approaches, the requirement of the irreducibility of a superfield representation of superalgebra results in differential constraints on superfields. The necessity of resolving these constraints via unconstrained superfields (prepotentials) entails certain difficulties in the analysis of quantum properties of the most interesting field-theoretic models with $\mathcal{N} = 2,\ldots,\mathcal{N} = 8$ extended supersymmetries and in the construction of the corresponding perturbation theory. In addition, still in seventies, several "no-go" theorems on the impossibility to describe certain important supermultiplets off the mass shell in terms of unconstrained superfields were formulated. As always, hidden constraints rooted in the conditions of "no-go" theorems had to be revealed in order to overcome such obstacles.

Since supersymmetry transformations bring bosons and fermions into each other, the fields of different spins are regarded as components of a certain supermultiplet. The simplest $\mathcal{N} = 1$ supermultiplets contain physical fields, such that their spins or helicities differ by 1/2, and auxiliary nonpropagating fields. The representation of $\mathcal{N} = 4$ supersymmetry on free equations of motion unifies massless physical fields with the helicities 1, 1/2, and 0 into a maximally extended vector supermultiplet. However, the set of auxiliary fields needed for ensuring the off-shell realization of complete $\mathcal{N} = 4$ supersymmetry still remains unknown, and it is not improbable that such representations, if exist, are infinite-dimensional. Further extensions $\mathcal{N} = 5,\ldots,8$ necessarily incorporate a graviton with the helicity 2, left- and right-handed gravitinos with the helicities $\pm 3/2$, and other fields. It is frequently sufficient to know the contents of the corresponding supermultiplets on the mass shell in order to write down supersymmetric Lagrangians in terms of the component fields. However, a detailed analysis of quantum properties of these field theory models requires to know the field contents of supermultiplets, as well as all symmetries of classical action, off the mass shell.

Extended supermultiplets are described most naturally in terms of superfields given on a superspace parametrized by the coordinates $z^M = \left\{x^m, \theta^{i\alpha}, \bar{\theta}^{\dot{\alpha}}_i\right\}$, where $\theta^{i\alpha}$, $\bar{\theta}^{\dot{\alpha}}_i$ are anticommuting ($\theta^2 = 0$) Weyl ($\alpha, \dot{\alpha} = 1,2$) spinors and $i = 1,2,\ldots\mathcal{N}$. Physical and auxiliary fields emerge as the components of expansion of a superfield in powers of Grassmann spinor coordinates. However, this expansion contains, in addition to the component fields of certain irreducible supermultiplet of interest, also a large number of superfluous components. Therefore, the corresponding superfields should necessarily satisfy the constraints that eliminate superfluous components. For complex $\mathcal{N} = 1$ superfields, such constraints are the conditions of Grassmann analyticity:

$$\bar{D}_{\dot{\alpha}}\Phi(x,\theta,\bar{\theta}) = 0, \quad D_\alpha \bar{\Phi}(x,\theta,\bar{\theta}) = 0.$$

Their solutions are chiral $\Phi(x_L,\theta)$ or antichiral $\bar{\Phi}(x_R,\bar{\theta})$ superfields depending on a half of Grassmann coordinates, such that supersymmetry transformations leave invariant conjugate chiral subspaces $\left\{x^m_L, \theta^\alpha\right\}$ and $\left\{x^m_R, \bar{\theta}^{\dot{\alpha}}\right\}$.

The method of $\mathcal{N} = 2$ harmonic superspace [51, 56–60] is an extension of the chirality concept to $\mathcal{N} = 2$ supersymmetry. In this case, it was found necessary to supplement the coordinates of $\mathcal{N} = 2$ superspace by harmonics $u^{\pm i}$, $u^{+i}u^-_i = 1$, $\overline{u^{+i}} = u^-_i$, that parametrize the coset space of $SU(2)/U(1)$ automorphism group of $\mathcal{N} = 2$ Poincaré superalgebra. It turned out that the constraints for matter hypermultiplets and $\mathcal{N} = 2$ gauge theory could be resolved after augmenting the standard $\mathcal{N} = 2$ superspace with harmonic sphere $SU(2)/U(1)$ and singling out, in this extended ("harmonic") superspace, an analytic subspace, which is closed under the $\mathcal{N} = 2$ supersymmetry transformations and is parametrized by a smaller number of Grassmann variables as compared to the standard $\mathcal{N} = 2$ superspace. The hypermultiplets and $\mathcal{N} = 2$ gauge multiplet are described by superfields which "live" on the analytic subspace and are not subject to any additional constraints. The prohibition (imposed by "nogo" theorems) against superfield description of representations of extended superalgebra off the mass shell is circumvented owing to the fact that analytic superfields contain an infinite number of auxiliary fields (in the case of a hypermultiplet) or an infinite number of purely gauge degrees of freedom (in the case of $\mathcal{N} = 2$ gauge multiplet). These components are eliminated either by the equations of motion or by choosing an appropriate gauge. The transformations of $\mathcal{N} = 2$ supersymmetry leave analytic subspace $\zeta^M \equiv \left\{x^m_A, \theta^{+\alpha}, \bar{\theta}^{+\dot{\alpha}}, u^{\pm i}\right\}$ invariant, and physical and auxiliary fields of $\mathcal{N} = 2$ supermultiplets emerge in the expansion of unconstrained superfields defined on analytical superspace ζ^M. The field content of supermultiplets becomes infinite-dimensional, since the coefficients of the expansion of superfields over harmonics are higher-isospin irreducible representations of group $SU(2)$. Therefore, for constructing the superfield actions one should lay down such rules that yield physical Lagrangians for a finite number of component fields after the elimination of auxiliary fields by their equations of motion. This is achieved by using a Berezin integral [62] over Grassmannian coordinates and an integral over harmonics on group $SU(2)$ defined by the following rules [51]:

$$\int d\theta = 0, \quad \int d\theta\theta = 1, \quad \int du = 1,$$
$$\int du u^+_{(i_1}\ldots u^+_{i_n} u^-_{j_1}\ldots u^-_{j_n)} = 0.$$

$\mathcal{N} = 2$ supersymmetric Yang–Mills theory in interaction with matter is formulated in terms of two $\mathcal{N} = 2$ multiplets. The vector multiplet is described by a real analytic superfield (prepotential $V^{++} = V^{++I}(\zeta)T_I$) that takes values in Lie algebra of the gauge group. It contains the vector gauge potential and its superpartners. Matter fields are accommodated by hypermultiplet $q^+(\zeta)$ and its conjugate $\tilde{q}^+(\zeta)$ that are transformed by a certain representation of the gauge group. The classical action of $\mathcal{N} = 2$ supersymmetric Yang–Mills theory in harmonic superspace can be written in a simple form:

$$S = \frac{1}{2g^2}\operatorname{tr}\int d^8z \mathcal{W}^2 + \frac{1}{2}\int d\zeta^{(-4)} q^+_a (D^{++} + igV^{++}) q^{+a}, \quad (a = 1,2). \tag{1.1}$$

Here $\mathcal{W} = -\frac{1}{4}\left(\bar{D}^+\right)^2 V^{--}$ is the gauge-invariant u-independent chiral field strength superfield, and V^{--} is the nonanalytic superfield coupled to the prepotential by zero curvature condition[2] $D^{++}V^{--} - D^{--}V^{++} + i\left[V^{++}, V^{--}\right] = 0$. Unfortunately, no such a construc-

[2] Here $D^{\pm\pm} = u^{\pm i}\frac{\partial}{\partial u^{\mp i}}$, $D^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}}$ are $SU(2)$-covariant derivatives.

tion for the most interesting $\mathcal{N} = 4$ supersymmetric Yang– Mills theory with all 16 supersymmetries realized off the mass shell exists.[3] One is forced to describe this theory using classical action (1.1) with a hypermultiplet in the adjoint representation and an additional hidden $\mathcal{N} = 2$ supersymmetry realized by the transformations

$$\begin{aligned} \delta V^{++} &= (\varepsilon^{\alpha a}\theta^{+}_{\alpha} + \bar{\varepsilon}^{a}_{\dot{\alpha}}\bar{\theta}^{+\dot{\alpha}})q^{+}_{a}, \\ \delta q^{+}_{a} &= -\frac{1}{2}(D^{+})^{4}[(\varepsilon^{\alpha}_{a}\theta^{-}_{\alpha} + \bar{\varepsilon}_{\dot{\alpha}a}\bar{\theta}^{-\dot{\alpha}})V^{--}]. \end{aligned} \tag{1.2}$$

Thus, the harmonic superspace formalism gives a natural description of $\mathcal{N} = 2$ supersymmetric theories off the mass shell and provides an opportunity to study the quantum aspects of such theories, with retaining manifest $\mathcal{N} = 2$ supersymmetry at all steps of calculation. The latter feature is of principal significance, since it allows one to verify the correctness of calculations and obtain results directly in terms of $\mathcal{N} = 2$ superinvariants. This is the reason why the development of general methods of covariant construction of effective action in $\mathcal{N} = 2$ supersymetric quantum theories of gauge fields and matter in harmonic superspace is an intensively developing research area.

Another circle of questions is related to the choice of field-theoretic description in terms of one or another tensor (super)fields describing massless and massive irreducible representations of the Poincaré (super)group. The dynamic equivalence of descriptions of a massless particle with zero spin in terms of scalar and tensor fields is the simplest example of this ambiguity [63]. The superfield realization of an irreducible representation of superspin 1/2 in terms of either an unconstrained real scalar $\mathcal{N} = 1$ superfield (prepotential) or a chiral spinor $\mathcal{N} = 1$ superfield supplies another example of equivalence that is valid on classical equations of motion. Being dynamically equivalent at the level of free Lagrangians, these different realizations of the same representation lead to different models of interaction with gravitational and gauge (super)fields. The $\mathcal{N} = 2$ vector-tensor multiplet with gauged central charge presents an important example of this sort of non-equivalence. One of the physical scalars of this multiplet is described by an antisymmetric tensor field; the interaction of the latter via the Chern–Simons term with $\mathcal{N} = 2$ vector multiplet and $\mathcal{N} = 2$ supergravity ensures the realization of the mechanism of Green–Schwarz anomalies cancellation in effective theory [64]. As a representation of $\mathcal{N} = 2$ supersymmetry, this multiplet is akin to a massless 8 + 8 Fayet–Sohnius hypermultiplet [65]; however, it does not possess an unconstrained formulation, which somewhat complicates the analysis of quantum properties of such systems.

Lately, there was a rebirth of interest in various aspects of field-theoretic description of $4D$ $\mathcal{N} = 1$ and $\mathcal{N} = 2$ massive tensor multiplets and their couplings to scalar and vector multiplets. This interest is driven basically by the fact that a massive 2-form emerges naturally in $\mathcal{N} = 2$ supergravity appearing as a result of compactification of type II string theory on Calabi–Yau manifolds in the presence of electric and magnetic fluxes (see, for example, [66]). This fact lays grounds for a more detailed investigation of massive $\mathcal{N} = 1$ and $\mathcal{N} = 2$ tensor multiplets [67]. A massive tensor multiplet was introduced (as a dual version of a massive vector multiplet) in $\mathcal{N} = 1$ supersymmetry 30 years ago, and this construction was mentioned in monographs [37, 39]. In contrast to standard tensor multiplets, the models of (massless) $\mathcal{N} = 1$ and $\mathcal{N} = 2$ improved tensor multiplets are superconformal in the case of global supersymmetry in Minkowski space and are invariant with respect to super-Weyl transformations in curved superspace. In addition, being constrained by the properly chosen harmonic constraints, the superfields of the $\mathcal{N} = 2$ tensor matter multiplets possess a finite set of auxiliary fields, and all models of self-interaction of these superfields are dual to special classes of q^{+}-hypermultiplet models [51]. There exist at least two reasons why the improved tensor multiplet bears a special interest: (i) it emerges as a superconformal compensator in the new minimal formulations of $\mathcal{N} = 1$ supergravity (see [37, 39]); (ii) it may play the role of a Goldstone multiplet for the partial breaking of $\mathcal{N} = 1$ superconformal symmetry associated with coset $SU(2,2|1)/(SO(4,1)\times U(1))$, involving AdS_5 as a bosonic subspace [68–71]. The improved tensor multiplet is remarkable in that its super-Weyl invariance is retained in the massive case.

One of the basic properties of the low-energy effective action in supersymmetric field theory is holomorphy [72]. This property implies that finite perturbative or nonperturbative quantum corrections to the classical action, in supersymmetric theories with complex superfields defined on a certain subspace of a full superspace, may emerge as holomorphic functions of these superfields integrated over the corresponding subspace. The nonperturbative holomorphic chiral potential with its property of non-renormalizability may serve as an example of holomorphy in $\mathcal{N} = 1$ supersymmetry [73, 74].

The remarkable demonstration of the possibilities provided by requirement of holomorphy is the exacr Seiberg–Witten solution for low-energy effective action in $\mathcal{N} = 2$ supersymmetric Yang-Milss theory [75], [76]. Proceeding from the assumption of holomorphic dependence of the effective action on $\mathcal{N} = 2$ chiral superfield strength $\mathcal{W}$ in the case of a theory

[3] A total of 12 out of 16 supersymmetries may be realized off the mass shell within $\mathcal{N} = 3$ harmonic superspace [51], but we do not consider such formulations in this review.

with gauge group $SU(2)$ spontaneously broken to $U(1)$ and using the concept of S-duality, Seiberg and Witten had determined this action exactly, with nonperturbative contributions of instantons taken into account. In the limit of unbroken $\mathcal{N} = 2$ supersymmetry, the theory has a moduli space [77, 78] on which gauge group $SU(2)$ is spontaneously broken to $U(1)$. The scale Λ of this breaking proves a dynamic parameter of $\mathcal{N} = 2$ supersymmetric Yang–Mills theory. All physical states may be classified according to the unbroken group $U(1)$. Such states as a photon and all its neutral superpartners remain massless, while the other (electrically charged) states, such as gauge bosons $W^{\pm}$ which belong to the coset $SU(2)/U(1)$, acquire mass $\sim\Lambda$. All charged states are massive and can be integrated out. Thus, Seiberg and Witten had presented an example of existence (in a judiciously chosen non-Abelian gauge theory) of the dual Meissner effect that leads to the linear confinement. This result sparked interest in the investigation of effective action of $\mathcal{N} = 2$ supersymmetric models with other gauge groups and matter multiplets.

Analytic effective potential $(q^{+})^4$ constructed as an integral over the analytic subspace of $\mathcal{N} = 2$ harmonic superspace is another striking example of holomorphy in $\mathcal{N} = 2$ supersymmetric models [79]. It should be noted that both holomorphic and analytic contributions to the effective action emerge only in the presence of central BPS charges.

Even more remarkable properties are featured by the maximally supersymmetric $\mathcal{N} = 4$ Yang–Mills theory. It is UV finite and conformally invariant, and there are strong grounds to believe that it is also self-dual with respect to nonperturbative $SL(2, Z)$ transformations [80, 81]. The effective action in this theory is a superfunctional of both $\mathcal{N} = 2$ superfield strength and $\mathcal{N} = 2$ hypermultiplet related to each other by additional "hidden" $\mathcal{N} = 2$ supersymmetry. It was demonstrated in [82–85] that the low-energy effective action depending on $\mathcal{N} = 2$ vector multiplet, takes the following form in the Coulomb branch of $\mathcal{N} = 2$ gauge theory:

$$\Gamma[\mathcal{W}, \bar{\mathcal{W}}] = c\int d^{12}z \ln\frac{\mathcal{W}^2}{\Lambda^2}\ln\frac{\bar{\mathcal{W}}^2}{\Lambda^2}. \tag{1.3}$$

It should be emphasized that the requirements of $\mathcal{N} = 4$ supersymmetry and superconformal invariance are so stringent that this nonholomorphic potential is defined unambiguously up to a numerical coefficient. Numerical coefficient c was determined from the direct one-loop quantum calculations and equals $(N-1)g^2/(4\pi)^2$ [86–89]. There are strong reasons to believe that this one-loop effect is not renormalized by higher loops or nonperturbative corrections; therefore, (1.3) is an exact low-energy effective action.

The effective action in theories with global and local symmetries that are not broken by anomalies should also respect these symmetries. Thus there arise the problem of finding such methods of constructing the effective action that retain the symmetries of classical action at all steps of calculations. It is common knowledge that an adequate formulation of four-dimensional $\mathcal{N} = 1$ supersymmetric field theories is obtained in terms of unconstrained superfields given on $\mathcal{N} = 1$ superspace. The corresponding quantum computational techniques that guarantee a manifest $\mathcal{N} = 1$ supersymmetry have been developed long ago and they are widely used [37, 39]. The structure of effective action in theories with $\mathcal{N} = 1$ supersymmetry (such as the Wess–Zumino model and $\mathcal{N} = 1$ supersymmetric Yang–Mills theory) has been studied fairly well [90–106]. Specifically, the superfield effective potential and the effective potential of auxiliary fields in the Wess–Zumino model [98] and the general Kähler sigma model [101] have been found, and the twoloop chiral effective superpotential has also been determined [99]. The Schwinger–DeWitt background field method in $\mathcal{N} = 1$ Yang–Mills theory was developed in [91]. This method is used to study renormalization properties and construct the effective action. The techniques for analysis of the structure of effective action in $\mathcal{N} = 1$ models have been improved greatly in recent years [107–111]; they allow one to sum up an infinite set of Feynman diagrams with an arbitrary number of free legs.

For the last years, the idea of unifying the Standard Model and gravity is considered within an approach that presupposes the existence of a unified nonperturbative superstring theory (the so-called M-theory). Perturbations in the vicinity of various vacua of M-theory are regarded as fundamental strings of one of the perturbative superstring theories inter-related by duality transformations. Duality transformations between different phases of M-theory generally relate, as a rule, the theories one of which is in the strong coupling regime. Depending on the values of parameters (or domains of the moduli space of M-theory), the same observed objects may be described as fundamental degrees of freedom of one of the perturbative theories and/or as collective excitations akin to soliton-like D-branes. The possibility to interpret closed strings as bound states of a theory of open strings is one of the implications of this phenomenon. In the field-theoretic limit, these two types of strings reproduce gravity and gauge theories of matter fields, respectively. Matter described in terms of open strings is associated with D-branes and "lives" on a certain surface, while gravity, which corresponds to massless excitations of closed strings, propagates within the bulk bounded by this surface. Gravity within the bulk is associated with the field-strength tensor of the boundary gauge theory. These statements constitute the essence of holographic duality.

It follows from the hypothesis of holographic duality that correlation functions for the strength tensor in the field theory may be associated with the amplitude of the graviton propagation between fixed points at the boundary. In the simplest examples of this correspondence, the geometry of multidimensional space has the form of direct product $AdS_5 \times S^5$. A five-dimensional anti-de Sitter space has a four-dimensional boundary, on which $\mathcal{N} = 4$ supersymmetric scale-invariant and finite Yang–Mills theory lives. This *AdS/CFT* correspondence had many physically relevant manifestations.[4] For example, the spectrum of anomalous dimensions of local operators of a conformal gauge theory should match with the spectrum of energies of a particle or, more precisely, a string mode propagating in *AdS* space. If the *AdS/CFT* correspondence is correct, it is desirable to derive *AdS* type geometries from first principles, proceeding from perturbative Feynman diagrams. A step in this direction was made in [117]. The author of this paper claimes that the one-loop one- and two-point functions in scalar theory can be described naturally in terms of propagators from the bulk to the boundary in AdS_5, integrated over the positions of the point in the bulk. It turned out that the Schwinger parameter in the first-quantized formulation can be identified with a radial coordinate in AdS_5 and, hence, the integration over this variable corresponds to integration over the intrinsic fifth dimension. However, the things get more complicated, starting from a four-point function, and a fully satisfactory formulation is still lacking. It was discussed in [118] how the background geometry can be revealed by studying the one-loop effective action in nonsupersymmetric theories in external Abelian fields. It was demonstrated that proper identification of Schwinger parameters in the Abelian Euler–Heisenberg effective action implies integration over AdS_3, S_3, and $T^\star S^3$ geometries, depending on the type of the external field..

From a more general point of view, the Fradkin–Tseytlin generating functional in string theory [119, 120] for massless external background fields, which defines certain generalizations of the Born–Infeld action, should match the generating functional for the effective action (which incorporates all quan tum corrections) of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory [121]. Certain α'-string corrections can be summed to all orders in the open string theory in an external constant Abelian vector field [119]. The known Dirac–Born–Infeld (DBI) action and its supersymmetric extensions emerge here as the effective action based on the representation of the generating functional for string amplitudes as a Polyakov integral with the Lagrangian of a covariant $2D$ sigma model in the exponent of the path integral [121]. BI action

$$L_{BI} = \sqrt{-\det(\eta_{mn} + F_{mn})} - 1 \tag{1.4}$$

was proposed in the 1930s as a solution to the problem of singularity of a point-like charge and infinity of its energy in the Maxwell theory by analogy with the square root $\sqrt{1 - v^2/c^2}$ of the action of a relativistic particle [122, 123]. The DBI theory has many intriguing properties; for example, it is causal and provides a nontrivial example of systems with electromagnetic duality. Supersymmetry is known to be compatible with the causality principle, positive energy density, and duality; therefore, supersymmetric (correctly generalized) DBI actions should have the same features [124–127]. This is indeed true for $\mathcal{N} = 1$ DBI action [129, 130] and should also hold true for $\mathcal{N} = 2$ DBI action [131–133]. It is common knowledge that $\mathcal{N} = 1$ DBI action is the Goldstone–Maxwell action for $\mathcal{N} = 1$ vector supermultiplet of Goldstone fields associated with partial spontaneous supersymmetry breaking $\mathcal{N} = 2 \to \mathcal{N} = 1$ [69–134]. Partial breaking $\mathcal{N} = 4 \to \mathcal{N} = 2$ in $\mathcal{N} = 2$ superspace is harder to implement, since a natural formulation of $\mathcal{N} = 4$ gauge theories with all supersymmetries off the mass shell is currently lacking.

The present interest in the DBI action and its numerous supersymmetric and non-Abelian modifications in the context of superstring theory is associated with the action of a probe D3-brane propagating in the curved background of an anti-de Sitter space and the background of the electric part of RR 4 form potential that are induced by a large number N of coincident branes [54]. This action takes the following form:

$$S = -T_3 \int d^4x H^{-1}(X) \tag{1.5}$$
$$\times \left[\sqrt{-\det(\eta_{mn} + H(X)\partial_m X^i \partial_n X^i + H^{1/2}(X) F_{mn})} - 1\right].$$

Here $i = 1,\ldots,6$, $m,n = 1,\ldots,4$, $T_3 = 1/2\pi g_s$, and $H = 1 + \frac{Q}{|X|^4}$, $Q \equiv \frac{1}{\pi} N g_s$. The full action incorporates also the Chern–Simons term characterizing the "magnetic" interaction part $S_{mag} = iN\int_5 \epsilon_{i_1\ldots i_6} \frac{1}{|X|^6} X^{i_1} dX^{i_2} \wedge \ldots \wedge dX^{i_6}$. The key idea of the hypothesis of *AdS/CFT* correspondence is the assumption of exact dual relationship between the IIB supergravity description of the interaction of parallel D-branes and the low-energy effective action of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in the Coulomb phase of spontaneous breaking of gauge symmetry (see [139, 140]). Specifically, it was found that the interaction potential of D3-branes (extended objects that are the solutions of classical equations of IIB supergrav-

[4] The *AdS/CFT* correspondence was proposed by Maldacena in the end of 1997 [116]. This correspondence was recently generalized to other (non-*AdS*) spaces and non-supersymmetric and nonconformal field theories dual to them.

ity), which is described by the DBI action, coincides with the leading term in the low-energy effective action of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory with gauge group $SU(N)$ (at large N) that is spontaneously broken to the maximal Abelian subgroup [141–143]. Thus, the calculation of subsequent contributions to the low-energy effective action of $\mathcal{N} = 4$ supergauge models turned out to be crucial for the investigation of interrelation between supersymmetric quantum field theory and string theory. The hypothesis of the presence of this strong interrelation motivates the researchers to analyze quantum corrections and their renormalization properties in supersymmetric Yang–Mills theories [107, 144, 145].[5]

Let us discuss these issues in more detail. We consider the case with $X^i = \text{const}$, i.e., all derivatives of scalar fields are neglected; it is also assumed that $\frac{Q}{|X|^4} \gg 1$, which allows us to omit unity in harmonic function H. Action S then becomes the same as the action of a probe D3-brane, which is oriented along the boundary of space $AdS_5 \times S^5$. Expanding in powers of F, we obtain the general structure of expansion

$$S = \frac{1}{g_s}\int d^4x \sum_{l=0}^{\infty} c_l (g_s N)^l \frac{F^{2l+2}}{|X|^{4l}} \tag{1.6}$$

with the first terms

$$S = T_3 \int d^4x \int\left(\frac{1}{4}F^2 + \frac{1}{8}\frac{Q}{|X|^4}\left[F^4 - \frac{1}{4}(F^2)^2\right] + \frac{1}{12}\frac{Q^2}{|X|^8}\left[F^6 - \frac{3}{8}F^4F^2 + \frac{1}{32}(F^2)^3\right] + \dots\right).$$

From the viewpoint of string theory in the weak coupling regime, the leading terms of the interaction between individual D-branes are described by diagrams of the "disk with holes" type. The small separation limit should be represented by loop corrections in supersymmetric Yang–Mills theory, while the large separation limit is described via interaction in classical supergravity. If the coefficient at a certain term in the string interaction potential turns out to be independent of distance (i.e., of the dimensionless ratio of separation and $\sqrt{\alpha'}$), this coefficient should remain the same in quantum supersymmetric Yang–Mills theory and in the interaction Lagrangian of classical supergravity. In terms of supersymmetric Yang–Mills theory, the calculation of the potential of interaction between a bunch of D3-branes and a parallel probe D3-brane, which carries constant background field F_{mn}, corresponds to the calculation of effective quantum action G on the constant scalar background Φ^i, which breaks symmetry $SU(N+1)$ to $SU(N) \times U(1)$, and with a constant $U(1)$ gauge field F_{mn}. In the case of interaction between D3-branes (i.e., in the case of a finite $\mathcal{N} = 4$, $D = 4$ supersymmetric Yang–Mills theory), the expansion of Γ in powers of dimensionless ratio $F^2/|\Phi|^4$ takes the following general form:

$$\Gamma = \frac{1}{g_{YM}^2}\int d^4x \sum_{l=0}^{\infty} f_l(g_{YM}^2, N)\frac{F^{2l+2}}{|\Phi|^{4l}}. \tag{1.7}$$

Functions f_l should depend only on λ in the flat limit (large N and fixed $\lambda \equiv g_{YM}^2 N$).

A naive comparison of this effective action with supergravity expansion (1.6) (with $g_{YM}^2 = 2\pi g_s$, $|\Phi| = T_3|X|$) leads to a hypothesis that the l-th term receives contributions only from the l-th order in the loop expansion of effective action of $\mathcal{N} = 4$, $D = 4$ supersymmetric Yang–Mills theory. This is indeed true for the leading $F^4/|X|^4$ term that appears only in the first loop and is lacking in higher orders owing to the nonrenormalization theorem. In addition, the one-loop coefficient of term F^4 in $\mathcal{N} = 4$ supersymmetric effective action agrees exactly with the supergravity expression. A similar correspondence for term F^6 was analyzed in [145] through the explicit calculation of its two-loop coefficient in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. The Lorentz structure of this term in the effective action is the same as in the DBI supergravity action (the form of Abelian term F^6 is, in fact, unambiguously fixed by $\mathcal{N} = 1$ supersymmetry). The flat ($N \gg 1$) part of the coefficient in front of this term turns out to be the same as the coefficient before the corresponding term in (1.6). Taken together with the known fact that Abelian term F^6 does not appear in the one-loop effective action, this observation may imply that the indicated two-loop coefficient should be exact (i.e., Abelian term F^6 does not obtain contributions from all higher loop ($l \geq 3$) orders).

This hypothesis was tested in [141] in the general non-Abelian case. It was demonstrated that a universal $Ng_{YM}^2 \frac{F^6}{|X|^8}$ expression, which reproduces the low-energy terms next to the leading ones in the supergravity potential between different configurations of coupled states of D-branes, exists in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. Since systems of branes with different fractions of supersymmetry are described by very different background configurations, the assumption that all the corresponding interaction potentials can be derived from a single universal expression in supersymmetric Yang–Mills theory imposes some rather non-trivial constraints on the possible structure of this expression.

[5] A more comprehensive list of references is found in [107–111, 145].

To summarize, one can expect the existence of a new nonrenormalization theorem for Abelian term F^6 in the low-energy effective action (similar to the known theorem for F^4). However, an analysis based on scale invariance and $\mathcal{N} = 2$ supersymmetry is not sufficient to prove the nonrenormalizability of the term F^6. Most likely, what should be used here to the full extent is the fact that $\mathcal{N} = 4$ Yang–Mills theory respects 16 supersymmetries realized in a "deformed" way [142]. Then it can be expected that $\mathcal{N} = 4$ supersymmetry demands the coefficient of the term F^6 to be strictly fixed through the coefficient of the F^4 term (be proprtional to its square). The fact that the F^4 term appears only at the one-loop level should imply that the term F^6 will be present only at the two-loop order.

The story of F^8 terms is even more complicated, since a large number of different invariants of order 8 can be constructed out of the strength F_{mn}. In contrast to terms of the F^4 and F^6 type, supersymmetry alone does not constrain unambiguously the form of invariants of the F^8 type: the F^8 terms in the DBI action and in the one-loop effective action of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory are different and even have different Lorentz structures. The following hypothesis concerning the effective action of supersymmetric Yang–Mills theory was proposed in [145]: (i) the coefficient in front of F^8 in expansion (1.6) of the DBI action receives contribution only from three loops, and this contribution precisely agrees with the supergravity action; (ii) the coefficient in front of the one-loop F^8 contribution to the effective action of $\mathcal{N} = 4$ theory receives corrections from all loops, and the flat part of the resulting nonvanishing function $f_3(Ng^2_{YM})$ in (1.7) tends to zero in the $Ng^2_{YM} \gg 1$ limit (as predicted by the *AdS*/*CFT* correspondence).

It was conjectured in [86] that "unprotected" non-Abelian tensor structures appear in the $SU(N)$, $N > 2$ case already in order $\sim v^4$; this would imply that the proof of the nonrenormalization theorem is applicable only to the $SU(2)$ case. As was found by calculation of the two-loop low-energy effective action (in the harmonic superspace approach), the coefficient before this action, in the limit of large N, exactly coincides with the coefficient in front of that term in the expansion of DBI action which corresponds to interaction of a bunch of N D3-branes with a parallel probe D3-brane. This result allows one to make a strong assertion that the correspondence between the low-energy effective action of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory and the potential of interaction of D3-branes can be promoted to higher orders. Higher order terms $F^{2l+2}/|X|^{4l}$ in the effective action should arise as the bosonic part of a combination of several $\mathcal{N} = 4$ (or $\mathcal{N} = 2$) superinvariants [144]. One of them (for each l) should have a "protected" coefficient that receives contributions only from the l-th loop. Just this term (its flat part) should survive in the strong coupling limit and should match with similar structures in the DBI action expansion, in accordance with the predictions of the *AdS*/*CFT* hypothesis.

There are reasons to believe that $\mathcal{N} = 4$ supersymmetric Yang–Mills theory is self-dual [80]. This property was initially formulated as the duality between ordinary and soliton sectors of the theory. Quite recently, there was put forward an assumption [124, 125], based in part on the ideas of the Seiberg–Witten theory and the *AdS*/*CFT* correspondence [116], that self-duality can be realized in terms of the low-energy effective action of the theory on the Coulomb branch. On this branch, gauge group $SU(N)$ is spontaneously broken down to $SU(N-1) \times U(1)$, and the dynamics is described by a single $\mathcal{N} = 2$ vector multiplet corresponding to the $U(1)$ factor of the unbroken subgroup. Two different scenarios of realization of the self-duality requirement for $\mathcal{N} = 4$ supersymmetric effective action in $\mathcal{N} = 2$ superspace were proposed: (i) self-duality with respect to the Legendre transformation [126, 127]; (ii) self-duality with respect to $U(1)$ rotations [125]. So far, neither of these scenarios have been deduced from first principles, and these propositions remain hypothetical. However, certain forms of self-duality of $\mathcal{N} = 4$ supersymmetric effective action look quite natural in the context of the *AdS*/*CFT* correspondence, and this issue has been actively discussed in literature [124–127]. These hypotheses have been verified to a certain extent at the one-loop level [125, 128]. It turned out that they can be further confirmed at the two-loop level as well; this was the main result of a series of papers [107–114]. If the effective action is indeed self-dual in the large N limit in the sense of scenarios (i) and (ii), an infinite number of nonrenormalization theorems should exist. This corollary of the possible self-duality is of great interest from the point of view of its verification in supersymmetric quantum field theory.

All the above-mentioned results regarding the structure of nonholomorphic potential were obtained only for a certain part of effective action, the one that depends on the fields of $\mathcal{N} = 2$ gauge multiplet. The problem of constructing the leading contribution to the full effective action, that depends on both $\mathcal{N} = 2$ gauge multiplet and the hypermultiplet fields was solved by Buchbinder and Ivanov [146]. The calculation in [146] was based on purely algebraic analysis proceeding from the requirement of additional hidden $\mathcal{N} = 2$ supersymmetries on the mass shell in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory formulated in $\mathcal{N} = 2$ superspace. Manifest $\mathcal{N} = 2$ supersymmetry off the mass shell and hidden supersymmetries on the equations of motion constitute a full set of supersymmetries of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory.

It was demonstrated that potential (1.3) and its generalizations for $SU(N)$ models could be completed to $\mathcal{N} = 4$ supersymmetric form by adding proper terms depending on the hypermultiplet superfields. The structure of these terms is unambiguously defined by hidden supersymmetry. The resulting effective action derived in [146] takes the following form:

$$\Gamma_q = c\int d^{12}z du\left\{(X-1)\frac{\ln(1-X)}{X} + [\mathrm{Li}_2(X) - 1]\right\}, \quad X = -2\frac{\overset{+a}{q}\overset{-}{q}_a}{{}^{\circ}W{}^{\circ}\bar{W}}, \tag{1.8}$$

where c is the same numerical coefficient as in nonholomorphic potential (1.3), and $\mathrm{Li}_2(X)$ is the Euler dilogarithm. In addition, any non-logarithmic terms in the low-energy effective action do not allow such a completion and so are excluded by the constraints imposed by $\mathcal{N} = 4$ supersymmetry. The construction of effective action (1.8) by direct calculation of superdiagrams within the framework of the quantum field theory remained an open problem for some time. This problem was solved in [147] and [215, 216].

The unification of gravity, supersymmetry, and gauge theories leads to supergravity theories, with the number of supersymmetries $\mathcal{N}$ varying from $\mathcal{N} = 1$ to $\mathcal{N} = 8$. When $\mathcal{N} \geq 4$, the minimal supergravity multiplet starts to include scalars; the nonlinearity of gravity action necessarily entails nonlinearity of the kinetic term of scalar fields. Thus, nonlinear sigma models (see [148]) are a part of the Lagrangian of extended supergravity. There is another strong reason for studying the classical and quantum aspects of non-linear sigma models. The principle of spontaneous symmetry breaking is the basic one for phenomenological applications of the quantum field theory. Spontaneously broken global symmetries are not realized as symmetry transformations of physical states, since they do not leave invariant the vacuum state. Then, according to the Goldstone theorem, the spectrum of physical states always includes massless particles for each broken symmetry generator (with a highly nonlinear effective action) [149]. In the general case, scalar fields of a sigma model take values in the Riemannian manifold with a positive-definite metric; this is necessary for the absence of states with a negative norm (the requirement of a positive-definite metric is met by compact (usually symmetric) spaces).

Sigma models with an extended number of supersymmetries are closely related to complex geometries. The following three results became canonical. (i) A Kähler manifold is the target space for supersymmetric sigma models with four supercharges ($D \leq 4$) [150]. In four dimensons ($D = 4$), such sigma models possess $\mathcal{N} = 1$ supersymmetry. (ii) Supersymmetric sigma models with eight supercharges ($D \leq 6$) "live" on hyper Kähler manifolds [151, 152]. In four dimensions ($D = 4$), such sigma models possess $\mathcal{N} = 2$ supersymmetry. (iii) Quaternion-Kähler manifolds are the superfield target spaces for locally supersymmetric sigma models with eight supercharges ($D \leq 6$) [153]. The superspace approach provides unique opportunities for the construction of general supersymmetric sigma models (see, for example, [154] for a review and a comprehensive list of references on this subject). In contrast to $2D$ models, $4D$ nonlinear supersymmetric sigma models (as well as their $4D$ non-supersymmetric counterparts) are nonrenormalizable by the divergence index counting. This is the basic reason why the quantum aspects of such models remain poorly studied.

The fact that supersymmetry has still not been confirmed experimentally suggests that it does not manifest itself at the currently probed energy scales. So the problem of seeking the theoretical mechanisms responsible for its breaking arises. The so-called soft supersymmetry breaking is one of the possible options. It is used in supersymmetric gauge theories and consists in adding to the action certain mass terms which preserve gauge invariance but break supersymmetry. If supersymmetry is broken spontaneously, auxiliary fields acquire nonzero vacuum values, and spinor massless Goldstone fields (goldstinos) emerge. However, standard methods of supersymmetry breaking can lead to the loss of remarkable quantum properties of supersymmetric theories, or at least narrow the scope of their applicability [155]. Thus, the search for alternative mechanisms of supersymmetry breaking and their analysis remain an important task.

It is clear from the said above, at present there are quite a few string-theory related hypotheses on the structure of low-energy effective action of supersymmetric Yang—Mills theory and nonlinear supersymmetric Kähler and hyper-Kähler sigma models. The verification of these hypotheses requires the explicit multiple-loop calculations. The hypothetical dualities discussed above allow one to make use of the powerful methods of supersymmetric quantum field theory to study string theory and, conversely, of the methods of string theory to study the effective action in the field theory.

The basic incentive of this review is to present the manifestly covariant methods of expansion of the superfield heat kernel in derivatives and the techniques (based on these methods) for calculation of one-loop contributions to the effective action in the framework of the background field method for various models of quantum field theory with extended supersymmetry. It should be emphasized that superfield formulations in terms of prepotentials reveal some specific features, such as their non-polynomiality, as well as certain complications associated, for example, with harmonic singularities, and the need of canceling them. In view of these circumstances, the naive construction of perturbation theory encounters incurable difficulties coming from the need to sum up an infinite number of supergraphs. Therefore, many new nontrivial approaches to the study of the structure of effective action had to be invented in order to efficiently handle $\mathcal{N} = 1$ and harmonic $\mathcal{N} = 2$ supergraphs. These tools are considered in the review.

The review is organized as follows.

The structure of the low-energy effective action in extended supersymmetric field models formulated in $\mathcal{N} = 1$ superspace is analyzed in the **second section**. The expansion in derivatives of one-loop effective action of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory containing both the fields of $\mathcal{N} = 2$ vector multiplet and the hypermultiplet fields is derived. The formulation of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in terms of $\mathcal{N} = 1$ superfields is discussed, and the one-loop effective action in the approximation of constant Abelian field strengths F_{mn} and constant hypermultiplet fields is constructed. The action obtained is presented in the form of an expansion over supercovariant derivatives and is rewritten in terms of $\mathcal{N} = 2$ superconformal invariants. Notably, full $\mathcal{N} = 4$ supersymmetric low-energy effective action found in [146] is reproduced in this manner, and the contributions next to the leading ones in this action are found.

It is noted that $\mathcal{N} = 2$ supersymmetric functionals with higher derivatives constructed from harmonic superfields of $\mathcal{N} = 2$ vector multiplet and hypermultiplet are not automatically invariant with respect to the initial transformations of hidden on-shell $\mathcal{N} = 2$ supersymmetry. The invariance of such functionals is attained by deforming the transformations of hidden supersymmetry by terms containing derivatives of hypermultiplets. Using the formulation of $\mathcal{N} = 4$ Yang–Mills theory in $\mathcal{N} = 2$ harmonic superspace and analyzing the possible deformations of transformations of hidden $\mathcal{N} = 2$ supersymmetry, we construct an $\mathcal{N} = 4$ extension of the F^6-type term in the effective action. This extension incorporates both the harmonic superfields with higher derivatives corresponding to $\mathcal{N} = 2$ gauge multiplet and the harmonic hypermultiplet superfields with higher derivatives. The proper deformation of transformations of hidden $\mathcal{N} = 2$ supersymmetry is found. A superfield functional that is invariant with respect to both the transformations of manifest $\mathcal{N} = 2$ supersymmetry and the deformed transformations of hidden $\mathcal{N} = 2$ supersymmetry and contains the F^6 term in the component expansion is constructed as a result.

The systematic $\mathcal{N} = 2$ harmonic superspace approach to the construction of one-loop effective action (incorporating the superfields of $\mathcal{N} = 2$ vector multiplet together with the background hypermultiplet fields) of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory is developed in the **third section**. The one-loop effective action is constructed using the technique of covariant $\mathcal{N} = 2$ harmonic supergraphs within the formulation of $\mathcal{N} = 4$ supersymmetric theory through $\mathcal{N} = 2$ harmonic superfields. Calculations are performed for a constant Abelian gauge field strength F_{mn} and the corresponding constant hypermultiplet fields. The effective action dependent on hypermultiplet fields is constructed and is written as an integral over the analytic subspace of harmonic superspace. It is demonstrated that each term of the low-energy effective action in the Schwinger–DeWitt expansion can be rewritten as an integral over full $\mathcal{N} = 2$ superspace.

The problems discussed in the second and the third sections partly overlap. The difference is that the second section is focused on developing an approach to the effective action in the case when the discussed models are formulated in terms of $\mathcal{N} = 1$ superfields, and the effective action has manifest $\mathcal{N} = 1$ supersymmetry. In the third section, models formulated in $\mathcal{N} = 2$ harmonic superspace are analyzed. The corresponding effective action has manifest $\mathcal{N} = 2$ supersymmetry. This provides an opportunity to evaluate the capacities and advantages of each of these two approaches.

The one-loop low-energy effective action in the hypermultiplet sector for $\mathcal{N} = 2$ superconformal models is studied in the **fourth section**. Any such model incorporates an $\mathcal{N} = 2$ vector multiplet and a certain number of hypermultiplets. It is assumed that gauge group G is broken to $\tilde{G} \times K$, where K is an Abelian subgroup, and the background vector multiplet belongs to a Cartan subalgebra corresponding to K. A general expression for the low-energy effective action in the form of a proper time integral is found. The leading (dependent on the space-time derivatives of superfields) contributions to the effective action are constructed, and their component structure in the bosonic sector is analyzed. The component action incorporates terms with three and four space-time derivatives of fields and is similar in form to the Chern–Simons action.

The **fifth section** is focused on the analysis of $\mathcal{N} = 2$ supersymmetric massive Yang–Mills field theory formulated in $\mathcal{N} = 2$ harmonic superspace. Various gauge-invariant forms of the mass term in action (including the ones with the Stueckelberg superfield) that lead to dual formulations of the theory involving a tensor multiplet are given. A gauge-invariant and manifestly supersymmetric scheme for loop expansion of superfield effective action off the mass shell is developed. Gauge-invariant and manifestly $\mathcal{N} = 2$ supersymmetric one-loop counterterms (including the counterterms depending on the Stueckelberg superfield) are calculated using this scheme. The component structure of such counterterms is analyzed.

The present paper is, in part, pedagogical in nature and aims to introduce the reader to a rapidly developing branch of modern theoretical physics. For the sake of convenience, each of the four sections of the review has its own introduction and discussion or summary that applies to the given section alone. Therefore, the sections are not interdependent and may be read separately. Naturally, all of them share common basic motivations, goals, and methods of research.

2. ONE-LOOP EFFECTIVE ACTION IN $\mathcal{N} = 4$ SUPERSYMMETRIC YANG–MILLS THEORY IN THE FORMALISM OF $\mathcal{N} = 1$ SUPERSPACE

2.1. Introduction

Various quantum aspects of low-energy string dynamics and the *AdS/CFT* correspondence are now being studied actively [116]. These studies resulted in a hypothesis (see [139, 145] for details) that the superconformal version of the Dirac–Born–Infeld action coincides with the sum of terms of expansion of the quantum effective action of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in the Coulomb branch, and the structure of certain terms of this expansion is defined by nonrenormalization theorems.

The aim of this section is to derive the expansion in derivatives of one-loop effective action (incorporating both the fields of $\mathcal{N} = 2$ vector multiplet and the hypermultiplet fields) of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. The formulation of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in terms of $\mathcal{N} = 1$ superfields is discussed. The one-loop effective action expressed through $\mathcal{N} = 1$ superfields is obtained in the approximation of constant Abelian strengths F_{mn} and constant hypermultiplet fields. The obtained action is presented as an expansion in supercovariant derivatives and may be rewritten equivalently in terms of $\mathcal{N} = 2$ superfields. Notably, full $\mathcal{N} = 4$ supersymmetric low-energy effective DBI action [146] can be reproduced in this manner, and the contributions to this action, next to the leading ones, can be found.

2.2. $\mathcal{N} = 4$ Supersymmetric Yang–Mills Theory

Our goal is to calculate the one-loop effective action of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory, such that it takes into account all fields of $\mathcal{N} = 4$ vector multiplet. The $\mathcal{N} = 2$ harmonic superspace approach is now considered to be the most convenient tool to describe the dynamics of $\mathcal{N} = 4$ vector multiplet. In this approach, $\mathcal{N} = 4$ supersymmetric Yang–Mills theory is presented as $\mathcal{N} = 2$ supersymmetric Yang–Mills theory supplemented by the minimal interaction with a hypermultiplet in the adjoint representation of the gauge group.

It is known that the exact quantum dynamics of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in the low-energy limit in the sector of $\mathcal{N} = 2$ vector multiplet is described by nonholomorphic effective potential $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$[6] that depends on $\mathcal{N} = 2$ strengths $\mathcal{W}, \bar{\mathcal{W}}$ (see [52, 86]). The exact form of the nonholomorphic potential for gauge group $SU(N)$ spontaneously broken to its maximal torus is as follows:

$$\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}}) = c\sum_{I<J}\ln\left(\frac{\mathcal{W}^I - \mathcal{W}^J}{\Lambda}\right)\ln\left(\frac{\bar{\mathcal{W}}^I - \bar{\mathcal{W}}^J}{\Lambda}\right). \quad (2.1)$$

Here Λ is the scale parameter, $I, J = 1\ldots N$, and $c = 1/(4\pi)^2$. Expression (2.1) defines the exact low-energy potential in the leading order of expansion in external momenta in the $\mathcal{N} = 2$ gauge superfield sector. It should be noted that expression (2.1) is a fairly general one and can be derived (up to a numerical factor) from symmetry considerations: the requirements of scale invariance and R-symmetry. In addition, potential (2.1) is renormalized neither by higher-loop perturbative corrections nor by instanton contributions. All these properties are essential to understanding the low-energy quantum dynamics of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in the Coulomb phase. Notably, effective potential (2.1) is the term following the leading one in the interaction of parallel D3-branes in superstring theory. It is assumed that full $\mathcal{N} = 4$ supersymmetric effective Yang–Mills theory action obtained by summing up all quantum corrections should reproduce (with certain reservations) the Dirac–Born–Infeld action [141], as it is predicted by the $\mathcal{N} = 4$ version of correspondence between supersymmetric Yang–Mills theory and supergravity. This correspondence was discussed and verified up to two loops in [145] (see also [107] for an analysis of a similar problem for non-Abelian background), and the general approach to the calculation of higher-loop corrections was presented in [40].

In order to reveal the structure of constraints on the effective action imposed by $\mathcal{N} = 4$ supersymmetry and analyze the "$\mathcal{N} = 4$ supersymmetric Yang–Mills theory/supergravity" correspondence in more detail, one should study the effective action not only in the $\mathcal{N} = 2$ vector multiplet sector, but take all the $\mathcal{N} = 4$ vector multiplet fields into account. This problem has long remained unsolved. In a relatively recent paper [146], the full exact low-energy effective action depending both on $\mathcal{N} = 2$ gauge superfield and the hypermultiplet was determined. It was demonstrated that the algebraic constraints on the structure of low-energy effective action in the framework of $\mathcal{N} = 2$ harmonic super-

[6] The low-energy effective action for an arbitrary $\mathcal{N} = 2$ supersymmetric Yang–Mills theory incorporates also the holomorphic effective potential [74] that is lacking in $\mathcal{N} = 4$ gauge theory.

space, which follow from the requirements of hidden $\mathcal{N} = 2$ supersymmetry, are so stringent that they allow one to reconstruct the dependence of effective action on the hypermultiplet fields based on the known nonholomorphic effective potential (2.1). As a result, the additional (depending on hypermultiplet fields) contributions, which incorporate $\mathcal{W}, \bar{\mathcal{W}}$ and hypermultiplet q^{ia} fields on the mass shell, are represented by effective action

$$\mathcal{L}_q = c\left\{(X-1)\frac{\ln(1-X)}{X} + [\mathrm{Li}_2(X) - 1]\right\},\qquad X = -\frac{q^{ia}q_{ia}}{\mathcal{W}\bar{\mathcal{W}}}, \tag{2.2}$$

where $\mathrm{Li}_2(X)$ is the Euler dilogarithm, and c is the same constant as in (2.1) (see [146] for notation and details). Effective Lagrangian (2.2) together with nonholomorphic effective potential (2.1) define the exact supersymmetric low-energy effective action of the $\mathcal{N} = 4$ Yang–Mills theory.

The leading low-energy effective Lagrangian (2.2) was derived in [146] through purely algebraic analysis. The determination of such a Lagrangian and further corrections to it in the expansion in external momenta by direct calculations in the framework of the quantum field theory was an interesting problem. This task is a fairly challenging one, since expression (2.2) contains powers of X and has a singularity at $\mathcal{W} = 0$. Therefore, it is not possible to obtain the desired result by analyzing Feynman diagrams with a fixed number of external lines of hypermultiplet and gauge fields; all such diagrams need to be summed. The covariant harmonic supergraphs technique [41] was used to solve the problem of calculation of effective Lagrangian (2.2) in [147]. A more general problem is to calculate, based on the quantum-field theory or algebraic approaches, the next-to-leading terms in the effective action, such that they would depend on all fields of $\mathcal{N} = 4$ supermultiplet, and represent these terms in a fully $\mathcal{N} = 4$ supersymmetric form.

This section is focused on solving the latter problem for one-loop effective action. The expansion in derivatives of one-loop effective Lagrangian $\mathcal{L}_{eff}$, which depends on both $\mathcal{N} = 2$ background gauge superfields (and their spinor derivatives up to a certain order) and background hypermultiplet superfields, is studied. The formulation of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in terms of $\mathcal{N} = 1$ superfields [37, 39] and the technique of expanding in derivatives in $\mathcal{N} = 1$ superspace [211, 213, 214] are used for this purpose. This approach allows one to obtain the exact coefficients at different powers of spinor covariant derivatives of the constant in space-time $\mathcal{N} = 2$ superfield Abelian strength $\mathcal{W}$. Note that this background strength belongs to a Cartan subalgebra of gauge group $SU(N)$ spontaneously broken to $U(1)^{n-1}$. Likewise, hypermultiplet fields q^{ia} are also chosen as constants in space-time. We thus obtain

$$\begin{gathered}\mathcal{W}| = \Phi = \text{const},\quad D^i_\alpha\mathcal{W}\big| = \lambda^i_\alpha = \text{const},\\ q^{ia}\big| = \text{const},\quad D^i_{(\alpha}D_{\beta)i}\mathcal{W}\big| = F_{\alpha\beta} = \text{const},\\ D^{\alpha(i}D^{j)}_\alpha\mathcal{W}\big| = 0,\quad D^i_\alpha q^{aj}\big| = 0,\quad D^i_{\dot\alpha}q^{aj}\big| = 0,\end{gathered} \tag{2.3}$$

where $\Phi = diag(\Phi^1, \Phi^2, \ldots, \Phi^n)$, $\sum\Phi^I = 0$. This background is the simplest one and allows one to perform exact calculations of one-loop effective action. It will be shown below that $\mathcal{N} = 1$ superfield effective action for this background can be determined unambiguously by a simple substitution of variables in the effective action (without taking into account hypermultiplet fields). Using the procedure detailed in [144], one can rewrite this result in $\mathcal{N} = 2$ supersymmetric form that restores full dependence on hypermultiplet fields. Crucially, background (2.3) is a special supersymmetric solution of classical equations of motion for $\mathcal{N} = 4$ supersymmetric Yang–Mills theory written in terms of $\mathcal{N} = 1$ superfields. Hence, the obtained effective action does not depend on the choice of gauge for $\mathcal{N} = 1$ gauge superfields. In addition, it can be shown that background (2.3) has an equivalent representation in terms of $\mathcal{N} = 2$ superfields; therefore, the final result can also be written in a manifestly $\mathcal{N} = 2$ supersymmetric form. However, this background is not invariant with respect to the transformations of hidden $\mathcal{N} = 2$ supersymmetry that completes manifest $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 4$. The transformations of $\mathcal{N} = 4$ supersymmetry on the mass shell mix physical fields from $\mathcal{N} = 2$ vector multiplet and hypermultiplet. However, background (2.3) does not involve physical spinor fields that belong to the hypermultiplet and should be transformed via physical scalar fields from $\mathcal{N} = 2$ vector multiplet under the transformations of hidden $\mathcal{N} = 2$ supersymmetry. Since manifest $\mathcal{N} = 2$ supersymmetry is present, while the initial hidden $\mathcal{N} = 2$ supersymmetry is broken, there is no reason to expect that the effective action put on this background should feature $\mathcal{N} = 4$ invariance.

In what follows, we will study that leading low-energy contribution to the effective action which includes no spinor derivatives of strength $\mathcal{W}$ and hypermultiplet fields, i.e coincides with the effective potential (2.2). It will be shown how the transformations of the initial hidden $\mathcal{N} = 2$ supersymmetry should be deformed in order to secure the full $\mathcal{N} = 4$ supersymmetry of this effective potential.

2.3. Superfield Formulations of $\mathcal{N} = 4$ Supersymmetric Yang–Mills Theory

No formulations of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory with manifest $\mathcal{N} = 4$ supersymmetry off the mass shell are known. Therefore, the studies of specific quantum aspects of this theory are often based on its other formulations in terms of physical component fields (all four supersymmetries are realized as hidden), $\mathcal{N} = 1$ superfields (one manifest supersymmetry and three hidden ones; see, for example, [37]), or $\mathcal{N} = 2$ harmonic superfields [51, 56] (two manifest supersymmetries and two hidden ones). In all formulations, some supersymmetries close only on the mass shell. In the context of quantum calculations, it is desirable to have as many manifest supersymmetries as possible in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. The use of $\mathcal{N} = 2$ harmonic superspace then seems to be the best choice. However, the formulation in terms of $\mathcal{N} = 1$ superfields has its own advantages attributable to the relatively simple structure of $\mathcal{N} = 1$ superspace and the fact that vast experience in working with $\mathcal{N} = 1$ supergraphs has already been accumulated.

The $\mathcal{N} = 4$ vector multiplet can be described on the mass shell in terms of $\mathcal{N} = 4$ superfields W^{AB}, $A, B = 1\ldots4$ satisfying the reality condition

$$W^{AB} = \frac{1}{2}\varepsilon^{ABCD}W_{CD}, \quad W_{AB} = \bar{W}^{AB}$$

and the mass shell conditions

$$\bar{D}_{A\dot{\alpha}}W^{BC}\frac{1}{3}\delta_A^{[B}\bar{D}_{E\dot{\alpha}}W^{EC]}, \quad D_\alpha^{(A}W^{B)C} = 0.$$

Superfield W^{AB} incorporates all physical fields of $\mathcal{N} = 4$ vector multiplet. Unfortunately, no manifestly $\mathcal{N} = 4$ supersymmetric action off the mass shell for $\mathcal{N} = 4$ supersymmetric Yang–Mills theory has been found yet.

2.4. Formulation of $\mathcal{N} = 4$ Supersymmetric Yang–Mills Theory in $\mathcal{N} = 1$ Superspace

The physical field content of superfield W^{AB} coincides with that of a set of three $\mathcal{N} = 1$ chiral superfields and one $\mathcal{N} = 1$ vector multiplet [37]. Six real scalars that are the lower components of W^{AB} superfield are identified with three complex scalar components of chiral $\mathcal{N} = 1$ superfields Φ^i. Three out of four Weyl fermions present in W^{AB} are included into Φ^i, and the remaining fermion is identified as a gaugino and, together with the real gauge vector field, is put into $\mathcal{N} = 1$ vector multiplet V. In the framework of this description, subgroup $SU(3)\otimes U(1)$ of group $SU(4)$ of R-symmetry remains manifest, and $SU(4)$ representations are decomposed into the representations of this subgroup in accordance with the following rule: $\mathbf{6} \to \mathbf{3} + \underline{\mathbf{3}}$, $\mathbf{4} \to \mathbf{3} + \mathbf{1}$. Thus, chiral superfields Φ^i are transformed as $\mathbf{3}$ of $SU(3)$ groups, antichiral $\bar{\Phi}_i$ superfields are transformed as $\bar{\mathbf{3}}$, and the vector multiplet is an $SU(3)$ singlet.

The action of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory is written in $\mathcal{N} = 1$ superspace in the following way:

$$\begin{aligned} S = \frac{1}{g^2}\mathrm{tr}\Big(&\int d^4xd^2\theta W^2 + \int d^4xd^4\theta\bar{\Phi}_i e^V\Phi^i e^{-V} \\ &+ \frac{1}{3!}\int d^4xd^2\theta i c_{ijk}\Phi^i[\Phi^j,\Phi^k] \\ &+ \frac{1}{3!}\int d^4xd^2\bar{\theta} i c^{ijk}\bar{\Phi}_i[\bar{\Phi}_j,\bar{\Phi}_k]\Big). \end{aligned} \tag{2.4}$$

The notation and conventions adopted in [37] are used here. All superfields are valued in the adjoint representation of the gauge group. In addition to manifest $\mathcal{N} = 1$ supersymmetry and $SU(3)$ symmetry, which acts on indices $i, j, k,\ldots$ of superfields Φ and $\bar{\Phi}$, the action has hidden global supersymmetry defined by the following transformations:

$$\begin{aligned} \delta W_\alpha &= -\epsilon^i_\alpha\bar{\nabla}^2\bar{\Phi}_{ci} + i\epsilon_i^{\dot{\alpha}}\nabla_{\alpha\dot{\alpha}}\Phi^i_c, \\ \delta\bar{W}_{\dot{\alpha}} &= -\bar{\epsilon}_{\dot{\alpha}i}\nabla^2\Phi^i_c + i\epsilon^{\alpha i}\nabla_{\alpha\dot{\alpha}}\bar{\Phi}_{ci}, \\ \delta\Phi^i_c &= \epsilon^{\alpha i}W_\alpha, \quad \delta\bar{\Phi}_{ci} = \bar{\epsilon}_i^{\dot{\alpha}}\bar{W}_{\dot{\alpha}}. \end{aligned} \tag{2.5}$$

In addition, action (2.4) is invariant with respect to transformations

$$\begin{aligned} \delta\Phi^i_c &= c^{ijk}\bar{\nabla}^2(\bar{\chi}_j\bar{\Phi}_{ck}) + i[\chi^j\bar{\Phi}_{cj},\Phi^i_c], \\ \delta\bar{\Phi}_{ci} &= c_{ijk}\nabla^2(\chi^j\Phi^k_c) + i[\bar{\chi}_j\Phi^j_c,\bar{\Phi}_{ci}]. \end{aligned} \tag{2.6}$$

Covariant spinor derivatives ∇_α, $\nabla_{\dot{\alpha}}$, ∇^2, and $\bar{\nabla}^2$ used here were defined in [37], and χ^i is an $\mathcal{N} = 1$ superfield parameter that forms, like the Φ^i superfield itself, an $SU(3)$ isospinor. This superfield parameter incorporates the parameters of transformations of the central charge, supersymmetry, and internal symmetry $SU(4)/SU(3)$ as its components. Transformations (2.6) are defined for background covariant superfields $\Phi_c = e^{\bar{\Omega}}\Phi e^{-\bar{\Omega}}$ and $\bar{\Phi}_c = e^{-\Omega}\bar{\Phi}e^{\Omega}$ [37]. Only these covariantly chiral superfields will be used below, and subscript c will be omitted. It is convenient to introduce new notation $\Phi^1 = \Phi$, $\Phi^2 = Q$, $\Phi^3 = \tilde{Q}$ and rewrite two last terms in (2.4) as follows:

$$i\int d^4xd^2\theta Q[\Phi,\tilde{Q}] + i\int d^4xd^2\bar{\theta}\bar{Q}[\bar{\Phi},\bar{\tilde{Q}}].$$

This is an $\mathcal{N} = 1$ superfield form of interaction between a hypermultiplet and the lower components of chiral $\mathcal{N} = 2$ vector field strength in $\mathcal{N} = 4$ theory.

In the case of an Abelian gauge group, the considered model is free. If the gauge group is non-Abelian, the model has a vacua space parameterized by the vacuum values of six real scalars. The vacua manifold is specified by the scalar potential vanishing conditions ("F-flatness" and "D-flatness") [74]. The solutions of these equations define the structure of the theory vacua and are classified according to the gauge theory phase to which these vacua belong. In a purely Coulomb branch, each scalar field may have a certain nonzero vacuum value. As a result, vacua form manifold $\mathcal{M} = R^{6r}/\mathcal{S}_r$, where $\mathcal{S}_r$ is the Weyl group of permutations of r elements. Torus $U(1)^r$ is the unbroken part of the gauge group. If several scalar fields develop the same vacuum value, a certain non-Abelian gauge subgroup $G \in SU(N)$ remains unbroken, and additional massless gauge fields emerge in the theory.

2.5. Formulation of $\mathcal{N} = 4$ Supersymmetric Yang–Mills Theory in $\mathcal{N} = 2$ Harmonic Superspace

With respect to $\mathcal{N} = 2$ supersymmetry, the $\mathcal{N} = 4$ vector multiplet is decomposed into a vector $\mathcal{N} = 2$ multiplet and a hypermultiplet. Thus, $\mathcal{N} = 4$ supersymmetric Yang–Mills theory can be regarded as a particular case of $\mathcal{N} = 2$ supersymmetric gauge theory with the full action being the sum of actions of $\mathcal{N} = 2$ supersymmetric Yang–Mills theory and hypermultiplet q^{ia} in the adjoint representation interacting minimally with the $\mathcal{N} = 2$ vector multiplet. This theory is formulated in $\mathcal{N} = 2$ harmonic superspace [51, 56], and the dynamic variables in this case are a real analytic gauge superfield V^{++} and the complex analytic hypermultiplet q^+ superfield. They are not subject to any additional constraints. The action of $\mathcal{N} = 4$ supersymmetric Yang–Mills field theory is as follows:

$$\begin{aligned} S[V^{++}, q^+, \breve{q}^+] &= \frac{1}{2g^2}\operatorname{tr}\int d^8z\,\mathcal{W}^2 \\ &- \frac{1}{2g^2}\operatorname{tr}\int d\zeta^{-4} q^{+a}\mathcal{D}^{++}q_a^+. \end{aligned} \tag{2.7}$$

The corresponding equations of motion are

$$\begin{aligned} \mathcal{D}^{++}q_a^+ &= D^{++}q^{+a} + ig[V^{++}, q^{+a}] = 0, \\ \mathcal{D}^{+\alpha}\mathcal{D}_\alpha^+\mathcal{W} &= [q^{+a}, q_a^+]. \end{aligned} \tag{2.8}$$

Here $a = 1, 2$ are the indices of global $SU(2)$ symmetry; $q_a^+ = (q^+, \breve{q}^+)$; $q^{+a} = \varepsilon^{ab}q_b^+ = (\breve{q}^+, -q^+)$; $\mathcal{W}$ is the strength of $\mathcal{N} = 2$ analytic gauge superfield connection V^{++} in the λ-basis; g is the interaction constant; and $d^8z = d^4xd^2\theta^+d^2\theta^-du$, $d\zeta^{-4} = d^4xd^2\theta^+d^2\bar\theta^+du$, and du are the measures of integration over the entire harmonic space, its analytic subspace, and $SU(2)$ harmonics $u^{\pm i}$, respectively. Derivatives $D^+_{\alpha(\dot\alpha)}$ do not contain connections in the λ-basis, where manifest G-analyticity takes place. Equations (2.8) are the equations of motion for $\mathcal{N} = 4$ supersymmetric Yang–Mills field theory that are written in terms of $\mathcal{N} = 2$ superfields. Action (2.7) allows manifestly $\mathcal{N} = 2$ supersymmetric generalizations. In addition, this action is invariant with respect to hidden $\mathcal{N} = 2$ supersymmetry transformations [51] that mix superfields $\mathcal{W}$ and $\bar{\mathcal{W}}$ with q_a^+ but close only on the mass shell. In the Abelian case, the transformations of this hidden $\mathcal{N} = 2$ supersymmetry take the following form:

$$\begin{aligned} \delta\mathcal{W} &= \frac{1}{2}\bar\varepsilon^{\dot\alpha a}\bar D^-_{\dot\alpha}q_a^+, \quad \delta\bar{\mathcal{W}} = \frac{1}{2}\varepsilon^{\alpha a}D^-_\alpha q_a^+, \\ \delta q_a^+ &= \frac{1}{4}(\varepsilon_a^\alpha D_\alpha^+\mathcal{W} + \bar\varepsilon_a^{\dot\alpha}\bar D_{\dot\alpha}^+\bar{\mathcal{W}}), \\ \delta q_a^- &= \frac{1}{4}(\varepsilon_a^\alpha D_\alpha^-\mathcal{W} + \bar\varepsilon_a^{\dot\alpha}\bar D_{\dot\alpha}^-\bar{\mathcal{W}}). \end{aligned} \tag{2.9}$$

Thus, the considered model has $\mathcal{N} = 4$ supersymmetry on the mass shell.

The vacuum structure of model (2.7) in the Abelian case is defined by the solutions of equations

$$(\mathcal{D}^+)^2\mathcal{W} = (\bar{\mathcal{D}}^+)^2\bar{\mathcal{W}} = 0, \quad D^{++}q^{+a} = 0, \tag{2.10}$$

which are basically the Abelian variant of general equations (2.8). For physical component fields of $\mathcal{N} = 4$ vector multiplet defined by expansions

$$\begin{aligned} q^+(\zeta, u) &= f^i(x)u_i^+ + \theta^{+\alpha}\psi_\alpha(x) + \bar\theta^+_{\dot\alpha}\bar\kappa^{\dot\alpha}(x) \\ &+ 2i\theta^+\partial\!\!\!/\bar\theta^+ f^i(x)u_i^-, \quad \mathcal{W} = \Phi(x) \\ &+ \theta^{-\alpha}\lambda_\alpha^+(x) + \theta^{(+\alpha}\theta^{-\beta)}F_{\alpha\beta}(x), \end{aligned} \tag{2.11}$$

superfield equations (2.10) lead to the following system of ordinary equations:

$$\partial\!\!\!/\psi = \partial\!\!\!/\bar\kappa = \Box f^i = \Box\Phi = \partial\!\!\!/\lambda^i = \partial_m F_{mn} = 0. \tag{2.12}$$

A set of constant background fields is the simplest solution to these equations of motion:

$$\begin{aligned} f^i &= \text{const}, \quad \psi = \text{const}, \\ \bar\kappa &= \text{const}, \quad \Phi = \text{const}, \quad F_{mn} = \text{const}. \end{aligned} \tag{2.13}$$

These fields are transformed *linearly* through each other with respect to hidden $\mathcal{N} = 2$ supersymmetry (2.9):

$$\begin{aligned} \delta\phi &= \frac{1}{2}\bar\varepsilon^{\dot\alpha a}\bar\kappa_{\dot\alpha a}, \quad \delta\bar\phi\frac{1}{2}\varepsilon^{\alpha a}\psi_{\alpha a}, \\ \delta\psi_{\alpha a} &= \frac{1}{2}\varepsilon_a^\beta F_{\alpha\beta}, \quad \delta\bar\kappa_{\dot\alpha a} = \frac{1}{2}\bar\varepsilon_a^{\dot\beta}F_{\dot\alpha\dot\beta}, \\ \delta f_a^i &= \frac{1}{4}\varepsilon_a^\alpha\lambda_\alpha^i + \frac{1}{4}\bar\varepsilon_{a\dot\alpha}\bar\lambda^{i\dot\alpha}, \quad \delta\lambda_\alpha^i 0, \delta F_{\alpha\beta} = 0. \end{aligned} \tag{2.14}$$

Solutions (2.13) are the simplest vacuum configuration realizing the representation of $\mathcal{N} = 4$ supersymmetry and allowing one to calculate $\mathcal{N} = 4$ supersymmetric low-energy effective action for $\mathcal{N} = 4$ super Yang–Mills theory.

It is instructive to compare $\mathcal{N} = 4$ supersymmetric background (2.13) with background (2.3). The latter contains components Φ and F present in $\mathcal{N} = 2$ vector multiplet, while components $\bar{\kappa}$ and ψ of the hypermultiplet are not included in it. As a result, background (2.3) is not invariant with respect to hidden $\mathcal{N} = 2$ supersymmetry transformations (2.14) and thus does not realize the representation of $\mathcal{N} = 4$ supersymmetry. However, background (2.3) is a representation of $\mathcal{N} = 2$ supersymmetry. Therefore, the effective action calculated using background (2.3) in the framework of the $\mathcal{N} = 2$ supersymmetric background field method should feature manifest $\mathcal{N} = 2$ supersymmetry and gauge invariance, but should not exhibit full $\mathcal{N} = 4$ supersymmetry. The results of calculations (see below) confirm this assumption.

A method proposed in [146] should be used to construct a fully $\mathcal{N} = 4$ invariant effective action. The idea is as follows: the effective action in the $\mathcal{N} = 2$ vector multiplet sector is analyzed, and the action depending on both $\mathcal{N} = 2$ vector multiplet and a hypermultiplet is then constructed in such a way that the resulting action is invariant with respect to the transformations of hidden $\mathcal{N} = 2$ supersymmetry (2.9).

2.6. Background Field Method in $\mathcal{N} = 1$ Superspace

When calculating the effective action, we can use the background field method in $\mathcal{N} = 1$ superspace and the proper time technique adapted to the $\mathcal{N} = 1$ superfield formalism. This provides an opportunity to preserve classical gauge invariance in the calculation of effective action and sum up an infinite set of contributions of Feynman diagrams to a single functional depending on background fields. It was already noted that the considered theory may be formulated in terms of component fields, through $\mathcal{N} = 1$ superfields, or through $\mathcal{N} = 2$ harmonic superfields. The use of the component formulation makes the problem an exceedingly complex one, since a large number of interacting fields are present, while manifest supersymmetry is lacking. To study the effective action of the model, the formulation in $\mathcal{N} = 2$ harmonic superspace can be also used. The background field method for theories in $\mathcal{N} = 2$ harmonic superspace was proposed in [41]. The proper time technique in superfield theories was discussed in several recent papers (see, for example, [107] and references therein). One can run into certain technical difficulties if trying to directly apply general methods for finding out the effective action; therefore, when performing the concrete calculations, the general methods should be supplemented with some specfic tricks.

The calculation of effective action in the considered theory with the proper time technique requires analyzing matrix differential operators in a superspace which mix the sectors of $\mathcal{N} = 2$ vector multiplet and hypermultiplet. The harmonic supergraphs technique can be used to study the effective action; however, while doing so, certain technical problems can arise.[7] To be able to overcome them, some special methods should be developed for determining those contributions to the effective action which are next to the leading one and so take the theory beyond the low-energy approximation. These problems can be evaded by working in terms of $\mathcal{N} = 1$ superfields and making use of the experience accumulated in the study of theories in $\mathcal{N} = 1$ superspace [37, 39, 107, 211].

It is common knowledge that the background field method implies decomposing fields into their background and quantum components and imposing gauge-fixing restrictions on quantum fields. It is obvious that full gauge invariance of classical (background) fields is preserved by this procedure, although gauge fixing can break some classical symmetries (see [112] for a detailed discussion of this problem).

We determine one-loop effective action Γ, which depends on background superfields (2.3), as a functional integral over the space of quantum fields

$$e^{i\Gamma} = \int \mathcal{D}v\mathcal{D}\varphi\mathcal{D}c\mathcal{D}c'\mathcal{D}\bar{c}\mathcal{D}\bar{c}' e^{i(S_{(2)}+S_{\mathrm{FP}})}. \qquad (2.15)$$

Here $S_{(2)}$ is the classical action part that is quadratic in quantum fields and includes the gauge fixing conditions, and S_{FP} is the corresponding ghost action. Formal calculation of path integral (2.15) leads to a representation of the effective action through the functional determinant (see (2.24)).

The choice of a multiparameter covariant gauge fixing condition is one of the key points of $\mathcal{N} = 1$ superfield calculations:

$$S_{\mathrm{GF}} = -\frac{1}{\alpha g^2}\int d^8z(F^A\bar{F}^A + b^A\bar{b}^A), \qquad (2.16)$$

where $b, \bar{b}$ are the Nielsen–Kallosh ghosts fields. The gauge fixing conditions imposed on quantum superfields v and φ are as follows:

$$\begin{aligned} \bar{F}^A &= \nabla^2 v^A + \lambda\left[\frac{1}{\Box_+}\nabla^2\varphi^i, \bar{\Phi}_i\right]^A, \\ F^A &= \bar{\nabla}^2 v^A - \bar{\lambda}\left[\frac{1}{\Box_-}\bar{\nabla}^2\bar{\varphi}_i, \Phi^i\right]^A, \end{aligned} \qquad (2.17)$$

where $\alpha, \lambda, \bar{\lambda}$ are arbitrary numerical parameters, and $\Box_+$, $\Box_-$ are standard symbols for Laplace-like operators in $\mathcal{N} = 1$ superspace. It is obvious that gauge conditions (2.17) are covariant with respect to background gauge transformations. Gauge conditions (2.17) may be regarded as a superfield generalization of R_ξ-type gauges (see [185, 211]) that are often used in spontaneously broken gauge theories. Since the Abelian background is the solution of classical equations of

[7] These methods are discussed in Section 3.

motion, no matter how gauge parameters are chosen. A gauge of the Fermi–DeWitt type, $\alpha = \lambda = 1$, well suits our purpose. If the parameters are chosen this way, the problem of calculation of mixed contributions with vector and chiral superfields in propagators may be evaded. Otherwise, for calculation of such contributions one would be forced to operate with involved expressions of the following type:

$$\begin{aligned}\mathrm{Tr}\ln\Bigg(&-\Box + iW^{\alpha}\nabla_{\alpha} + i\bar{W}^{\dot{\alpha}}\bar{\nabla}_{\dot{\alpha}} + M\\ &- \bar{X}\frac{1}{\Box_- - \mu\bar{\mu}}X\bar{\nabla}^2\nabla^2 - X\frac{1}{\Box_+ - \bar{\mu}\mu}\bar{X}\nabla^2\bar{\nabla}^2\\ &+ \bar{X}\frac{1}{\Box_- - \bar{\mu}\mu}\bar{\mu}\bar{X}\bar{\nabla}^2 + X\frac{1}{\Box_+ - \mu\bar{\mu}}\mu X\nabla^2\Bigg).\end{aligned} \tag{2.18}$$

It should be emphasized once more that not all global symmetries of classical action are necessarily retained in an explicit form in gauge theories at he quantum level (even in the absence of anomalies). The reason for this is related to the necessity of gauge fixing. It will be shown later that this fixing breaks certain symmetries (the breaking of classical conformal symmetry was discussed in [112]). This phenomenon is a fairly general one. In the case under consideration, gauge fixing (2.16) breaks global classical $\mathcal{N} = 4$ symmetry (2.5), (2.6), since this gauge is covariant only with respect to the transformations of $\mathcal{N} = 1$ supersymmetry. Therefore, one should expect that the calculated effective action is invariant only with respect to the appropriate quantum deformation of hidden transformations (2.5). The proper deformation can be found in each given order of the loop expansion.

After splitting fields into quantum and background parts (i.e., $e^{V_{tot}} = e^{\Omega}e^{gv}e^{\bar{\Omega}}$, $\Phi \to \Phi + \varphi$, $\bar{\Phi} \to \bar{\Phi} + \bar{\varphi}$, $Q \to Q + q$, $\tilde{Q} \to \tilde{Q} + \tilde{q}$, $\bar{Q} \to \bar{Q} + \bar{q}$, $\bar{\tilde{Q}} \to \bar{\tilde{Q}} + \bar{\tilde{q}}$), we can write down the quadratic part of classical action (2.4) and the term that fixes gauge (2.16):

$$\begin{aligned}S_{(2)} = -\frac{1}{2}\sum_{I<J}\int d^4x d^4\theta\big[&\mathcal{F}^{IJ}\mathbf{H}_{IJ}\mathcal{F}^{\dagger IJ}\\ &+ \bar{v}^{IJ}(O_V - M)_{IJ}v^{IJ}\big].\end{aligned} \tag{2.19}$$

Here $\mathcal{F} = (\bar{\varphi}, \varphi, \bar{q}, q, \bar{\tilde{q}}, \tilde{q})$, $\mathcal{F}^{\dagger} = (\varphi, \bar{\varphi}, q, \bar{q}, \tilde{q}, \bar{\tilde{q}})^T$,

$$\begin{aligned}M_{IJ} &= (\bar{\Phi}_{IJ}\Phi_{IJ} + \bar{Q}_{IJ}Q_{IJ} + \bar{\tilde{Q}}_{IJ}\tilde{Q}_{IJ}),\\ O_V &= \Box - iW^{\alpha}_{IJ}\nabla_{\alpha} - i\bar{W}^{\dot{\alpha}}_{IJ}\bar{\nabla}_{\dot{\alpha}},\end{aligned} \tag{2.20}$$

$W^{\alpha}_{IJ} = W^{\alpha}_I - W^{\alpha}_J$, $\bar{W}^{\dot{\alpha}}_{IJ} = \bar{W}^{\dot{\alpha}}_I - \bar{W}^{\dot{\alpha}}_J$ are the strengths of background fields lying in a Cartan subalgebra, and $\Phi_{IJ} = \Phi_I - \Phi_J$. The Weyl basis in the space of Hermitian traceless matrices of algebra $su(N)$ was used in the derivation of (2.19). In the case under study, we limit ourselves to gauge group $SU(N)$ broken to its maximal torus $U(1)^{N-1}$. The $I < J$ constraint emerges due to the fact that the quantum superfield components valued in a Cartan subalgebra do not interact with background fields and are thus fully disentangled.

Operator $\mathbf{H}$ depends on covariant derivatives and background fields. The exact form of its matrix is as follows:

$$\begin{pmatrix} G_+(\phi)\nabla^2\bar{\nabla}^2 & 0 & -\phi\bar{f}\frac{\nabla^2\bar{\nabla}^2}{\Box} & i\bar{v}\nabla^2 & -\phi f\frac{\nabla^2\bar{\nabla}^2}{\Box} & -i\bar{f}\nabla^2 \\ 0 & G_-(\phi)\bar{\nabla}^2\nabla^2 & iv\bar{\nabla}^2 & \bar{\phi}f\frac{\bar{\nabla}^2\nabla^2}{\Box} & -if\bar{\nabla}^2 & -\bar{\phi}v\frac{\bar{\nabla}^2\nabla^2}{\Box} \\ -f\bar{\phi}\frac{\nabla^2\bar{\nabla}^2}{\Box} & i\bar{v}\nabla^2 & G_+(f)\nabla^2\bar{\nabla}^2 & 0 & f\bar{v}\frac{\nabla^2\bar{\nabla}^2}{\Box} & i\bar{\phi}\nabla^2 \\ -iv\bar{\nabla}^2 & -\bar{f}\phi\frac{\bar{\nabla}^2\nabla^2}{\Box} & 0 & G_-(f)\bar{\nabla}^2\nabla^2 & i\phi\bar{\nabla}^2 & -\bar{f}v\frac{\bar{\nabla}^2\nabla^2}{\Box} \\ -v\bar{\phi}\frac{\nabla^2\bar{\nabla}^2}{\Box} & \bar{f}\nabla^2 & -v\bar{f}\frac{\nabla^2\bar{\nabla}^2}{\Box} & -i\bar{\phi}\nabla^2 & G_+(v)\nabla^2\bar{\nabla}^2 & 0 \\ if\bar{\nabla}^2 & -\bar{v}\phi\frac{\bar{\nabla}^2\nabla^2}{\Box} & -i\phi\bar{\nabla}^2 & -\bar{v}f\frac{\bar{\nabla}^2\nabla^2}{\Box} & 0 & G_-(v)\bar{\nabla}^2\nabla^2 \end{pmatrix}, \tag{2.21}$$

where

$$G_{\pm}(a) = 1 - \frac{(a\bar{a})}{\Box_{\pm}}, \quad \phi = \Phi_{IJ}, \quad \bar{\phi} = \bar{\Phi}_{IJ},$$

$$f = Q_{IJ}, \quad \bar{f} = \bar{Q}_{IJ}, \quad v = \tilde{Q}_{IJ}, \quad \bar{v} = \bar{\tilde{Q}}_{IJ},$$

and $\Box_{\pm}$ denote operators $\nabla^2\bar{\nabla}^2$ and $\bar{\nabla}^2\nabla^2$, respectively. Operators $\Box_{\pm}$ act in the space of chiral and antichiral superfields in the following way:

$$\nabla^2\bar{\nabla}^2 := \Box_+ = \Box - i\bar{W}^{\dot{\alpha}}\bar{\nabla}_{\dot{\alpha}} - \frac{i}{2}(\bar{\nabla}\bar{W}),$$

$$\bar{\nabla}^2\nabla^2 := \Box_- = \Box - iW^{\alpha}\nabla_{\alpha} - \frac{i}{2}(\nabla W).$$

The calculation of the second variational derivative of classical action generally yields a 7×7 matrix. However, the chosen gauge conditions (2.17) lead to partial diagonalization. The matrix assumes a block form $1\times 1 \oplus 6\times 6$ as a result, and the kinetic operator of vector fields is separated, thus simplifying the problem of calculation of functional traces. The chosen gauge parameters ensure that no vertex of interaction between quantum matter fields and quantum vector fields is present, but produce vertices of interaction between quantum chiral fields and ghosts.

Let us consider the structure of contributions of Faddeev–Popov ghosts to one-loop effective action. Ghost action $S_{\rm FP}$ for the term that fixes gauge (2.17) is

$$S_{\rm FP} = \mathrm{tr}\int d^8z\Bigg[(\bar{c}'c - c'\bar{c}) - \left(c'\left[\Phi^i, \frac{\lambda}{\Box_+}[\bar{c},\bar{\Phi}_i]\right] + \bar{c}'\left[\frac{\bar{\lambda}}{\Box_-}[c,\Phi^i], \bar{\Phi}_i\right]\right)\Bigg], \qquad (2.22)$$

which gives the following contribution of ghosts to effective action:

$$\ln \mathrm{Det}\mathbf{H}_{FP} = 2\sum_{I<J}\mathrm{Tr}\ln\begin{pmatrix} 0 & \left(1-\frac{M}{\Box_+}\right)\nabla^2\bar{\nabla}^2 \\ -\left(1-\frac{M}{\Box_-}\right)\bar{\nabla}^2\nabla^2 & 0\end{pmatrix}_{IJ}, \qquad (2.23)$$

where matrix M_{IJ} is defined in (2.20).

The result of integration in functional integral (2.15) over all quantum superfields is given by the formal representation through functional determinants

$$e^{i\Gamma} = \prod_{I<J}[\mathrm{Det}^{-1}(O_V - M)](\mathrm{Det}^{-1}\mathbf{H})(\mathrm{Det}^2\mathbf{H}_{FP}). \qquad (2.24)$$

Since superfields Φ and W_α belong to a Cartan subalgebra, only a half of roots should be taken into account when integrating over quantum fields, and the effective action acquires the form

$$\Gamma = \sum_{I<J}\Gamma_{IJ}.$$

Our next goal is to calculate functional determinant (2.24).

2.7. Calculation of Functional Traces and One-Loop Effective Action

This section presents the basic stages of calculation of functional traces of differential operators in a superspace that define the background-dependent contribution to effective action (2.24). It can be seen from (2.21) that if background superfields $Q, \tilde{Q}$ take zero values, matrix operator $\mathbf{H}$ includes only the background-dependent inverse propagators G_+ and G_- and the vertices of background fields Φ that interact with the hypermultiplet quantum fields. It should be noted that the form of $\mathbf{H}$ containing full inverse propagators is fixed completely by the choice of the R_ξ-type gauge (2.17).

At the first stage, we decompose matrix $\mathbf{H}$ into a sum of two matrices, $\mathbf{H} = \mathbf{H}_\Box + \mathbf{H}_\nabla$, where matrix $\mathbf{H}_\Box$ contains all blocks with $\nabla^2\bar{\nabla}^2$, $\bar{\nabla}^2\nabla^2$, and matrix $\mathbf{H}_\nabla$ contains only the blocks with $\bar{\nabla}^2$ and ∇^2. The logarithm of matrix $\mathbf{H}$ can be presented in the following way:

$$\ln\mathbf{H} = \ln\mathbf{H}_\Box + \ln(1 - \mathbf{H}_\Box^{-1}\mathbf{H}_\nabla).$$

Using the known Frobenius formula for the inversion of block-type matrices

$$H = \begin{pmatrix} A & B \\ C & D\end{pmatrix},$$

$$H^{-1}\begin{pmatrix} A^{-1} + A^{-1}BE^{-1}CA^{-1} & -A^{-1}BE^{-1} \\ -E^{-1}CA^{-1} & E^{-1}\end{pmatrix},$$

where $E = D - CA^{-1}B$, one can calculate the inverse of matrix $\mathbf{H}_\Box$:

$$\mathbf{H}_\Box^{-1} = \begin{pmatrix}
g_+(\phi)\frac{\nabla^2\bar{\nabla}^2}{\Box_+^2} & 0 & \frac{\phi\bar{f}}{\Box_{+M}}\frac{\nabla^2\bar{\nabla}^2}{\Box_+^2} & 0 & \frac{\phi\bar{v}}{\Box_{+M}}\frac{\nabla^2\bar{\nabla}^2}{\Box_+^2} & 0 \\
0 & g_-(\phi)\frac{\bar{\nabla}^2\nabla^2}{\Box_-^2} & 0 & \frac{\bar{\phi}f}{\Box_{-M}}\frac{\bar{\nabla}^2\nabla^2}{\Box_-^2} & 0 & \frac{\bar{\phi}v}{\Box_{-M}}\frac{\bar{\nabla}^2\nabla^2}{\Box_-^2} \\
\frac{f\bar{\phi}}{\Box_{+M}}\frac{\nabla^2\bar{\nabla}^2}{\Box_+^2} & 0 & g_+(f)\frac{\nabla^2\bar{\nabla}^2}{\Box_+^2} & 0 & \frac{f\bar{v}}{\Box_{+M}}\frac{\nabla^2\bar{\nabla}^2}{\Box_+^2} & 0 \\
0 & \frac{\bar{f}\phi}{\Box_{-M}}\frac{\bar{\nabla}^2\nabla^2}{\Box_-^2} & 0 & g_-(f)\frac{\bar{\nabla}^2\nabla^2}{\Box_-^2} & 0 & \frac{\bar{f}v}{\Box_{-M}}\frac{\bar{\nabla}^2\nabla^2}{\Box_-^2} \\
\frac{v\bar{\phi}}{\Box_{+M}}\frac{\nabla^2\bar{\nabla}^2}{\Box_+^2} & 0 & \frac{v\bar{f}}{\Box_{+M}}\frac{\nabla^2\bar{\nabla}^2}{\Box_+^2} & 0 & g_+(v)\frac{\nabla^2\bar{\nabla}^2}{\Box_+^2} & 0 \\
0 & \frac{\bar{v}\phi}{\Box_{-M}}\frac{\bar{\nabla}^2\nabla^2}{\Box_-^2} & 0 & \frac{\bar{v}f}{\Box_{-M}}\frac{\bar{\nabla}^2\nabla^2}{\Box_-^2} & 0 & g_-(v)\frac{\bar{\nabla}^2\nabla^2}{\Box_-^2}
\end{pmatrix},$$

where

$$g_{\pm}(\phi) := 1 + \frac{\phi\bar{\phi}}{\Box_{\pm M}}, \quad \Box_{\pm M} := \Box_{\pm} - M.$$

It can be seen that matrix M defined in (2.20) emerges naturally upon inversion. Surprisingly, the $\mathbf{H}_{\Box}^{-1}\mathbf{H}_{\nabla}$ product acquires a fairly simple form

$$\mathbf{H}_{\Box}^{-1}\mathbf{H}_{\nabla} = \begin{pmatrix} 0 & 0 & 0 & i\bar{v}\frac{\nabla^2}{\Box_-} & 0 & -i\bar{f}\frac{\nabla^2}{\Box_-} \\ 0 & 0 & iv\frac{\bar{\nabla}^2}{\Box_+} & 0 & -if\frac{\bar{\nabla}^2}{\Box_+} & 0 \\ 0 & -i\bar{v}\frac{\nabla^2}{\Box_-} & 0 & 0 & 0 & i\bar{\phi}\frac{\nabla^2}{\Box_-} \\ -iv\frac{\bar{\nabla}^2}{\Box_+} & 0 & 0 & 0 & i\phi\frac{\bar{\nabla}^2}{\Box_+} & 0 \\ 0 & i\bar{f}\frac{\nabla^2}{\Box_-} & 0 & -i\bar{\phi}\frac{\nabla^2}{\Box_-} & 0 & 0 \\ if\frac{\bar{\nabla}^2}{\Box_+} & 0 & -i\phi\frac{\bar{\nabla}^2}{\Box_+} & 0 & 0 & 0 \end{pmatrix}. \tag{2.25}$$

At the next stage, we turn to calculating matrix traces. Let us expand $\mathrm{Tr}\ln(1 - \mathbf{H}_{\Box}^{-1}\mathbf{H}_{\nabla})$ into a series in powers of $\mathbf{H}_{\Box}^{-1}\mathbf{H}_{\nabla}$. Nonzero contributions to the trace are produced only by even expansion powers that are grouped into the following expression:

$$\begin{aligned} &\mathrm{Tr}_{6\times6}\ln(1 - \ln\mathbf{H}_{\Box}^{-1}\mathbf{H}_{\nabla}) \\ &= \mathrm{Tr}\left[\ln\left(1 - \frac{M}{\Box_+}\right)\frac{\nabla^2\bar{\nabla}^2}{\Box_+}\right] + \mathrm{Tr}\left[\ln\left(1 - \frac{M}{\Box_-}\right)\frac{\bar{\nabla}^2\nabla^2}{\Box_-}\right], \end{aligned} \tag{2.26}$$

where matrix M is the same as in (2.20), and Tr stands for a functional trace. Let us then consider the trace of matrix $\ln\mathbf{H}_{\Box}$. Using the technique described above, we decompose matrix $\mathbf{H}_{\Box}$ into a sum of a diagonal matrix and all the rest (i.e., present it as $\mathbf{H}_{\Box} = \mathbf{H}_0 + \mathbf{\Delta}$). Hence,

$$\mathrm{Tr}\ln\mathbf{H}_{\Box} = \mathrm{Tr}\ln\mathbf{H}_0 + \mathrm{Tr}\ln(1 + \mathbf{H}_{\Box}^{-1}\mathbf{\Delta}). \tag{2.27}$$

Matrix $\mathbf{H}_0$ contains only the $\nabla^2\bar{\nabla}^2$ and $\bar{\nabla}^2\nabla^2$ operators in the case of zero background fields $\Phi, Q, \tilde{Q}$ and can thus be discarded. Matrix elements $\mathbf{H}_{\Box}^{-1}\mathbf{\Delta}$ are blocks with chiral $\frac{\nabla^2\bar{\nabla}^2}{\Box_+}$ and antichiral $\frac{\bar{\nabla}^2\nabla^2}{\Box_-}$ projectors. After a permutation of rows and columns, the trace of logarithm of matrix $1 + \mathbf{H}_0^{-1}\mathbf{\Delta}$ can be written as

$$\begin{aligned} &\mathrm{Tr}_{6\times6}\ln(1 + \mathbf{H}_0^{-1}\mathbf{\Delta}) \\ &= \mathrm{Tr}_{3\times3}\ln\left[1 - \begin{pmatrix} (\phi\bar{\phi}) & (\phi\bar{f}) & (\phi\bar{v}) \\ (f\bar{\phi}) & (f\bar{f}) & (f\bar{v}) \\ (v\bar{\phi}) & (v\bar{f}) & (v\bar{v}) \end{pmatrix}\frac{\nabla^2\bar{\nabla}^2}{\Box_+^2}\right] \\ &\quad + \left(\frac{\nabla^2\bar{\nabla}^2}{\Box_+^2} \to \frac{\bar{\nabla}^2\nabla^2}{\Box_-^2}\right). \end{aligned} \tag{2.28}$$

Direct calculations of matrix traces for the first terms of the Taylor series show that one can write

$$\begin{aligned} &\mathrm{Tr}_{6\times6}\ln(1 + \mathbf{H}_0^{-1}\mathbf{\Delta}) \\ &= \mathrm{Tr}\left[\ln(1 - \frac{M}{\Box_+})\frac{\nabla^2\bar{\nabla}^2}{\Box_+}\right] + \mathrm{Tr}\left[\ln\left(1 - \frac{M}{\Box_-}\right)\frac{\bar{\nabla}^2\nabla^2}{\Box_-}\right]. \end{aligned} \tag{2.29}$$

Taken together with (2.26), the latter expression yields

$$\begin{aligned} \ln\mathrm{Det}^{-1}\mathbf{H} = &-2\mathrm{Tr}\left[\ln\left(1 - \frac{M}{\Box_+}\right)\frac{\nabla^2\bar{\nabla}^2}{\Box_+}\right] \\ &- 2\mathrm{Tr}\left[\ln\left(1 - \frac{M}{\Box_-}\right)\frac{\bar{\nabla}^2\nabla^2}{\Box_-}\right]. \end{aligned} \tag{2.30}$$

The contribution of Faddeev–Popov ghosts is defined by (2.23). Having isolated and discarded expression $\ln\begin{pmatrix} 0 & \nabla^2\bar{\nabla}^2 \\ -\bar{\nabla}^2\nabla^2 & 0 \end{pmatrix}$, we obtain the following form of the contribution of ghosts to the effective action:

$$\begin{aligned} \ln\mathrm{Det}^2\mathbf{H}_{FP} = &\,2\mathrm{Tr}\left[\ln\left(1 - \frac{M}{\Box_-}\right)\frac{\bar{\nabla}^2\nabla^2}{\Box_-}\right] \\ &+ 2\mathrm{Tr}\left[\ln\left(1 - \frac{M}{\Box_+}\right)\frac{\nabla^2\bar{\nabla}^2}{\Box_+}\right], \end{aligned} \tag{2.31}$$

which coincides with (except for sign) expression (2.30). Therefore, the second and the third determinants in (2.24) cancel each other. This surprising cancellation of contributions of ghosts and chiral fields in the one-loop effective action of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory was first noted in [52] in calculations in harmonic superspace. However, this result holds true only for the background of constant chiral superfields.

Owing to the cancellation of (2.31) and (2.30), we can finally determine the full one-loop contribution to effective action (2.24) in a simple form which is determined exclusively by contributions from vector fields

$$\Gamma = i\sum_{I<J} \mathrm{Tr}\ln(O_V - M)_{IJ}, \tag{2.32}$$

while the whole background dependence is accommodated by matrix M. The expansion of the functional trace in powers of the strength of gauge fields for the $(O_V - M)_{IJ}$ operator in the above formula has already been calculated several times by different authors, but only in the case of a single chiral superfield (see [96, 144, 211, 213] and references therein). The theory with hypermultiplets differs only in the structure of matrix $M = (\bar{\Phi}\Phi + \bar{Q}Q + \bar{\tilde{Q}}\tilde{Q})$, which is defined in (2.20) and is invariant under the transformations of R-symmetry group of $\mathcal{N} = 4$ supersymmetry. Therefore, we can use the results obtained earlier and generalize them to the considered model just by replacing matrix M.

Functional trace (2.32) can be expressed as an expansion in powers of dimensionless superfield combinations Ψ and $\bar{\Psi}$ defined as

$$\bar{\Psi}^2 = \frac{1}{M^2}\nabla^2 W^2, \quad \Psi^2 \frac{1}{M^2}\bar{\nabla}^2\bar{W}^2. \tag{2.33}$$

In the constant background approximation, one can sum up these expressions and obtain the following correction to the full one-loop action (see [144] for details):

$$\Gamma = \frac{1}{8\pi^2}\int \mathrm{d}^8 z\int_0^\infty \mathrm{d}t t e^{-t}\frac{W^2\bar{W}^2}{M^2}\omega(t\Psi, t\bar{\Psi}),$$
$$\omega(t\Psi, t\bar{\Psi}) = \frac{\cosh(t\Psi)-1}{t^2\Psi^2} \tag{2.34}$$
$$\times\frac{\cosh(t\bar{\Psi})-1}{t^2\bar{\Psi}^2}\frac{t^2(\Psi^2-\bar{\Psi}^2)}{\cosh(t\Psi)-\cosh(t\bar{\Psi})}.$$

It is worth repeating that effective actions that take or do not take the contribution of hypermultiplet fields into account differ only in the structure of matrix M, which is defined in the present case by (2.20). In the component form, the expression for correction (2.34) to the one-loop effective action admits natural expansion of the Schwinger type in powers of F^2/M^2. This expansion does not contain an F^6-type term, which is attributable to the features of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. Function ω, which is introduced in (2.34) (see [144]), is expanded as

$$\omega(x, y) = \frac{1}{2} + \frac{x^2y^2}{4\cdot 5!} - \frac{5}{12\cdot 7!}(x^4y^2 + x^2y^4)$$
$$+\frac{1}{34500}(x^2y^6 + x^6y^2) + \frac{1}{86400}x^4y^4 + \dots. \tag{2.35}$$

Expression (2.35) allows one to expand effective action (2.34) into a series in powers of Ψ^2 and $\bar{\Psi}^2$:

$$\Gamma = \Gamma_{(0)} + \Gamma_{(2)} + \Gamma_{(3)} + \dots, \tag{2.36}$$

where term $\Gamma_{(n)}$ contains $c_{m,l}\Psi^{2m}\bar{\Psi}^{2l}$ with $m + l = n$. In the bosonic sector, this expansion corresponds to the expansion in powers of strength F, namely, $\Gamma_{(n)} \sim F^{4+2n}/M^{2+2n}$, $M = (\Phi\bar{\Phi} + f^{ia}f_{ia})$, where $\Phi, \bar{\Phi}$ and f^{ia} are physical bosonic fields of $\mathcal{N} = 2$ vector multiplet and hypermultiplet.

2.8. Transformation of $\mathcal{N} = 1$ Supersymmetric Effective Action into a Manifestly $\mathcal{N} = 2$ Supersymmetric Form

Effective action (2.34) and its expansion (2.36) are written in terms of $\mathcal{N} = 1$ superfields. In the present section, Eq. (2.36) is rewritten in a manifestly $\mathcal{N} = 2$ supersymmetric form. To this end, we single out the $\mathcal{N} = 1$ superfield argument $X = -\frac{\bar{Q}Q + \bar{\tilde{Q}}\tilde{Q}}{\bar{\Phi}\Phi}$ (defined in (2.2) in terms of $\mathcal{N} = 2$ superfields) in matrix M defined in (2.20), represent M as $M = \Phi\bar{\Phi}(1 - X)$, and expand denominators $(1/M)^k$ appearing in (2.34) in a power series in X. This expansion yields the following expression for the general term of series (2.36):

$$\int d^8 z\frac{W^2\bar{W}^2}{(\Phi\bar{\Phi})^{2(m+l+k+1)}}$$
$$\times(\nabla^2 W^2)^m(\bar{\nabla}^2\bar{W}^2)^l\left[-(\bar{Q}Q + \bar{\tilde{Q}}\tilde{Q})\right]^k. \tag{2.37}$$

Making use of the fact that $\int \mathrm{d}^{12}z = \int \mathrm{d}^8 z(\nabla_2)^2(\bar{\nabla}^2)^2$ and the definition of $\mathcal{N} = 1$ projections of $\mathcal{N} = 2$ vector multiplet on the mass shell as $\mathcal{W}| = \Phi$, $\nabla_{2\alpha}\mathcal{W}| = -W_\alpha$, $\nabla_2^2\mathcal{W}| = 0$, we then transform the general term into $\mathcal{N} = 2$ form (2.2). One should keep in mind that this procedure features ambiguities off the mass shell even in the absence of hypermultiplet fields (see [107]); however, these possible ambiguities are irrelevant to our purpose.

The known nonholomorphic potential appears as the first term in the expansion of effective action (2.34) in derivatives (see (2.42)). Using the $\mathcal{N} = 1$ expression on background (2.3), one can uniquely rewrite it in $\mathcal{N} = 2$ supersymmetric form. This is the only term that is automatically invariant under the transformations of $\mathcal{N} = 4$ supersymmetry, since it contains no derivatives of hypermultiplet superfields and $\mathcal{N} = 2$ superfield strength. It is less clear how the $\mathcal{N} = 4$ invariance can be restored in the other terms of expansion of the effective action. This requires further analysis.

All calculations of the effective action were performed on constant background (2.3). Unfortunately, this background is insufficient for the restoration of $\mathcal{N}=2$ supersymmetric form, since the derivatives of $\mathcal{N}=1$ hypermultiplet fields need to be taken into account. The procedure of restoration of $\mathcal{N}=2$ supersymmetric expressions based on their corresponding $\mathcal{N}=1$ reductions involves, first and foremost, the restoration of $\mathcal{N}=2$ integration measure $\int d^{12}z = \int d^8z(\nabla_2)^2(\bar{\nabla}^2)^2$. Thus, in order to restore an integral over $\mathcal{N}=2$ superspace from that over $\mathcal{N}=1$ superspace, one should extract derivatives $(\nabla_2)^2(\bar{\nabla}^2)^2$ from the original superfield expression under the $\mathcal{N}=1$ integral. For gaining such total derivatives in the expression under the integral in (2.37), one should manually add all missing terms containing $\nabla_{\alpha j}q^{ia}$ derivatives with correct numerical coefficients to the initial $\mathcal{N}=1$ superfield expression, since such terms cannot emerge in the course of calculation. If the effective action would be calculated on the proper background (2.13) instead of a special (2.3) one, these terms would emerge automatically. In this case, one could at once single out derivatives $\nabla_2^2\bar{\nabla}_2^2$ in the expression under the $\mathcal{N}=1$ integral and transform the latter into an integral over $\mathcal{N}=2$ superspace.

Some rather evident assumptions regarding the properties of effective action are used below. The effective action should be manifestly $\mathcal{N}=2$ supersymmetric; consequently, each term of its expansion should be written as an integral over $\mathcal{N}=2$ superspace of a function that depends on $\mathcal{N}=2$ superfield strengths, hypermultiplet superfields, and their spinor derivatives. Therefore, integrating by parts in integrals over $\mathcal{N}=2$ superspace and step-by-step properly transforming the terms of expansion in derivatives, we transfer all derivatives acting on hypermultiplet superfields from these superfields on $\mathcal{N}=2$ superfield strengths, and then perform the reduction to $\mathcal{N}=1$ superfields. The obtained result demonstrates that all terms of the expansion in derivatives can be written in a form similar to the series general term $\Gamma_{(n)}$ that is defined in (2.36) (i.e., without the derivatives of hypermultiplet superfields). Thus, one can start with contributions to the effective action that are written in an $\mathcal{N}=1$ supersymmetric form and then transform the obtained expressions into the corresponding $\mathcal{N}=2$ supersymmetric form. It should also be kept in mind that the terms of expansion in derivatives in the limit of zero hypermultiplet fields are expressed through $\mathcal{N}=2$ superconformal scalars [144]

$$\bar{\mathbf{\Psi}}^2 = \frac{1}{\bar{\mathcal{W}}^2}\nabla^4 \ln \mathcal{W}, \quad \mathbf{\Psi}^2 = \frac{1}{\mathcal{W}^2}\bar{\nabla}^4 \ln \bar{\mathcal{W}}. \tag{2.38}$$

Therefore, one should look for such a dependence on hypermultiplets that is compatible with these features.

We then demonstrate how the described procedure can be used to write functionals $\Gamma_{(0)}, \Gamma_{(2)}, \Gamma_{(3)}, \ldots$ (2.36) in terms of $\mathcal{N}=2$ superfields. Let us consider functional $\Gamma_{(0)} = \frac{1}{(4\pi)^2}\int d^8z \frac{W^2\bar{W}^2}{M^2}$ (proportional to F^4) and bring it to the form (2.37) using expansion $\frac{1}{(1-X)^2} = \sum_{k=0}^{\infty}(k+1)X^k$:

$$\Gamma_{(0)} = \frac{1}{(4\pi)^2}\int d^8z\left(\frac{W^2\bar{W}^2}{\Phi^2\bar{\Phi}^2} + \sum_{k=1}^{\infty}(k+1) \times \frac{W^2\bar{W}^2}{\Phi^{2+k}\bar{\Phi}^{2+k}}\cdot[-(\bar{Q}Q+\bar{\tilde{Q}}\tilde{Q})^k]\right). \tag{2.39}$$

It is natural to identify the quadratic combination of $\mathcal{N}=1$ superfields $(\bar{Q}Q+\bar{\tilde{Q}}\tilde{Q})$ with $\mathcal{N}=1$ projection of the quadratic combination of hypermultiplet $q^{ia}q_{ia}$. This identification can be verified through a direct comparison of components. We then use relations

$$\begin{aligned}\nabla_2^2 \ln \mathcal{W}\big| &= -\frac{W^\alpha W_\alpha}{\Phi^2} + \ldots,\\ \nabla_2^2\frac{1}{\mathcal{W}^m}\bigg| &= \frac{m(m+1)}{\Phi^m}\frac{W^\alpha W_\alpha}{\Phi^2} + \ldots,\end{aligned} \tag{2.40}$$

where ellipses denote the terms with Φ derivatives that can be ignored in an analysis on the mass shell. The $\mathcal{N}=1$ expression under the integral (2.39) can then be written in terms of superfields of $\mathcal{N}=2$ vector multiplet and hypermultiplet:

$$\nabla_2^2 \ln \mathcal{W}\bar{\nabla}_2^2 \ln \bar{\mathcal{W}} + \sum_{k=1}^{\infty}\frac{1}{k^2(k+1)}\nabla_2^2\frac{1}{\mathcal{W}^k}\bar{\nabla}_2^2\frac{1}{\bar{\mathcal{W}}^k}\cdot(-q^{ia}q_{ia}) + \ldots. \tag{2.41}$$

Here ellipses denote the terms containing derivatives of hypermultiplet superfields of the following kind:

$$\nabla_2^\alpha\frac{1}{\mathcal{W}^k}\nabla_{2\alpha}(-q^{ia}q_{ia})\bar{\nabla}_2^2\frac{1}{\bar{\mathcal{W}}^k},$$
$$\frac{1}{\mathcal{W}^k}\nabla_2^2(-q^{ia}q_{ia})\bar{\nabla}_2^2\frac{1}{\bar{\mathcal{W}}^k}.$$

In accordance with the procedure described above, they should be added to the integral over $\mathcal{N}=1$ superspace (2.37) in order to restore full $\mathcal{N}=2$ integration measure $\nabla_2^2\bar{\nabla}_2^2$. The following expression is obtained as a result:

$$\Gamma_{(0)} = \frac{1}{(4\pi)^2} \times \int d^{12}z\left[\ln \mathcal{W}\ln \bar{\mathcal{W}} + \sum_{k=1}^{\infty}\frac{1}{k^2(k+1)}X^k\right], \tag{2.42}$$

where superfield $X = \left(-\frac{q^{ia}q_{ia}}{\mathcal{W}\bar{\mathcal{W}}}\right)$ is defined in (2.2). The second term in (2.42) can be transformed into form (2.2) with the use of the Euler dilogarithm expansion into a power series and the relation $\frac{1}{k^2(k+1)} = \frac{1}{k^2} - \frac{1}{k} + \frac{1}{k+1}$. It becomes clear after these transformations that expression (2.42) is nothing but the effective Lagrangian (2.2) derived in [146].

The $\mathcal{N} = 2$ representation of the next term ($\sim F^8$) of series (2.36) is derived using (2.40) and the expansion of $(1/M)^6$ in X. Direct analysis (similar to the one performed for the first term) yields the following expression for $\Gamma_{(2)}$ in (2.36):

$$\Gamma_{(2)} = \frac{1}{2(4\pi)^2}\int d^{12}z\mathbf{\Psi}^2\bar{\mathbf{\Psi}}^2 \times\left[\frac{1}{36}+\frac{1}{5!}\sum_{k=1}^{\infty}\frac{(k+5)(k+4)(k+1)}{(k+3)(k+2)}X^k\right]. \tag{2.43}$$

The X-independent part of this term was obtained in [144]. The series in (2.43) is summed to the exact expression in the following way:

$$\sum_{k=1}^{\infty}\frac{(k+5)(k+4)(k+1)}{(k+3)(k+2)}X^k = \frac{1}{(1-X)^2} + \frac{4}{(1-X)} + \frac{6X-4}{X^3}\ln(1-X) + 4\frac{X-1}{X^2} - \frac{10}{3}. \tag{2.44}$$

Applying the same procedure to the third term ($\sim F^{10}$) in (2.36), one obtains

$$\Gamma_{(3)} = -\frac{5}{6(4\pi)^2}\int d^{12}z(\mathbf{\Psi}^4\bar{\mathbf{\Psi}}^2 + \mathbf{\Psi}^2\bar{\mathbf{\Psi}}^4) \times\left[-\frac{1}{5!}+\frac{1}{7!}\sum_{k=1}^{\infty}(k+7)(k+6)(k+1)X^k\right], \tag{2.45}$$

where the sum on the right-hand side yields

$$\sum_{k=1}^{\infty}(k+7)(k+6)(k+1)X^k = \frac{2X}{(1-X)^4}(56-116X+84X^2-21X^3). \tag{2.46}$$

Thus, we have obtained hypermultiplet-dependent additions to $\Gamma_{(0)}$, $\Gamma_{(2)}$, and $\Gamma_{(3)}$ for the effective action derived in [144] in the $\mathcal{N} = 2$ vector multiplet sector. It is evident that each term of the expansion of effective action (2.36) can be transformed into an $\mathcal{N} = 2$ supersymmetric form. For example, the X-dependent part of the fourth term ($\sim F^{12}$) in (2.36) comprises two parts. The first one is expressed as

$$\Gamma_{(4_1)} = \frac{1}{(4\pi)^2}\frac{1}{17250}\int d^{12}z(\mathbf{\Psi}^2\bar{\mathbf{\Psi}}^6 + \mathbf{\Psi}^6\bar{\mathbf{\Psi}}^2) \times\frac{12X}{(1-X)^6}(450-1545X+2284X^2 - 1779X^3+720X^4-120X^5), \tag{2.47}$$

and the second one is given by

$$\Gamma_{(4_2)} = \frac{1}{5\cdot 6!(4\pi)^2}\int d^{12}z\mathbf{\Psi}^4\bar{\mathbf{\Psi}}^4\left[\frac{12(5X-4)}{X^5}\ln(1-X) - \frac{1}{5X^4(1-X)^6}(240-1620X+4610X^2 - 7120X^3+6363X^4-4878X^5+6135X^6 - 7560X^7+5670X^8-2268X^9+378X^{10})\right]. \tag{2.48}$$

Thus, we arrive at the conclusion that the outlined technique for the restoration of $\mathcal{N} = 2$ superfield form of effective action (2.34) for $\mathcal{N} = 4$ supersymmetric Yang–Mills theory written initially in terms of $\mathcal{N} = 1$ superfields can be applied to any term of expansion (2.36). In order to do that, one should add, to the terms in (2.36), new terms with hypermultiplet superfields. This technique does not guarantee that the restored $\mathcal{N} = 2$ form bears $\mathcal{N} = 4$ invariance. This should be verified separately. In this analysis, the $\mathcal{N} = 1$ superfield representation of hidden $\mathcal{N} = 4$ supersymmetry transformations (2.5), (2.6) or the $\mathcal{N} = 2$ superfield form (2.9) of these transformations in harmonic superspace can be used.

The low-energy effective action of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory should be self-dual [124] and invariant with respect to superconformal transformations (possibly deformed, see [112]). However, these requirements are insufficient to fix uniquely the form of the $\mathcal{N} = 4$ effective action. So far, we have used the constant field approximation (2.3) that implies the absence of derivatives of hypermultiplet fields. This approximation is sufficient for restoring manifest $\mathcal{N} = 2$ supersymmetry of the effective action from its $\mathcal{N} = 1$ form (2.34) which was obtained by using the background (2.3) involving $\mathcal{N} = 2$ gauge multiplet scalars [144]. However, the calculation of $\mathcal{N} = 4$ supersymmetric effective action requires using the background with $Dq \neq 0$. The terms with derivatives Dq^{ia}, which are needed to restore the $\mathcal{N} = 4$ supersymmetric form of effective action, can also be obtained through algebraic analysis similar to the one performed in [146].

2.9. Analysis of the Deformation of $\mathcal{N} = 4$ Supersymmetry for the Effective Action in $\mathcal{N} = 2$ Superspace

Global and local (in supergravity) $\mathcal{N} = 2$ supersymmetries off the mass shell are realized linearly on physical fields and on an infinite set of auxiliary fields. Supersymmetry transformations do not depend on the specific form of action in the theory in this case. However, symmetry transformations are realized nonlinearly on the mass shell. When obtaining such higher-derivative contributions to the effective action that satisfy the requirement of preservation of extended supersymmetry, one should deform classical transformations systematically and self-consistently and, at the same time, construct supersymmetrically invariant terms with higher-order corrections to the action:

$$\left(\delta_0 + \sum_n \delta_n\right)\left(S_0 + \sum_n S_n\right) = 0.$$

Here δ_0 are classical supersymmetry transformations, S_0 is the classical action, and δ_n, S_n are quantum deformations and higher-order corrections to the action, respectively. It is hardly possible to determine the full derivative dependence of the effective action in a closed form. The only thing one can do in this situation is to write down, relying on the known particular results, all the supersymmetric invariants and deformed transformations with a given number of derivatives. The problem of calculating the full $\mathcal{N} = 4$ supersymmetric invariant for the leading potential $\propto F^4$ in the sector of $\mathcal{N} = 2$ superfield strength was solved in [147] (see [52, 107, 145] for a review of recent progress in this field).

Many different approaches to the construction of supersymmetric corrections with higher derivatives to string effective action are known. An $\mathcal{N} = 3$ extension of the Abelian $D = 4$ Born–Infeld action off the mass shell was found in [124] proceeding from the action of supersymmetric Maxwell theory in $\mathcal{N} = 3$ harmonic superspace [51]. The $\mathcal{N} = 3$ superfield strength contains a specific combination of tensor auxiliary fields and the gauge field strength. The nonlinearity in the ordinary gauge field strength emerges in the component Lagrangian as a result of elimination of these auxiliary fields by nonlinear equations of motion. This mechanism of derivation of the bosonic Born–Infeld Lagrangian from supersymmetric action differs from the case of $\mathcal{N} = 1$, $D = 4$ supersymmetric Born–Infeld theory in which each term of the expansion of bosonic action in powers of the gauge potential strength is supersymmetrized independently.

In this section, we present one possible self-consistent way of finding hypermultiplet-dependent complements and appropriately deformed hidden supersymmetry transformations which are needed for ensuring manifest $\mathcal{N} = 4$ supersymmetry of the next-to-leading terms of the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory effective action. The method proposed may prove useful in solving the problem of construction of invariants with higher derivatives, since the direct search for these invariants deals with a great number of alternative options.

In what follows, we focus on the problem of invariance of the F^6 term in the effective action of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory under the transformations of hidden $\mathcal{N} = 2$ supersymmetry.

In order to construct next-to-leading terms in the derivative expansion of the effective action, such that they are invariant with respect to hidden $\mathcal{N} = 2$ transformations and involve dependence on all fields of the supersymmetric $\mathcal{N} = 4$ Yang–Mills field multiplet, one can use the Noether procedure for classical supersymmetry transformations defined up to surface terms and free equations of motions:

$$\begin{gathered}\delta_0\mathcal{W} = \frac{1}{2}\bar{\varepsilon}^{\dot{\alpha}a}\bar{D}^-_{\dot{\alpha}}q^+_a, \quad \delta_0\bar{\mathcal{W}} = \frac{1}{2}\varepsilon^{\alpha a}D^-_{\alpha}q^+_a,\\ \delta_0 q^{\pm}_a = \frac{1}{4}(\varepsilon^{\alpha}_a D^{\pm}_{\alpha}\mathcal{W} + \bar{\varepsilon}^{\dot{\alpha}}_a\bar{D}^{\pm}_{\dot{\alpha}}\bar{\mathcal{W}}).\end{gathered} \tag{2.49}$$

Let us discuss the possible hypermultiplet completions for the $\propto F^6$-type two-loop term which was obtained in [145] using the supergraph techniques in harmonic superspace:

$$\begin{aligned}\Gamma_{(6|0)} &= c_2\int d^{12}z\left[\frac{1}{\bar{\mathcal{W}}^2}\ln\mathcal{W}D^4\ln\mathcal{W}\right.\\ &\left.+\frac{1}{\mathcal{W}^2}\ln\bar{\mathcal{W}}\bar{D}^4\ln\bar{\mathcal{W}}\right] = c_2\int d^{12}zL_{(6|0)} + c.c.,\end{aligned} \tag{2.50}$$

where $c_2 = N^2 g^2_{YM}\dfrac{1}{48\cdot(4\pi)^4}$ and $D^4 = (D^+)^2(D^-)^2$, $\bar{D}^4 = (\bar{D}^2)^2(\bar{D}^-)^2$. Expression (2.50) has two parts with fourth powers of different spinor derivatives (D^4 and $\bar{D}^4$). These parts can be considered separately, since transformations (2.49) do not mix them. The variation of the first part of (2.50) induced by transformations (2.49) is given by

$$\begin{aligned}\delta_0\mathcal{L}_{(6|0)} &= -\frac{2q^{+a}}{\bar{\mathcal{W}}^3\mathcal{W}}(\bar{\varepsilon}^{\dot{\alpha}}_a\bar{D}^-_{\dot{\alpha}}\bar{\mathcal{W}}\\ &+\varepsilon^{\alpha}_a D^-_{\alpha}\mathcal{W})D^4\ln\mathcal{W} + \frac{q^{+a}\varepsilon^{\alpha}_a D^-_{\alpha}W}{\mathcal{W}\bar{\mathcal{W}}^3}D^4\ln\mathcal{W}.\end{aligned} \tag{2.51}$$

The problem consists in that classical variations (2.51) contain terms that are nonsymmetric with respect to the $\varepsilon \leftrightarrow \bar{\varepsilon}$ substitution.

Next, we consider classical transformations δ_0 defined in (2.49) together with their deformations $\delta_{(n|k)}$. Full deformed transformations are sought in the form of an expansion in powers of $D, \bar{D}$ and superfield $X = \dfrac{-2q^{+a}q^-_a}{\mathcal{W}\bar{\mathcal{W}}}$ (i.e., $\delta = \delta_0 + \delta_1(D^4) + \delta_2(D^8) + \ldots$). The subscript k denotes the degree in X (i.e., $\delta_1 = \sum \delta_{(1|k)}$). Let us define the first complement to (2.50):

$$\mathcal{L}_{(6|1)} = d_1\left[X\frac{1}{\bar{\mathcal{W}}^2}D^4\ln\mathcal{W} + X\frac{1}{\mathcal{W}^2}\bar{D}^4\ln\bar{\mathcal{W}}\right]. \quad (2.52)$$

Its variation $\delta_0^{(q)}\mathcal{L}_{(6|1)}$ with respect to $q^{\pm}$ cancels the first term in (2.51) under the choice $d_1 = -2$; however, the second term in (2.51) is not cancelled out. Since the structure of functionals (2.50) is nonsymmetric with respect to the $\mathcal{W} \leftrightarrow \bar{\mathcal{W}}$ substitution, while the $\delta^{(q)}$-variation is symmetric, a remainder (the difference between variations (2.50) and (2.51)) will be produced at each step of the variational procedure. One-loop term $\Gamma_{(4)} \propto F^4$ with the known complement to it [146] can be used to cancel this remainder:

$$\Gamma_{(4)} = c_1\int d^{12}z \times\left(\ln\mathcal{W}\ln\bar{\mathcal{W}} + \frac{1}{2}X + \frac{1}{4\cdot 3}X^2 + \ldots\right), \quad (2.53)$$

where $c_1 = N\frac{1}{(4\pi)^2}$. It is known that this term is not renormalized either by higher loops or by instanton corrections. Let us assume that the classical transformations of hidden symmetry are deformed as

$$\begin{aligned}\delta_{(1|0)}\mathcal{W} &= \frac{\bar{A}}{2}\bar{\varepsilon}^{\dot{\alpha}a}\bar{D}^-_{\dot{\alpha}}q^+_a\frac{1}{\mathcal{W}^2}\bar{D}^4\ln\bar{\mathcal{W}},\\ \delta_{(1|0)}\bar{\mathcal{W}} &= \frac{A}{2}\varepsilon^{\alpha a}D^-_{\alpha}q^+_a\frac{1}{\bar{\mathcal{W}}^2}D^4\ln\mathcal{W},\end{aligned} \quad (2.54)$$

$$\begin{aligned}\delta_{(1|0)}q^{\pm}_a = \frac{1}{4}\bigg(&B\varepsilon^{\alpha}_a D^{\pm}_{\alpha}\mathcal{W}\frac{1}{\bar{\mathcal{W}}^2}D^4\ln\mathcal{W}\\ &+ \bar{B}\bar{\varepsilon}^{\dot{\alpha}}_a\bar{D}^{\pm}_{\dot{\alpha}}\bar{\mathcal{W}}\frac{1}{\mathcal{W}^2}\bar{D}^4\ln\bar{\mathcal{W}}\bigg).\end{aligned} \quad (2.55)$$

If conditions

$$c_2 + c_1\frac{(\bar{A}-\bar{B})}{2} = 0, \quad c_2 + c_1\frac{(A-B)}{2} = 0 \quad (2.56)$$

for the coefficients introduced in (2.54) and (2.55) are satisfied, the deformed variation of the first two terms in $\Gamma_{(4)}$ cancels out the last term in (2.51). The variation of $\mathcal{W}$ in the first complement $\delta_0^{(\mathcal{W})}\mathcal{L}_{(6|1)}$ (2.52) under classical transformations (2.49) is given by

$$\begin{aligned}&\delta_0^{(\mathcal{W})}\mathcal{L}_{(6|1)}\\ &= 4\cdot\frac{5}{3}\frac{q^+q^-}{\mathcal{W}^2\bar{\mathcal{W}}^4}q^{+a}(\bar{\varepsilon}^{\dot{\alpha}}_a\bar{D}^-_{\dot{\alpha}}\bar{\mathcal{W}} + \varepsilon^{\alpha}_a D^-_{\alpha}\mathcal{W})D^4\ln\mathcal{W}\\ &\quad+\frac{4}{3}\frac{-2q^+q^-}{\mathcal{W}^2\bar{\mathcal{W}}^4}q^{+a}\varepsilon^{\alpha}_a D^-_{\alpha}\mathcal{W}D^4\ln\mathcal{W}\\ &\quad+\frac{4}{3}\frac{\bar{\varepsilon}^{\dot{\alpha}}_a\bar{D}^+_{\dot{\alpha}}q^{-a}}{\mathcal{W}^2\bar{\mathcal{W}}^3}(q^+D^+q^-)\frac{1}{16}D^+_{\alpha}D^{-2}\ln\mathcal{W}.\end{aligned} \quad (2.57)$$

Let us then introduce the second complement:

$$\begin{aligned}\mathcal{L}_{(6|2)} = d_2\bigg(&X^2\cdot\frac{1}{\bar{\mathcal{W}}^2}D^4\ln\mathcal{W}\\ &+ X^2\cdot\frac{1}{\mathcal{W}^2}\bar{D}^4\ln\bar{\mathcal{W}}\bigg).\end{aligned} \quad (2.58)$$

Its variation with respect to $q^{\pm}$ coincides exactly with the first term of (2.57) and cancels out the variation induced by $\bar{\varepsilon}$ if $d_2 = -\frac{5}{3}$ is chosen. At the same time, a part of variation (2.57) is retained. In order to cancel this remainder, one should consider the variation of action (2.53) with respect to the following deformed transformations:

$$\begin{aligned}\delta_{(1|1)}\bar{\mathcal{W}} &= \frac{A_1}{2}\cdot X\varepsilon^{\alpha a}D^-_{\alpha}q^+_a\cdot\frac{1}{\bar{\mathcal{W}}^2}D^4\ln\mathcal{W},\\ \delta_{(1|1)}\mathcal{W} &= \frac{\bar{A}_1}{2}\cdot X\bar{\varepsilon}^{\dot{\alpha}a}\bar{D}^-_{\dot{\alpha}}q^+_a\cdot\frac{1}{\mathcal{W}^2}\bar{D}^4\ln\bar{\mathcal{W}},\end{aligned} \quad (2.59)$$

$$\begin{aligned}\delta_{(1|1)}q^-_a = &\frac{B_1}{4}\cdot X\varepsilon^{\alpha}_a D^-_{\alpha}\mathcal{W}\frac{1}{\bar{\mathcal{W}}^2}D^4\ln\mathcal{W}\\ &+\frac{\bar{B}_1}{4}\cdot X\bar{\varepsilon}^{\dot{\alpha}}_a\bar{D}^-_{\dot{\alpha}}\bar{\mathcal{W}}\frac{1}{\mathcal{W}^2}\bar{D}^4\ln\bar{\mathcal{W}}.\end{aligned} \quad (2.60)$$

Let us apply these deformed transformations for the calculation of variation of expression (2.53): the first term is varied by $\delta^{(\bar{\mathcal{W}})}_{(1|1)}$; the second one, by $\delta^{(\bar{\mathcal{W}})}_{(1|0)}$ and $\delta^{(q)}_{(1|1)}$; and the third one, by $\delta^{(q)}_{(1|0)}$. The variation part obtained this way takes the following form:

$$\begin{aligned}\delta\mathcal{L}_{(4)} = c_1&\left(-\frac{2}{3}A_1 + B_1 + \frac{B-A}{3}\right)\\ &\times(q^+q^-)q^{+a}\varepsilon^{\alpha}_a D^-_{\alpha}\mathcal{W}\frac{D^4\ln\mathcal{W}}{W^2\bar{\mathcal{W}}^4}.\end{aligned} \quad (2.61)$$

The requirement of mutual cancellation of the second term in $\Delta\mathcal{L}_{(6|1)}$ (2.57) and the corresponding term in $\delta\mathcal{L}_{(4)}$ (2.61) leads to the following equations:

$$\begin{aligned}4\frac{2}{3}c_2 &= c_1\left(\frac{B-A}{3} + B_1 - \frac{2}{3}A_1\right),\\ c_1 &= \left(B_1 - \frac{2}{3}A_1\right)c_2.\end{aligned} \quad (2.62)$$

These relations define the coefficients of expansion of the deformed supersymmetry transformations in powers of X. The values of A and B are chosen arbitrarily; certain additional requirements are needed to get rid of this ambiguity. In order to cancel out the second part in (2.57), we introduce a new type of complement

$$\mathcal{L}'_{(6|1)} = -\frac{1}{3}\cdot X\cdot\frac{q^+D^{+\alpha}q^-}{\mathcal{W}\bar{\mathcal{W}}}\frac{1}{\bar{\mathcal{W}}^2}D^+_{\alpha}D^{-2}\ln\mathcal{W}$$

and analyze its variation with respect to transformations (2.49). One part of the transformed expression for this variation $\sim D^4\ln W$ cancels the second line in

(2.57); however, complete cancellation requires incorporating additional structures into both the action and the hidden symmetry transformations. In what follows we will show which terms should be added to the action and the hidden supersymmetry transformations in order to secure invariance up to the terms of order F^8.

The above example demonstrates that the complement to $\mathcal{L}_{(6|0)} \propto D^4$ (2.50) is basically defined by classical transformations (2.49) induced by parameter $\bar{\varepsilon}$, and the discrepancy associated with ε is removed by a proper modification of hidden symmetry transformations $\delta_{(1|n)} \propto X^n$ in each order. Thus, the problem is split into two separate tasks. This is the basic idea behind the proposed method for constructing expressions which contain derivatives of the vector multiplet strengths and are invariant with respect to the transformations of hidden $\mathcal{N} = 2$ supersymmetry.

Let us now try to determine the leading F^6-type term in the full effective action using the method described above. To this end, one should choose a general term in the series of complements to (2.50) of the form

$$\Gamma_{(6|n)} = d_n \int d^{12}z \left(\frac{-2q^+q^-}{\mathcal{W}\bar{\mathcal{W}}}\right)^n \frac{1}{\bar{\mathcal{W}}^2} D^4 \ln W + c.c. \quad (2.63)$$

Classical variation $\delta_0 \mathcal{L}_{(6|n)}$ induced by parameter $\bar{\varepsilon}$ for terms $\propto D^4 \ln \mathcal{W}$ is given by

$$\left[-d_n \cdot n \cdot \frac{(-2q^+q^-)^{n-1}}{\mathcal{W}^n \bar{\mathcal{W}}^{n+2}} + d_n \cdot \frac{n(n+4)}{n+2} \cdot \frac{(-2q^+q^-)^n}{\mathcal{W}^{n+1}\bar{\mathcal{W}}^{n+3}}\right] \quad (2.64)$$
$$\times (q^{+a}\bar{\epsilon}^{\dot{\alpha}}_a \bar{D}^-_{\dot{\alpha}} \bar{\mathcal{W}}) D^4 \ln \mathcal{W}.$$

The requirement of cancellation of variations $\delta_0 \mathcal{L}_{(6|n)}$ and $\delta_0 \mathcal{L}_{(6|n+1)}$ is satisfied if

$$d_n = d\frac{(n+2)(n+3)}{n}. \quad (2.65)$$

Summing all complements $\mathcal{L}^q_{(6)} = \sum_{n=0}^{\infty} \mathcal{L}_{(6|n)}(X)$, one obtains

$$\Gamma^q_{(6)} = -\frac{c_2}{6}\int d^{12}z$$
$$\times \left[\frac{X}{(1-X)^2} + \frac{5X}{1-X} - 6\ln(1-X)\right]\frac{1}{\bar{\mathcal{W}}^2} D^4 \ln \mathcal{W}. \quad (2.66)$$

This result is not complete in the sense that contributions involving the derivatives of hypermultiplet fields should be added to it. However, the general term with derivatives $q^+D^+q^-$ can be determined. Let us introduce the following new type of complement:

$$\mathcal{L}'_{(6|n)} = p_n \frac{(-2q^+q^-)^n}{\mathcal{W}^{n+1}\bar{\mathcal{W}}^{n+3}} q^+ D^+_\alpha q^- D^{+\alpha} D^{-3} \ln \mathcal{W}. \quad (2.67)$$

The required cancellation of its variation $\delta^q_0(\bar{\varepsilon})$ and the corresponding term in variation (2.63) leads to an inhomogeneous recurrence relation for p_n with the following solution:

$$p_n = -\frac{1}{6} \cdot \frac{(n+2)(n+3)}{n+1} - \frac{1}{12} \cdot \frac{(n+2)^2(n+3)}{n+7} H^{(2)}_{n+1}, \quad (2.68)$$

where $H^{(2)}_n = \sum_{k=1}^{n} \frac{1}{k^6}$ are harmonic numbers [n, 2]. For the coefficients of a series with the general term of a different type, $\mathcal{L}''_{(6|n)} = h_n X^n (q^+D^+q^-)^2 D^{-2} \ln \mathcal{W}$, another type of recurrence relation between the coefficients p_n, d_n, h_n arises.

Thus, for solving the problem of $\mathcal{N} = 4$ supersymmetrization of the next-to-leading terms of effective action, one has to consider transformations that mix the terms of different powers in the expansion of effective action in derivatives. Let us consider the variation of classical action in order to make sure that transformations (2.54) and (2.55) are meaningful. The variation of the hypermultiplet action is proportional to equations of motion $D^{++}q^+ = 0$, while the variation of $\mathcal{N} = 2$ gauge superfield strength according to the rule (2.54), generates the following variation of action:

$$\delta_{(1|0)}\Gamma_0 = \frac{1}{8}\int d^{12}z$$
$$\times \left(\int \bar{A} \bar{\varepsilon}^{\dot{\alpha}a} \bar{D}^-_{\dot{\alpha}} q^+_a \frac{1}{\mathcal{W}} \ln \bar{\mathcal{W}} + A \varepsilon^{\alpha a} D^-_\alpha q^+_a \frac{1}{\bar{\mathcal{W}}} \ln \mathcal{W}\right). \quad (2.69)$$

The structure of this variation is the same as that of the variation of expression $\int d^{12}z \ln \mathcal{W} \ln \bar{\mathcal{W}}$ with respect to (2.49); therefore, this expression should be added to the classical action in order to establish invariance with respect to deformed transformations (2.54).

Let us consider the first deformed variation of a term of the F^6 type (2.50). We obtain

$$\delta_{(1|0)}\mathcal{L}_{(6|0)} = c_2 \left[\frac{\bar{A}}{\mathcal{W}^3 \bar{\mathcal{W}}^2} \bar{\varepsilon}^{\dot{\alpha}a} \bar{D}^-_{\dot{\alpha}} q^+_a D^4 \ln \mathcal{W} \bar{D}^4 \ln \bar{\mathcal{W}}\right.$$
$$\left. - \frac{A}{\bar{\mathcal{W}}^5} \varepsilon^{\alpha a} D^-_\alpha q^+_a \ln \mathcal{W} (D^4 \ln \mathcal{W})^2\right]. \quad (2.70)$$

The first term in brackets is similar to the classical variation of a one-loop term of the F^8 type: $\mathcal{L}_{(8|0)} = \mathbf{\Psi}^2 \bar{\mathbf{\Psi}}^2 = \frac{1}{\mathcal{W}^2} \bar{D}^2 \ln \bar{\mathcal{W}} \frac{1}{\bar{\mathcal{W}}^2} D^4 \ln \mathcal{W}$; however, it has a different coefficient c_2. This implies that one-loop coefficient $\frac{1}{2(24\pi)^2}$ in front of the F^8-type term [145] should be renormalized by two-loop corrections. The second term in (2.70) is of a new type. Its cancellation requires a term of a new structure

$$\Gamma'_{(8|0)} = c_8 \int d^{12}z \ln \mathcal{W} \left(\frac{1}{\bar{\mathcal{W}}^2} D^4 \ln \mathcal{W}\right)^2, \quad (2.71)$$

with a typical $\propto \frac{1}{(4\pi)^4}$ two-loop coefficient $c_8 = -c_2 \frac{A}{2}$. Therefore, one might assume that an F^8-type term should emerge in two-loop calculations of the effective action. On the one hand, exactly this term is needed to cancel the second term in (2.70); on the other hand, it should have a coefficient typical for two-loop corrections.[8]

Thus, the self-consistent approach allows one to obtain proper complements, which incorporate hypermultiplet superfields, and deformed supersymmetry transformations that, in their turn, provide an opportunity to retrieve information concerning the renormalization of higher contributions to the effective action.

2.10. Summary

The derivative expansion of one-loop effective action (incorporating both the $\mathcal{N} = 1$ vector multiplet superfields and the superfields of matter hypermultiplets) of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory was constructed. The obtained action was rewritten in terms of the expansion in powers of $\mathcal{N} = 2$ superconformal invariants the leading term of which coincides with the $\mathcal{N} = 4$ supersymmetric effective potential constructed earlier in [146]. Next-to-leading contributions to this action were found. It was emphasized that all the obtained manifestly $\mathcal{N} = 2$ supersymmetric contributions (except for the leading one) are not necessarily invariant with respect to classical hidden $\mathcal{N} = 2$ supersymmetries, which is attributable to the choice of background and the gauge-fixing procedure. The possible self-consistent deformations of hidden $\mathcal{N} = 2$ supersymmetries and the subleading terms of the effective action of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory off the mass shell were analyzed in the formulation using $\mathcal{N} = 2$ harmonic superspace. There was cnstructed a complement (depending on the superfields of hypermultiplets) for two-loop term $\propto F^6$ in the Schwinger–DeWitt expansion of the effective action.

3. ONE-LOOP EFFECTIVE ACTION OF $N = 4$ SUPERSYMMETRIC YANG–MILLS THEORY IN HARMONIC SUPERSPACE

3.1. Introduction

The $\mathcal{N} = 4$ supersymmetric Yang–Mills theory attracts particular attention owing to its several unique quantum properties such as finiteness and exact superconformal invariance and due to its intimate relationship with string and brane theory (see, for example, [54, 139]). The discovery of *AdS/CFT* correspondence stimulated another surge in interest in the study of various aspects of $\mathcal{N} = 4$ supergauge theory. It was already noted that the *AdS/CFT* correspondence [116] is a hypothesis of duality between IIB superstring theory compactified on $AdS_5 \times S^5$ and four-dimensional $\mathcal{N} = 4$ supergauge theories in the t'Hooft limit. It is assumed that the low-energy properties of string theory in a 5-dimensional bulk are connected to $\mathcal{N} = 4$ supersymmetric gauge quantum field theory on a 4-dimensional boundary. More specifically, the theory in a bulk is reduced in the considered limit to 5-dimensional classical supergravity that contains complete information regarding the correlation functions of gauge-invariant composite operators in quantum $\mathcal{N} = 4$, $D = 4$ theory. It also follows from this hypothetical correspondence between $\mathcal{N} = 4$ gauge theory and string theory that the interactions of D3-branes in the static limit should be described completely in terms of the low-energy effective action of $\mathcal{N} = 4$, $D = 4$ supersymmetric Yang–Mills theory in the Coulomb branch [141, 144, 145]. Thus, $\mathcal{N} = 4$ supersymmetric gauge theory should be regarded as an integral part of superstring theory.

It was already noted that the formulation of $\mathcal{N} = 4$ gauge theory with manifest $\mathcal{N} = 4$ supersymmetry off the mass shell still remains unknown. A superfield description of this theory on the mass shell is attained in terms of scalar superfield $W_{AB} = -W_{BA}$, $A, B = 1,\ldots,4$ that forms a six-dimensional real representation of the internal symmetry ("R-symmetry") group $SU(4)$: $\bar{W}^{AB} = \frac{1}{2}\epsilon^{ABCD}W_{CD}$ [186]. Superfield W_{AB} is subject to constraints

$$\mathcal{D}_{\alpha A}W_{BC} = \mathcal{D}_{\alpha[A}W_{BC]},$$
$$\bar{\mathcal{D}}^A_{\dot{\alpha}}W_{BC} = -\frac{2}{3}\delta^A_{[B}\bar{\mathcal{D}}^E_{\dot{\alpha}}W_{C]E}.$$

These constraints lead to the equations of motion for component fields. The set of the latter corresponds to a vector $\mathcal{N} = 4$ multiplet containing six real scalar fields, four Majorana spinor fields, and one vector gauge field.[9]

In the framework of $\mathcal{N} = 2$ supersymmetric field theory, an $\mathcal{N} = 4$ vector multiplet is a direct sum of an $\mathcal{N} = 2$ vector multiplet and a hypermultiplet [51]. Therefore, the $\mathcal{N} = 4$ gauge theory may be regarded as a certain extension of $\mathcal{N} = 2$ supersymmetric gauge theory, such that its ordinary action is supplemented by the action of $\mathcal{N} = 2$ hypermultiplet in the adjoint representation, interacting with the $\mathcal{N} = 2$ vector mul-

[8] The possibility that an F^8-type term emerges in calculations of two-loop corrections to the effective action is discussed in [145].

[9] The same multiplet may be described off the mass shell in $\mathcal{N} = 3$ harmonic superspace [51, 56]. Quantum aspects of $\mathcal{N} = 3$ gauge theory in the harmonic fornulation were discussed in [61]. The effective action structure of this theory was analyzed in [41].

tiplet via minimal coupling. This model has an additional hidden $\mathcal{N} = 2$ supersymmetry which, together with manifest $\mathcal{N} = 2$ supersymmetry, close on $\mathcal{N} = 4$ supersymmetry. Both $\mathcal{N} = 2$ theories constituting the $\mathcal{N} = 4$ theory are naturally formulated in $\mathcal{N} = 2$ harmonic superspace [51, 56]. Owing to the presence of manifest $\mathcal{N} = 2$ supersymmetry in the harmonic approach, this formulation simplifies quantum analysis.

Direct calculations of multi-loop contributions to the effective action in a closed form is highly complicated technical problem even in the harmonic formalism. The analysis of effective action is simplified and clarified greatly in the framework of the method of background harmonic superfields [41, 52]. For recent years, an essential progress was achieved in developing the technique of the multi-loop calculations in harmonic superspace for background vector multiplet [107]. It appears instructive and important to study the general structure of possible higher-order corrections to the effective action in the background superfield method.

In the present section, the $\mathcal{N} = 4$ supergauge theory is regarded as a certain special case of general $\mathcal{N} = 2$ gauge models. All such models may be formulated in an explicitly $\mathcal{N} = 2$ supersymmetric way in $\mathcal{N} = 2$ harmonic superspace. The $\mathcal{N} = 4$ theory is specific in that it has an additional hidden $\mathcal{N} = 2$ supersymmetry. The formulation in terms of $\mathcal{N} = 2$ superspace allows one to use the known classification of ground states in $\mathcal{N} = 2$ gauge models (see, for example, [74]). In accordance with this classification, the phase of the theory in which both scalars from the $\mathcal{N} = 2$ vector multiplet and scalars from the hypermultiplet have nonzero vacuum values is called the mixed one. This version of spontaneous gauge symmetry breaking corresponds exactly to the problem under consideration. It is evident that vacuum values of fields of both the $\mathcal{N} = 2$ vector multiplet and the hypermultiplet should differ from zero in order to retain hidden $\mathcal{N} = 2$ supersymmetry of the ground state. The explicit form of transformations of hidden $\mathcal{N} = 2$ supersymmetry on the constant values of ground-state fields and the application of these transformations are detailed in [215].[10]

It is now well known that the exact low-energy quantum dynamics of $\mathcal{N} = 4$ gauge theory in the $\mathcal{N} = 2$ vector multiplet sector for gauge group $SU(N)$ broken spontaneously to its maximal torus $U(1)^{N-1}$ is described by nonholomorphic effective potential $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$ which depends on $\mathcal{N} = 2$ superfield strengths $\mathcal{W}, \bar{\mathcal{W}}$ [86, 41]. The structure of $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$ is so unique that it can be determined (up to a numerical coefficient) based just on the requirements of scale invariance and R-symmetry. In addition, potential $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$ is not subject to perturbative quantum corrections beyond one loop and to any nonperturbative instanton corrections (the nonholomorphic potential in $\mathcal{N} = 2$ gauge theory was discussed in [82], and the next-to-leading two-loop contributions to the effective action were analyzed in [107, 145]). All these features are crucial for understanding the low-energy quantum dynamics of $\mathcal{N} = 4$ gauge theory in the context of its correspondence with superstring theory. Specifically, it turns out that effective potential $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$ characterizes, in the framework of this correspondence, leading low-energy terms in momentum expansion of interaction between parallel D3-branes.

In order to reveal the constraints imposed by $\mathcal{N} = 4$ supersymmetry on effective action, identify the complete structure of effective action, and gain a thorough insight into the "$\mathcal{N} = 4$ gauge theory/supergravity" correspondence, one should determine the dependence of effective action on all fields of the $\mathcal{N} = 4$ multiplet (including the hypermultiplet fields). The problem of constructing the effective action of $\mathcal{N} = 4$ Yang–Mills theory in the mixed phase has remained unexplored for a long time. A full exact expression for low-energy effective potential $\mathcal{L}_q(X)$ $\left(X = -\dfrac{q^{ia}q_{ia}}{\mathcal{W}\bar{\mathcal{W}}}\right)$ with the dependence on both $\mathcal{N} = 2$ gauge superfields and hypermultiplet superfields has been constructed relatively recently in [146]. It was demonstrated that the algebraic constraints imposed by hidden $\mathcal{N} = 2$ supersymmetry on the structure of low-energy effective action in the $\mathcal{N} = 2$ harmonic superspace approach are so stringent that they allow one to reconstruct the hypermultiplet dependence of low-energy effective action starting from the known nonholomorphic effective potential $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$. As a result, the additional hypermultiplet-dependent contribution to the low-energy effective action was found. This contribution involves both the superfields of strengths $\mathcal{W}, \bar{\mathcal{W}}$ of $\mathcal{N} = 2$ gauge multiplet and the superfields of hypermultiplet q^{ia}, all subject to the mass shell conditions.

The leading low-energy effective Lagrangian $\mathcal{L}_q(X)$ was constructed in [146] by purely algebraic methods. It was later demonstrated in [147] how the effective Lagrangian $\mathcal{L}_q(X)$ can be calculated using the technique of harmonic superdiagrams and the background field method in harmonic superspace. The structure of one-loop effective action in the next-

[10] It is worth mentioning that in the literature there is no commonly accepted terminology concerning the ground state of $\mathcal{N} = 4$ Yang–Mills theory for which the notions of Coulomb, Higgs, and mixed phases lose their meaning. Since the $\mathcal{N} = 4$ theory is a special case of $\mathcal{N} = 2$ gauge theories, it is a viable option to use the notation pertinent to these theories. Here we follow this convention.

to-leading approximation was found in [215] based on the formulation of $\mathcal{N} = 4$ gauge theory in terms of $\mathcal{N} = 1$ superfields and the technique of expansion in derivatives in $\mathcal{N} = 1$ superspace. This formulation preserves a smaller number of manifest supersymmetries than the harmonic superspace approach does. Nevertheless, the choice of supersymmetric R_ξ-gauge and special techniques for restoring $\mathcal{N} = 2$ supersymmetric form of the action allow one to construct an effective action containing the dependence on arbitrary powers of Abelian strengths F_{mn} and special R-symmetric combinations of constant scalar fields Φ, f^{ia} from the vector multiplet and hypermultiplet.

This section is focused on the analysis of the hypermultiplet dependence of low-energy effective action of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in $\mathcal{N} = 2$ harmonic superspace. We calculate the low-energy effective action for the space-time independent background $\mathcal{N} = 2$ gauge superfield strengths and the hypermultiplet superfields under the following assumptions: (i) $\mathcal{W}|_{\theta=0} = \text{const}$, $D^\pm_\alpha \mathcal{W}|_{\theta=0} = \text{const}$, $D^-_\alpha D^+_\beta \mathcal{W}|_{\theta=0} = \text{const}$; (ii) the background hypermultiplet is on the mass shell (i.e., $q^{\pm a} = q^{ia} u^\pm_i$, where q^{ia} does not depend on harmonics and is considered to be constant). This means that we analyze the effective action in the hypermultiplet sector as a series in powers of spinor superfield derivatives of q^{+a} and study the simplest approximation corresponding to those contributions to the effective action that do not depend on any derivatives of q^{+a}. As compared to [147], the effective action beyond the leading low-energy approximation in the $\mathcal{N} = 2$ vector multiplet sector is considered (i.e., all powers of Abelian strength F_{mn} are taken into account). On the other hand, in contrast to [215], only the $\mathcal{N} = 2$ harmonic superspace is used at all stages of the analysis. This gives justification for the special heuristic algorithm that was used for the reconstruction of manifestly $\mathcal{N} = 2$ supersymmetric form of effective action in [215] and was discussed in detail in the previous section. The key result of the analysis undertaken is the proper time representation for low-energy effective action in the form of an integral over the analytic subspace of harmonic superspace.

The section is organized as follows. Subsection 3.2 is devoted to the construction of $\mathcal{N} = 4$ gauge theory in $\mathcal{N} = 2$ harmonic superspace; the structure of the corresponding perturbation theory is also outlined there. The general procedure for the determination of one-loop effective action in the hypermultiplet sector is described in Subsection 3.3. Subsection 3.4 presents a method of summing up an infinite series of covariant harmonic superdiagrams with an arbitrary number of external hypermultiplet lines on the nontrivial background of $\mathcal{N} = 2$ vector multiplet. These techniques yield the effective action in a form which is most suitable for its further analysis within the proper time method. Subsection 3.5 is concerned with the calculation of one-loop effective action using a generalization of the operator symbols technique to the case of $\mathcal{N} = 2$ harmonic superspace. The final result for one-loop effective action in the form of an integral over the analytic subspace of harmonic superspace is obtained in Subsection 3.6. It is demonstrated that this result implies expanding the effective action in spinor covariant derivatives, and the first two terms of such an expansion are given explicitly. Each term in this expansion can be rewritten as an integral over the full $\mathcal{N} = 2$ superspace. The basic results are summarized in Subsection 3.7.

3.2. $\mathcal{N} = 4$ Supersymmetric Yang–Mills Theory in $\mathcal{N} = 2$ Harmonic Superspace

The harmonic superspace is a universal tool for constructing $\mathcal{N} = 2$ supersymmetric theories with preserving manifest $\mathcal{N} = 2$ supersymmetry off the mass shell at all stages of the analysis. This new type of superspaces was introduced in [56] and was developed and advanced by many authors. The basics and applications of the harmonic superspace method are discussed in detail in [51]. In the present section, we use the notation and follow the conventions adopted in this book.

The harmonic superspace is an extension of ordinary $\mathcal{N} = 2$ superspace $z = (x^m, \theta^\alpha_i, \bar\theta^{\dot\alpha i})$ by internal sphere $S^2 = SU(2)/U(1)$, where $SU(2)$ is the automorphism group of $\mathcal{N} = 2$ Poincaré superalgebra. This sphere is described in a parameterization-independent way by harmonics u^{+i} and conjugate harmonics u^-_i subject to the $u^{+i}u^-_i = 1$ condition. This extended superspace provides an opportunity to single out analytic superspace with half as many Grassmannian coordinates by switching to a new (analytic) basis $z_A = (\zeta, \theta^-_\alpha, \bar\theta^-_{\dot\alpha})$. Variables $\zeta = (x^m_A, \theta^{+\alpha}, \bar\theta^+_{\dot\alpha}, u^+_i, u^-_i)$, form the analytic subspace of interest the key feature of which is its closedness with respect to coordinate transformations of $\mathcal{N} = 2$ supersymmetry. The relationship with the initial (central) basis is given by

$$x^m_A = x^m - 2i\theta^{(i}\sigma^m\bar\theta^{j)}u^+_i u^-_j,\quad \theta^\pm_\alpha = u^\pm_i\theta^i_\alpha,\quad \bar\theta^\pm_{\dot\alpha} u^\pm_i \bar\theta^i_{\dot\alpha}.$$

Both general harmonic superfields and superfields defined on an analytic subspace ("analytic harmonic superfields") can carry nonzero external ("harmonic") $U(1)$ charge, where $U(1)$ is the subgroup in the denominator of harmonic coset space $S^2 = SU(2)/U(1)$. This is the same charge that is carried by harmonic coordinates $u^\pm_i$ and the corresponding projections of Grassmannian coordinates in the analytic basis. Strict conservation of harmonic $U(1)$ charge at all stages of analysis is one of the key principles of the harmonic superspace method.

Spinor covariant derivatives in the central and analytic bases are related as $D^{\pm}_{\alpha} = u^{\pm}_i D^i_{\alpha}$, $\bar{D}^{\pm}_{\dot{\alpha}} = u^{\pm}_i D^i_{\dot{\alpha}}$, and derivatives D^{+}_{α}, $\bar{D}^{+}_{\dot{\alpha}}$ become "short" in the analytic basis:

$$D^{+}_{\alpha} = \frac{\partial}{\partial\theta^{-\alpha}}, \quad \bar{D}^{+}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{-\dot{\alpha}}}. \tag{3.1}$$

Keeping in mind the evident basis-independent anticommutativity properties

$$\left\{D^{+}_{\alpha}, D^{+}_{\beta}\right\} = \left\{\bar{D}^{+}_{\dot{\alpha}}, \bar{D}^{+}_{\dot{\beta}}\right\} = \left\{D^{+}_{\alpha}, \bar{D}^{+}_{\dot{\beta}}\right\} = 0 \tag{3.2}$$

(these follow from the form of anticommutators of the initial spinor derivatives after the projection of the latter onto harmonics u^{+i}), one can define covariantly the analytic harmonic superfields $\Phi^{(p)}$, where (p) is the harmonic $U(1)$ charge, as a subclass of general harmonic superfields that is distinguished by constraints

$$D^{+}_{\alpha}\Phi^{(p)} = \bar{D}^{+}_{\dot{\beta}}\Phi^{(p)} = 0. \tag{3.3}$$

Owing to "shortness" (3.1), constraints (3.3) take the form of Grassmannian Cauchy–Riemann conditions in the analytic basis and are readily solved through unconstrained superfields "living" on an analytic superspace:

$$\Phi^{(p)} = \Phi^{(p)}(\zeta) = \Phi^{(p)}(x_A, \theta^{+}, \bar{\theta}^{+}, u^{\pm}). \tag{3.4}$$

The hypermultiplet and the $\mathcal{N} = 2$ vector multiplet are two essential $\mathcal{N} = 2$ multiplets that form the basis for the $\mathcal{N} = 4$ supersymmetric gauge theory we are interested in. In the harmonic superspace, the hypermultiplet is described off the mass shell by an analytic superfield with a $U(1)$ charge of $+1$:

$$D^{+}_{\alpha}q^{+} = \bar{D}^{+}_{\dot{\alpha}}q^{+} = 0 \Rightarrow q^{+} = q^{+}(x_A, \theta^{+}, \bar{\theta}^{+}, u).$$

Its free action is given by an integral over the analytic subspace

$$S_{hyper} = -\int d\zeta^{(-4)}\breve{q}^{+}D^{++}q^{+}, \tag{3.5}$$

where $d\zeta^{(-4)} = d^4x d^2\theta^{+} d^2\bar{\theta}^{+} du$ is the corresponding integration measure (charged, since Grassmannian coordinates carry harmonic charges), and

$$D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}} - 2i(\theta^{+}\sigma^m\bar{\theta}^{+})\frac{\partial}{\partial x^m_A}$$

is one of the three harmonic derivatives in the analytic basis. This derivative is distinguished by the fact that it commutes with spinor derivatives D^{+}_{α}, $\bar{D}^{+}_{\dot{\beta}}$ and, consequently, retains harmonic analyticity. The other two derivatives (D^{--} and D^{0}) form, together with D^{++}, algebra $su(2)$,

$$[D^{++}, D^{--}] = D^{0}, \quad [D^{0}, D^{\pm\pm}] = \pm 2D^{\pm\pm},$$

where $D^{\pm\pm}$ serve as the raising and lowering operators, and D^{0} is the harmonic $U(1)$ charge operator that takes a constant value on harmonic superfields, $D^{0}\Phi^{(p)} = p\Phi^{(p)}$. In addition to the integration over coordinates x^m_A and θ^{+}_{α}, $\bar{\theta}^{+}_{\dot{\beta}}$, measure $d\zeta^{(-4)}$contains harmonic integration measure du. The rules for integration over harmonics are given in [51].

The hypermultiplet equation of motion, which follows from action (3.5), also has the form of an analyticity condition (now with respect to harmonic $SU(2)$ coordinates):

$$D^{++}q^{+} = 0.$$

This equation "cuts" an infinite "tail" of auxiliary fields, which are present in the expansion of superfield q^{+} in harmonic coordinates off the mass shell, and leads to the ordinary mass-shell conditions for physical fields (a quartet of scalars and two Weyl fermions). Its solution forms the so-called ultrashort superfield.

The vector (gauge) $\mathcal{N} = 2$ multiplet is described by a real doubly charged analytic superfield $V^{++}(x, \theta^{+}, \bar{\theta}^{+}, u)$, $D^{0}V^{++} = 2V^{++}$. This superfield serves as the gauge connection in covariantized harmonic derivative

$$\mathcal{D}^{++} = D^{++} + igV^{++},$$

that, in common with flat D^{++}, commutes with $D^{+}_{(\alpha,\dot{\alpha})}$: $[\mathcal{D}^{++}, \mathcal{D}^{+}_{\alpha,\dot{\alpha}}] = 0$. In the Wess–Zumino gauge, superfield V^{++} incorporates as its components all fields of $\mathcal{N} = 2$ vector multiplet off the mass shell (complex scalar field, gauge vector field, $SU(2)$ doublet of Weyl fermions, and real $SU(2)$ triplet of auxiliary fields). Thus, V^{++} is a basic object of $\mathcal{N} = 2$ gauge theory (its unconstrained analytic prepotential). All other objects of this theory (e.g., the superfield strength W) are expressed through V^{++} [51, 56]. More specifically, the field-strength superfield is expressed directly through nonanalytic superfield V^{--},

$$\mathcal{W} = -\frac{1}{4}(\bar{\mathcal{D}}^{+})^2V^{--}, \quad \bar{\mathcal{W}} = -\frac{1}{4}(\mathcal{D}^{+})^2V^{--}. \tag{3.6}$$

The latter is related to V^{++} by the "zero harmonic curvature" condition

$$D^{++}V^{--} - D^{--}V^{++} + i[V^{++}, V^{--}] = 0. \tag{3.7}$$

This condition has a solution in the form of a power series

$$V^{--} = \sum_{n=1}^{\infty} \int d^{12}z du_1 \ldots du_n (-i)^{n+1} \times \frac{V^{++}(z,u_1)\ldots V^{++}(z,u_n)}{(u^+u_1^+)(u_1^+u_2^+)\ldots(u_n^+u^+)}, \tag{3.8}$$

that is a nonlocal functional of V^{++} in the harmonic variables sector. The covariant u-independence of strength $\mathcal{W}$,

$$\mathcal{D}^{\pm\pm}\mathcal{W} = 0,$$

follows from representation (3.6), condition (3.7), and the analyticity of V^{++}. Other important properties of superstrengths $\mathcal{W}, \overline{\mathcal{W}}$ following from definition (3.6) are their chirality (antichirality), $\mathcal{D}^{\pm}_{\alpha}\overline{\mathcal{W}} = \overline{\mathcal{D}}^{\pm}_{\dot\alpha}\mathcal{W} = 0$, and the Bianchi identity: $\mathcal{D}^{\alpha i}\mathcal{D}^{j}_{\alpha}\mathcal{W} = \overline{\mathcal{D}}^{i}_{\dot\alpha}\overline{\mathcal{D}}^{\dot\alpha j}\overline{\mathcal{W}}$. The full algebra of covariant derivatives should also be written, since it will be used later:

$$\left\{\mathcal{D}^+_{\alpha}, \mathcal{D}^+_{\beta}\right\} = \left\{\overline{\mathcal{D}}^+_{\dot\alpha}, \overline{\mathcal{D}}^+_{\dot\beta}\right\} = \left\{\mathcal{D}^+_{\alpha}, \overline{\mathcal{D}}^+_{\dot\alpha}\right\} = 0, \tag{3.9}$$

$$\left\{\mathcal{D}^+_{\alpha}, \mathcal{D}^-_{\beta}\right\} = -2i\varepsilon_{\alpha\beta}\overline{\mathcal{W}}, \quad \left\{\overline{\mathcal{D}}^+_{\dot\alpha}, \overline{\mathcal{D}}^-_{\dot\beta}\right\} 2i\varepsilon_{\dot\alpha\dot\beta}\mathcal{W},$$
$$\left\{\overline{\mathcal{D}}^+_{\dot\alpha}, \mathcal{D}^-_{\alpha}\right\} = -\left\{\mathcal{D}^+_{\alpha}, \overline{\mathcal{D}}^-_{\dot\alpha}\right\} = 2i\mathcal{D}_{\alpha\dot\alpha},$$
$$[\mathcal{D}^{++}, \mathcal{D}^-_{\alpha}] = \mathcal{D}^+_{\alpha}, \quad [\mathcal{D}^{++}, \overline{\mathcal{D}}^-_{\dot\alpha}] = \mathcal{D}^+_{\dot\alpha},$$
$$[\mathcal{D}^{\pm}_{\alpha}, \mathcal{D}_{\beta\dot\beta}]\varepsilon_{\alpha\beta}\overline{\mathcal{D}}^{\pm}_{\dot\beta}\overline{\mathcal{W}}, \quad [\overline{\mathcal{D}}^{\pm}_{\dot\alpha}, \mathcal{D}_{\beta\dot\beta}] = \varepsilon_{\dot\alpha\dot\beta}D^{\pm}_{\beta}\mathcal{W}, \tag{3.10}$$
$$[\mathcal{D}_{\alpha\dot\alpha}, \mathcal{D}_{\beta\dot\beta}]\frac{1}{2i}\left\{\varepsilon_{\alpha\beta}\overline{D}^+_{\dot\alpha}\overline{D}^-_{\dot\beta}\overline{\mathcal{W}} + \varepsilon_{\dot\alpha\dot\beta}D^-_{\alpha}D^+_{\beta}\mathcal{W}\right\}.$$

As has already been noted a few times, the action of $\mathcal{N} = 4$ gauge theory in $\mathcal{N} = 2$ harmonic superspace is a sum of actions of the vector multiplet and the hypermultiplet in the adjoint representation of the gauge group which interact with V^{++} via minimal coupling:

$$S_{(\mathcal{N}=4SYM)} = \frac{1}{2g^2}\mathrm{tr}\int d^8z\mathcal{W}^2 - \frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}q^{+a}(D^{++} + iV^{++})q^+_a. \tag{3.11}$$

Here $a = 1, 2$ is the index of global $SU(2)$ Pauli–Gürsey symmetry group: $q^+_a = (q^+, \breve{q}^+)$, $q^{+a} = \varepsilon^{ab}q^+_a = (\breve{q}^+, -q^+)$ (it commutes with the generators of $\mathcal{N} = 2$ superalgebra, covariant derivatives, and $SU(2)$ automorphisms), and $d^8z = d^4xd^2\theta^+d^2\theta^-du$ is the measure of integration over the chiral superspace.

Action (3.11) allows one to formulate the rules of manifestly $\mathcal{N} = 2$ supersymmetric quantization of the theory. This action is invariant with respect to the transformations of hidden $\mathcal{N} = 2$ supersymmetry [51] realized on superfields of the vector multiplet and the hypermultiplet in the following way:

$$\delta V^{++} = (\varepsilon^{\alpha a}\theta^+_a + \overline{\varepsilon}^a_{\dot\alpha}\overline{\theta}^{+\dot\alpha})q^+_a,$$
$$\delta q^+_a = -\frac{1}{2}(D^+)^4[(\varepsilon^{\alpha}_a\theta^-_{\alpha} + \overline{\varepsilon}_{\dot\alpha a}\overline{\theta}^{+\dot\alpha})V^{--}]. \tag{3.12}$$

Although these transformations have a proper closure with themselves and the transformations of manifest $\mathcal{N} = 2$ supersymmetry only on the mass shell, action (3.11) is invariant with respect to (3.12) off the mass shell. Thus, the model has a second (hidden) $\mathcal{N} = 2$ supersymmetry that, together with manifest $\mathcal{N} = 2$ supersymmetry, closes on $\mathcal{N} = 4$ supersymmetry on the equations of motion.

The structure of the model on the mass shell is defined in terms of solutions of the corresponding equations of motion

$$D^{++}q^{+a} + ig[V^{++}, q^{+a}] = 0, \quad (\mathcal{D}^+)^2\mathcal{W} = [q^{+a}, q^+_a]. \tag{3.13}$$

The simplest solution of these equations in the Abelian case consists of a set of constant background fields that are transformed through each other under the transformations of hidden $\mathcal{N} = 2$ supersymmetry, mixing $\mathcal{W}, \overline{\mathcal{W}}$ with q^+_a [41]:

$$\delta\mathcal{W} = \frac{1}{2}\overline{\varepsilon}^{\dot\alpha a}\overline{D}^-_{\dot\alpha}q^+_a, \quad \delta\overline{\mathcal{W}} = \frac{1}{2}\varepsilon^{\alpha a}D^-_{\alpha}q^+_a,$$
$$\delta q^+_a = \frac{1}{4}(\varepsilon^{\alpha}_a D^+_{\alpha}\mathcal{W} + \overline{\varepsilon}^{\dot\alpha}_a\overline{D}^+_{\dot\alpha}\overline{\mathcal{W}}). \tag{3.14}$$

The ground (vacuum) state of any $\mathcal{N} = 2$ superconformal model, including the $\mathcal{N} = 4$ gauge theory (a special case of such models), can incorporate only a massless $U(1)$ vector multiplet and massless neutral hypermultiplets, since charged hypermultiplets acquire mass via the Higgs mechanism. The vacua manifold is specified by the conditions of scalar potential vanishing ("F-flatness" plus "D-flatness"). The set of vacua incorporating only massless neutral hypermultiplets forms the "Higgs branch" of the theory, while the set incorporating only massless $U(1)$ vector multiplets forms the "Coulomb branch." The "mixed phase" of the theory is formed by vacua with coexisting multiplets of both kinds. Thus, massless neutral scalars, spinors, and $U(1)$ vector bosons, which are included into superfields $\mathcal{W}, \overline{\mathcal{W}}$, q^{+a}, q^{-a} on the mass shell with the following properties

$$(D^{\pm})^2\mathcal{W} = (\overline{D}^{\pm})^2\overline{\mathcal{W}} = 0, \quad D^{++}q^{+a} = D^{--}q^{-a} = 0,$$
$$q^{-a} \equiv D^{--}q^{+a}, \quad D^-_{\alpha}q^{-a} = \overline{D}^-_{\dot\alpha}q^{-a} = 0.$$

are the propagating low-energy fields in the mixed phase. In what follows, we consider the low-energy effective action in $\mathcal{N} = 4$ supergauge theory on the mixed branch in the above sense. On the other hand, since the $\mathcal{N} = 2$ vector multiplet and the hypermultiplet form an $\mathcal{N}=4$ vector multiplet in the case under consideration, we can assume that all vacuum states form a unique Coulomb branch of the $\mathcal{N}=4$ model.[11]

Manifestly $\mathcal{N}=2$ supersymmetric Feynman rules in harmonic superspace were developed in [56] (see also [51]). The calculation of quantum corrections can feature potentially dangerous harmonic singularities that emerge in harmonic distributions at coincident points. The problem of coincident harmonic singularities in the framework of the harmonic Feynman rules for superdiagrams was first discussed in [51], and a certain solution to it was proposed. The background field method for the construction of effective action in a harmonic superspace was developed in [41] (see also [46], where the same method was applied in an ordinary $\mathcal{N}=2$ superspace). This method allows one to calculate the effective action for an arbitrary $\mathcal{N}=2$ supersymmetric gauge model in the form that preserves manifest $\mathcal{N}=2$ supersymmetry and classical gauge invariance at the quantum level.

The background field method implies splitting the initial superfields into classical superfields V^{++}, q^{+} and quantum superfields v^{++}, Q^{+}; gauge conditions are imposed only on the quantum fields. The Feynman rules are derived from quantum action of the $S_{quant}=S_2+S_{int}$ type, where S_2 is the quadratic form of quantum fields and ghosts, and S_{int} characterizes interaction. Both of these actions depend on background fields. This procedure is detailed in [41].

Action S_2 defines the propagators that depend on background fields. Let us now use the background-covariant gauge condition $D^{++}v^{++}=0$. The quantum gauge superfield propagator is in this case given by

$$G^{(2,2)}(1|2)=\frac{1}{2\hat{\Box}_1\hat{\Box}_2} \times(\mathcal{D}_1^{+})^4(\mathcal{D}_2^{+})^4\left\{\delta^{12}(z_1-z_2)(D_2^{--})^2\delta^{(-2,2)}(u_1,u_2)\right\}. \tag{3.15}$$

Note that this propagator is an analytic superfield with respect to each of its arguments. The propagator of Faddeev–Popov ghosts b and c is as follows:

$$G^{(0,0)}(1|2)=i\left\langle b(1)c^T(2)\right\rangle =\frac{1}{\hat{\Box}_1}(\mathcal{D}_1^{+})^4(\mathcal{D}_2^{+})^4\left\{\delta^{12}(z_1-z_2)\frac{(u_1^-u_2^-)}{(u_1^+u_2^+)^3}\right\}. \tag{3.16}$$

The propagator of the q^+-hypermultiplet described by action (3.11) in external field V^{++} is given by

$$G^{(1,1)}(1|2)=i\left\langle q^{+a}(1)\breve{q}_b^{+}(2)\right\rangle =-\delta_b^a\frac{1}{\hat{\Box}_1}(\mathcal{D}_1^{+})^4(\mathcal{D}_2^{+})^4\left\{\delta^{12}(z_1-z_2)\frac{1}{(u_1^+u_2^+)^3}\right\}. \tag{3.17}$$

[11]It bears repeating that the use of different terms for vacuum states in this theory is a matter of convention.

All these propagators contain operator $\hat{\Box}=-\frac{1}{2}(\mathcal{D}^{+})^4(\mathcal{D}^{--})^2$ that transforms each covariant-analytic superfield into a covariant-analytic one. Operator $\hat{\Box}$ on the space of such superfields is represented by a second-order differential operator

$$\begin{aligned}\hat{\Box}&=\frac{1}{2}\mathcal{D}^{\dot{\alpha}\alpha}\mathcal{D}_{\alpha\dot{\alpha}}+\frac{i}{2}(\mathcal{D}^{+\alpha}\mathcal{W})\mathcal{D}_{\alpha}^{-}\\&+\frac{i}{2}(\bar{\mathcal{D}}_{\dot{\alpha}}^{+}\bar{\mathcal{W}})\bar{\mathcal{D}}^{-\dot{\alpha}}-\frac{i}{4}(\bar{\mathcal{D}}_{\dot{\alpha}}^{+}\bar{\mathcal{D}}^{+\dot{\alpha}}\bar{\mathcal{W}})\mathcal{D}^{--}\\&+\frac{i}{8}[\mathcal{D}^{+\alpha},\mathcal{D}_{\alpha}^{-}]\mathcal{W}+\frac{1}{2}\{\mathcal{W},\bar{\mathcal{W}}\}.\end{aligned} \tag{3.18}$$

This expression is derived from the algebra of covariant derivatives (3.9). It is noteworthy that the differential part of operator $\hat{\Box}$ is defined unambiguously from the requirements that (i) $\hat{\Box}$ is expressed through covariant derivatives alone; (ii) operator $\hat{\Box}$ transforms each covariant-analytic superfield into a covariant-analytic one. It seems natural to call operator $\hat{\Box}$ an analytic d'Alembertian. The preservation of analyticity, $[D^{+}_{(\alpha,\dot{\alpha})},\hat{\Box}]=0$, is one of its significant features.

Owing to a nontrivial dependence on harmonics, $\mathcal{N}=2$ propagators have a complex structure. However, it was shown recently in [107] that the harmonic dependence of $\mathcal{N}=2$ propagators is simplified greatly if the background vector multiplet satisfies classical equations of motion $\mathcal{D}^{ij}\mathcal{W}=\bar{\mathcal{D}}^{ij}\bar{\mathcal{W}}=0$. The harmonic dependence of propagators is factorized completely in this case, thus allowing one to keep the harmonic dependence of $\mathcal{N}=2$ superdiagrams under control.

The proper time technique or the heat kernel technique is often used to calculate the effective action in the framework of the background field method. These techniques provide an opportunity to sum effectively an infinite set of Feynman diagrams with an ever-increasing number of background field insertions and construct an expansion of effective action in background-field derivatives in a manifestly gauge-covariant way. The background field method and the heat kernel methods for $\mathcal{N}=1$ gauge theories are well-developed (see, for example, [37]). The background field method in harmonic superspace was developed in [41], and certain examples of its important applications were analyzed in [107]. However, until relatively recently, the heat kernel method in $\mathcal{N}=2$ superspace was not studied in certain respects. In Section 5, we will use the $\mathcal{N}=2$ superfield heat kernel method to construct the one-loop effective action of $\mathcal{N}=4$ gauge theory in $\mathcal{N}=2$ harmonic superspace. The one-loop results [147] for the leading quantum low-energy corrections to the effective action which depend on both the $\mathcal{N}=2$ vector multiplet and the $\mathcal{N}=2$ hypermultiplet will be extended to the next-to-leading contributions.

3.3. One-Loop Effective Action in the Hypermultiplet Sector

Let us consider $\mathcal{N} = 4$ Yang–Mills theory with gauge group $SU(2)$ formulated in terms of $\mathcal{N} = 2$ harmonic superfields. The gauge symmetry of the model is broken to $U(1)$; hence, background fields V^{++}, q^+ belong to a Cartan subalgebra of the gauge group. Just as in [147], we start with background-quantum splitting $q^{+a} \to q^{+a} + Q^{+a}$, $V^{++} \to V^{++} + gv^{++}$, where q^{+a}, V^{++} are background fields and Q^{+a}, v^{++} are quantum ones. In one-loop calculations, it is sufficient to consider only a part of quantum action S_{quant} that is quadratic in quantum superfields:

$$S_2 = -\frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}[v^{++}\hat{\Box}v^{++} + Q^{+a}(D^{++} + iV^{++})Q_a^+ \\ + [Q^{+a}, igv^{++}]q_a^+ + q^{+a}ig[v^{++}, Q_a^+]] + \ldots. \tag{3.19}$$

The ellipsis denotes the contribution of ghosts. Operator $\hat{\Box}$ contains background $\mathcal{N} = 2$ strength $\mathcal{W}, \bar{\mathcal{W}}$ superfields, acts in the adjoint representation of the gauge group, and has the form (3.18). The set of background superfields in (3.19) satisfies the equations $(\mathcal{D}^+)^2\mathcal{W} = 0$, $\mathcal{D}q^{+a} = 0$ and $\mathcal{D}^{+\alpha}\mathcal{D}_\alpha^-\mathcal{W} = 0$, $\mathcal{D}^{-\alpha}\mathcal{D}_\alpha^-\mathcal{W} = 0$. It is convenient to redefine $gQ^+ \to Q^+$ and present background fields as $V^{++} = \tau_3 V_3^{++}$, $q^+ = \tau_3 q_3^+$, and $\mathcal{W} = \tau_3\mathcal{W}_3$. Here $\tau_i = \frac{1}{\sqrt{2}}\sigma_i$ are the $su(2)$ algebra generators:

$$[\tau_i, \tau_j] = i\sqrt{2}\epsilon_{ijk}\tau_k, \quad \mathrm{tr}(\tau_i\tau_j) = \delta_{ij}.$$

In the case of a background that takes values in the Abelian algebra, the $\mathcal{D}^{\pm\alpha}\mathcal{W} = D^{\pm\alpha}\mathcal{W}$ requirement and a similar one for $\bar{\mathcal{W}}$ are also satisfied (i.e., covariant spinor derivatives become "flat" $D, \bar{D}$ derivatives when acting upon such constrained $\mathcal{W}$ and $\bar{\mathcal{W}}$). Taking into account the equations of motion and the fact that all quantum superfields belong to the adjoint representation, $v^{++} = v_i^{++}\tau_i$, $Q^{+a} = Q_i^{+a}\tau_i$, we then obtain the following:

$$\hat{\Box}v^{++} = \Box_{cov}v^{++} + \frac{i}{2}[D^{+\alpha}\mathcal{W}, \mathcal{D}_\alpha^- v^{++}] \\ + \frac{i}{2}[\bar{D}^{+\dot{\alpha}}\bar{\mathcal{W}}, \bar{\mathcal{D}}_{\dot{\alpha}}^- v^{++}] + [\mathcal{W}, [\bar{\mathcal{W}}, v^{++}]]. \tag{3.20}$$

Here $\Box_{cov} = \frac{1}{2}\mathcal{D}^{\alpha\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}}$. We also get $\mathcal{D}^{++}q_1 = D^{++}q_1 + \sqrt{2}V^{++}q_2$, $\mathcal{D}^{++}q_2 = D^{++}q_2 - \sqrt{2}V^{++}q_1$, $\mathcal{D}^{++}q_3 = D^{++}q_3$, and $\mathcal{D}_m = D_m + \sqrt{2}A_m$.

The quadratic action part is then rewritten as

$$S_2 = -\frac{1}{2}\int d\zeta^{(-4)}\{v_1^{++}(\Box_{cov} + 2\mathcal{W}\bar{\mathcal{W}})v_1^{++} \\ + v_2^{++}(\Box_{cov} + 2\mathcal{W}\bar{\mathcal{W}})v_2^{++} + \frac{1}{\sqrt{2}}v_1^{++}((D^{+\alpha}\mathcal{W})D_\alpha^- \\ + (\bar{D}_{\dot{\alpha}}^+\bar{\mathcal{W}})\bar{D}^{-\dot{\alpha}})v_2^{++} - \frac{1}{\sqrt{2}}v_2^{++}((D^{+\alpha}\mathcal{W})D_\alpha^- \\ + (\bar{D}_{\dot{\alpha}}^+\bar{\mathcal{W}})\bar{D}^{-\dot{\alpha}})v_1^{++} + v_3^{++}\Box v_3^{++} + Q_i^{+a}D^{++}Q_{ia}^+ \\ + Q_1^{+a}(\sqrt{2}V^{++})Q_{2a}^+ + Q_2^{+a}(-\sqrt{2}V^{++})Q_{1a}^+ + v_2^{++}\sqrt{2}(q^{+a}Q_{a1}^+ \\ - Q_1^{+a}q_a^+) + v_1^{++}\sqrt{2}(Q_2^{+a}q_a^+ - q^{+a}Q_{a2}^+)\} + \ldots \tag{3.21}$$

The index 3 associated with background fields is hereafter omitted, and the ellipsis denotes, as before, the contribution of ghosts that do not depend on the background hypermultiplet. It can be seen from (3.21) that only the components of quantum superfields with subscripts 1 and 2 have nontrivial background-dependent propagators. The quantum superfields Q_3^{+a} and v_3^{++} do not interact with the background and is detached completely.

Passing to new complex quantum superfields

$$\chi^{++} = \frac{1}{\sqrt{2}}(v_1^{++} + iv_2^{++}), \quad \bar{\chi}^{++} = \frac{1}{\sqrt{2}}(v_1^{++} - iv_2^{++}),$$

$$\eta^{+a} = \frac{1}{\sqrt{2}}(Q_1^{+a} + iQ_2^{+a}), \quad \bar{\eta}^{+a} = \frac{1}{\sqrt{2}}(Q_1^{+a} - iQ_2^{+a}),$$

action (3.21) takes the following form:

$$S_2 = -\int d\zeta^{(-4)}\Big\{\bar{\chi}^{++}\Big(\Box_{cov} + 2\mathcal{W}\bar{\mathcal{W}} \\ - \frac{i}{\sqrt{2}}[(D^{+\alpha}\mathcal{W})D_\alpha^- + (\bar{D}_{\dot{\alpha}}^+\bar{\mathcal{W}})\bar{D}^{-\dot{\alpha}}]\Big)\chi^{++} \\ + \bar{\eta}^{+a}(D^{++} - i\sqrt{2}V^{++})\eta_a^+ - i\chi^{++}\sqrt{2}q^{+a}\bar{\eta}_a^+ \\ + i\bar{\chi}^{++}\sqrt{2}q^{+a}\eta_a^+ + \frac{1}{2}v_3^{++}\Box v_3^{++} + \frac{1}{2}Q_3^{+a}D^{++}Q_{3a}^+\Big\}. \tag{3.22}$$

This form of action is the most convenient for perturbative calculations. The action is a sum of diagonal part S_0, which defines propagators, and nondiagonal part V, which is responsible for the interaction. The terms with noninteracting fields Q_3^{a+} and v_3^{++} can be omitted.

Let us first introduce an operator

$$\hat{\Box}_{short} = \Box_{cov} + 2\mathcal{W}\bar{\mathcal{W}} \\ - \frac{i}{\sqrt{2}}[(D^{+\alpha}\mathcal{W})\mathcal{D}_\alpha^- + (\bar{D}_{\dot{\alpha}}^+\bar{\mathcal{W}})\bar{\mathcal{D}}^{-\dot{\alpha}}], \tag{3.23}$$

which can be derived from (3.18) by retaining only the $\mathcal{W}, \bar{\mathcal{W}}$ superfields associated with the unbroken $U(1)$ subgroup in this expression and placing them on the mass shell. Note that operators (3.18) and (3.23) on the mass shell are related to each other by substitutions

$\mathcal{W} \to -\frac{\mathcal{W}}{\sqrt{2}}$, $\bar{\mathcal{W}} \to -\frac{\bar{\mathcal{W}}}{\sqrt{2}}$. The Feynman rules for the calculation of effective action can now be written down easily in much the same way as in [41, 51, 56]. Expressions (3.15) and (3.17) are used for the propagators of gauge fields χ^{++} and $\bar{\chi}^{++}$ and hypermultiplets η_a^+ and $\bar{\eta}^{+a}$, respectively. The vertices are taken directly from interaction V in the form

$$V = -i\chi^{++}\sqrt{2}q^{+a}\bar{\eta}_a^+ + i\bar{\chi}^{++}\sqrt{2}q^{+a}\eta_a^+.$$

The resulting Feynman rules have a standard form. One significant feature to emphasize is that we can always take operator $(D^+)^4$ off one of the propagators in each vertex containing an integral over the analytic subspace and transform an integral over $d\zeta^{(-4)}$ into an integral over the full measure of $\mathcal{N} = 2$ superspace $d^{12}z$.

Note now that functional substitutions of variables in (3.22)

$$\chi^{++}(1) \to \chi^{++}(1) - i\int d\zeta_2^{(-4)} G^{(2,2)}(1|2)q^{+a}(2)\eta_a^+(2),$$
$$\bar{\chi}^{++}(1) \to \bar{\chi}^{++}(1) + i\int d\zeta_2^{(-4)} G^{(2,2)}(1|2)q^{+a}(2)\bar{\eta}_a^+(2),$$

where propagator $G^{2,2}(1|2)$ is defined in (3.15), allow one to diagonalize operator

$$\int d\zeta_1^{(-4)} d\zeta_2^{(-4)}\bar{\eta}^{+a}(1)\left\{\delta_a^b\mathcal{D}_1^{++}\delta^{(1,3)}(1|2) + q_a^+(1)G^{(2,2)}(1|2)q^{+b}(2)\right\}\eta_b^+(2).$$

One-loop effective action $\Gamma[V^{++}, q^+]$, defined by functional integral

$$e^{i\Gamma[V^{++},q^+]} = \int \mathcal{D}\bar{\eta}^+\mathcal{D}\eta^+\mathcal{D}\bar{\chi}^{++}\mathcal{D}\chi^{++} e^{iS_2[\eta^+,\bar{\eta}^+,\bar{\chi}^{++},\chi^{++},V^{++},q^+]},$$

can then be written down formally as

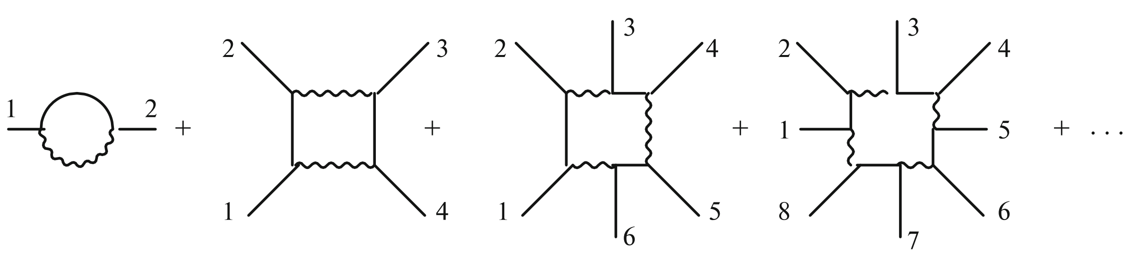
$$\Gamma[V^{++}, q^+, \breve{q}^+] = i\mathrm{Tr}\ln\left\{\delta_a^b\mathcal{D}_1^{++}\delta^{(1,3)}(1|2) + q_a^+(1)G^{(2,2)}(1|2)q^{+b}(2)\right\} + \Gamma[V^{++}]. \quad (3.24)$$

The last term $\Gamma[V^{++}]$ in this expression is a part of the full one-loop effective action that depends only on the $\mathcal{N} = 2$ gauge superfield. We focus on the first term, since it contains the complete dependence on the hypermultiplet.

Expression (3.24) written in the form of an analytical nonlocal superfunctional is the starting point for our calculations of one-loop effective action in the hypermultiplet sector. Expression (3.24) shows that the effective action is well defined in the framework of the perturbation theory in powers of nonlocal interaction $q_a^+(1)G^{(2,2)}(1|2)q^{+b}(2)$. An infinite series in these powers leads to an effective action in the form of $\Gamma[V^{++}, \breve{q}^+, q^+] = \sum_{n=1}^{\infty}\Gamma_{2n}[V^{++}, \breve{q}^+, q^+]$, where $2n$-th term is defined by a superdiagram with $2n$ external $\breve{q}^+, q^+$ lines and an arbitrary number of external V^{++} lines. Since action $\Gamma[V^{++}, \breve{q}^+, q^+]$ is gauge-invariant by construction, it is reasonable to expect that each coefficient Γ_{2n} should depend on the V^{++} background superfield only through superfield strengths $\mathcal{W}, \bar{\mathcal{W}}$ and their covariant derivatives. It is necessary to stress that the superdiagrams emerging during this procedure contain background-dependent superpropagators.

3.4. Analysis of Superdiagrams for Hypermultiplet-Dependent Contributions to the Effective Action

Hypermultiplet-dependent contributions to the one-loop effective action are represented by the following infinite series of superdiagrams:



Wavy lines here represent the propagator of $\mathcal{N} = 2$ gauge superfield, and solid external and internal lines denote the hypermultiplet background superfields and quantum propagators, respectively. Numbers $1, 2, \ldots$ identify the (ζ_A, u) arguments of external hypermultiplet lines. It was already noted that the whole dependence of the supergraph contributions on the gauge superfield is incorporated into background-dependent propagators.

An arbitrary supergraph with $2n$ external hypermultiplet lines is akin to a ring composed of n segments of the $\langle\bar{\eta}^+\eta^+\rangle\langle\chi^{++}\bar{\chi}^{++}\rangle$ form or n segments of the

$\langle\eta^+\bar{\eta}^+\rangle\langle\bar{\chi}^{++}\chi^{++}\rangle$ form. The full contribution of a $2n$-point superdiagram is given by the following general expression:

$$
\begin{aligned}
i\Gamma_{2n} &= \frac{4}{n}\int d\zeta_1^{(-4)}d\zeta_2^{(-4)}\dots d\zeta_{2n}^{(-4)}\frac{(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4}{(u_1^+u_2^+)^3}\\
&\times\left\{\frac{1}{\widehat{\square}_1}\delta^{12}_{1|2}\right\}\frac{(\mathcal{D}_2^+)^4(\mathcal{D}_3^+)^4}{\widehat{\square}_2\widehat{\square}_3}(\mathcal{D}_3^{--})^2\left\{\delta^{12}_{2|3}\delta^{(-2,2)}(u_2,u_3)\right\}\\
&\times\frac{(\mathcal{D}_3^+)^4(\mathcal{D}_4^+)^4}{(u_3^+u_4^+)^3}\left\{\frac{1}{\widehat{\square}_3}\delta^{12}_{3|4}\right\}\dots\frac{(\mathcal{D}_{2n-2}^+)^4(\mathcal{D}_{2n-1}^+)^4}{\widehat{\square}_{2n-2}\widehat{\square}_{2n-1}}\\
&\times(\mathcal{D}_{2n-1}^{--})^2\left\{\delta^{12}_{2n-2|2n-1}\delta^{(-2,2)}(u_{2n-2},u_{2n-1})\right\} \qquad (3.25)\\
&\times\frac{(\mathcal{D}_{2n-1}^+)^4(\mathcal{D}_{2n}^+)}{(u_{2n-1}^+u_{2n}^+)^3}\left\{\frac{1}{\widehat{\square}_{2n-1}}\delta^{12}_{2n-1|2n}\right\}\frac{(\mathcal{D}_{2n}^+)^4(\mathcal{D}_1^+)^4}{\widehat{\square}_{2n}\widehat{\square}_1}\\
&\times(\mathcal{D}_1^{--})^2\left\{\delta^{12}_{2n|1}\delta^{(-2,2)}(u_{2n},u_1)\right\}q_a^+(z_1,u_1)\\
&\times q^{+a}(z_2,u_2)q_b^+(z_3,u_3)\dots q_c^+(z_{2n-1},u_{2n-1})q^{+c}(z_{2n},u_{2n}).
\end{aligned}
$$

In order to evade the problem of harmonic singularities, we use hereafter the propagator of $\mathcal{N}=2$ gauge field in the form that is manifestly analytic with respect to both arguments.

Factor $4/n$ originates from the following. The contribution of a superdiagram akin to a ring composed of n repeating segments $\langle\bar{\eta}^+\eta^+\rangle\langle\chi^{++}\bar{\chi}^{++}\rangle$ emerges with a symmetry factor $2/n$. The same factor $2/n$ emerges from a supergraph composed of n repeating segments $\langle\eta^+\bar{\eta}^+\rangle\langle\bar{\chi}^{++}\chi^{++}\rangle$. Each vertex then introduces factor $-i$, and each $\langle\eta^+\bar{\eta}^+\rangle$ and $\langle\bar{\chi}^{++}\chi^{++}\rangle$ introduces factors i and $i/2$, respectively. Therefore, the full number n of segments introduces factor 2^{-n}. Any vertex also supplies a coefficient of $\sqrt{2}$. All this results in a coefficient of 2^n. Taking all these contributions into account, we obtain exactly the coefficient $4/n$.

We start with the direct calculation of the contribution of superdiagram $\Gamma_2[V^{++},q^{+a}]$. In an analytic basis, it is given by

$$
\begin{aligned}
i\Gamma_2 &= \int d\zeta_1^{(-4)}d\zeta_2^{(-4)}du_1du_2\\
&\times\left\{\frac{(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4}{(u_1^+u_2^+)^3}\frac{1}{\widehat{\square}_1}\delta^{12}(1|2)\right\}\\
&\times\left\{\frac{(\mathcal{D}_2^+)^4(\mathcal{D}_1^+)^4}{\widehat{\square}_2\widehat{\square}_1}\delta^{12}(2|1)(\mathcal{D}_1^{--})^2\delta^{-2,2}(u_2,u_1)\right\} \qquad (3.26)\\
&\quad q_a^+(z_1,u_1)q^{+a}(z_2,u_2).
\end{aligned}
$$

In accordance with the general rules for handling these superdiagrams, we should first restore the full Grassmannian integration measure in vertices according to $d^{12}z_1d^{12}z_2 = d^{(-4)}\zeta_1d^{(-4)}\zeta_2(D_1^+)^4(D_2^+)^4$. Since we are interested in the contributions to the effective action that do not depend on space-time derivatives $\mathcal{D}_mq_a^+$ and spinor derivatives $\mathcal{D}^-_{(\alpha,\dot{\alpha})}q_a^+$ of the background hypermultiplet, it is sufficient to use the following superfield constraints:

$$
\mathcal{D}^-_{(\alpha,\dot{\alpha})}q_a^+ = 0,\quad \mathcal{D}^+_{(\alpha,\dot{\alpha})}q^{-a} = 0. \qquad (3.27)
$$

Integrating by parts and using the delta function, we shrink the loop to the point of superspace. However, the following nonlocal expression featuring a harmonic distribution $(u_1^+u_2^+)^{-3}$ with a singularity at coincident points will remain:

$$
\begin{aligned}
i\Gamma_2 &= \int\frac{dzdu_1du_2}{(u_1^+u_2^+)^3}\\
&\times\frac{(\mathcal{D}_2^+)^4(\mathcal{D}_1^+)^4}{\widehat{\square}_1^2\widehat{\square}_2}\delta^{12}(z)\Big|\left\{(\mathcal{D}_1^{--})^2\delta^{-2,2}(u_2,u_1)\right\} \qquad (3.28)\\
&\times q_a^+(z_1,u_1)q^{+a}(z_1,u_2).
\end{aligned}
$$

If we were dealing with flat covariant derivatives, we could use the equality $(D_1^+)^4(D_2^+)^4\delta^8(\theta_1-\theta_2) = (u_1^+u_2^+)^4$ to get rid of this singularity. However, expression (3.28) incorporates covariant spinor derivatives that complicate further analysis. It is theoretically possible to employ the idea suggested in [107]; i.e., we could try to express covariant derivatives $\mathcal{D}_2^{+(\alpha,\dot{\alpha})}$ through covariant derivatives $\mathcal{D}_1^{+(\alpha,\dot{\alpha})}$ when calculating the two-point function of the $(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4\delta^{12}(z_1-z_2)\frac{1}{(u_1^+u_2^+)^q}$ type. This indeed can help in calculating Γ_2 (3.28), but the calculation procedure still remains a technically complicated one.

In our opinion, it is more convenient to start from representation (3.26) and act in the analytic subspace. Instead of calculating contribution (3.28) in the full $\mathcal{N}=2$ superspace, one then should obtain an equivalent expression

$$
\int d\zeta^{-4}(D^+)^4\mathcal{L}(\mathcal{W},\bar{\mathcal{W}},q^+)
$$

with the same $\mathcal{L}$. In order to do that, we switch back to the analytic subspace in (3.28) and use (twice) the relation for the two-point function $\left(\mathcal{D}_1^+\right)^4\left(\mathcal{D}_2^+\right)^4$ [107], as well as harmonic equalities $(u_1^+u_2^+)\big|_{1=2} = 0$, $D_1^{--}(u_1^+u_2^+) = (u_1^-u_2^+)$, $(u_1^+u_2^-)\big|_{1=2} = 1$, and

$$
\begin{aligned}
&\frac{1}{(u_1^+u_2^+)^3}(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4(\mathcal{D}_1^+)^4\\
&= (\mathcal{D}_1^+)^4\left\{\dots+(u_1^+u_2^+)(u_2^-u_1^+)^2(u_1^-u_2^+)^2\widehat{\square}_1\widehat{\square}_2+\dots\right\}.
\end{aligned} \qquad (3.29)
$$

The result is as follows:

$$i\Gamma_2 = \int d\zeta^{(-4)} du_1 du_2 (u_1^+ u_2^+)(q_a^+(u_1) q^{+a}(u_2)) \times (D_1^{--})^2 \delta^{(-2,2)}(u_2 u_1) \frac{(D^+)^4}{\hat{\Box}_2} \delta^{12}(z)\Big|. \tag{3.30}$$

In order to remove the $(D^{--})^2$ factor from the harmonic delta function and be able to apply harmonic equalities

$$(D_1^{--})^2 \delta^{(-2,2)}(u_2 u_1) = (D_2^{--})^2 \delta^{(2,-2)}(u_2 u_1),$$
$$D^{--} q^+ = q^-, \quad (u_1^+ u_2^+)\big|_{1=2} = 0,$$

we restore temporarily the full integration measure and switch back to the analytic measure in the end. Constraints (3.27) should be taken into account in the process.

The integration over one set of harmonics yields the final result for Γ_2

$$i\Gamma_2 = \int d\zeta^{(-4)} du (-4 q_a^- q^{+a}) \frac{(\mathcal{D}^+)^4}{\hat{\Box}} \delta^{12}(z)\Big|. \tag{3.31}$$

At the next step, the contribution with four external lines of hypermultiplet $\Gamma_4[q^+]$ is to be calculated. We start from general expression (3.25) for $n = 2$ and repeat the same operations that were performed in the previous case. This yields

$$\begin{aligned}
i\Gamma_4 = {}& \int d\zeta_1^{(-4)} \ldots d\zeta_4^{(-4)} du_1 \ldots du_4 \frac{(\mathcal{D}_1^+)^4 (\mathcal{D}_2^+)^4}{(u_1^+ u_2^+)^3} \\
&\times \left\{ \frac{1}{\hat{\Box}_1} \delta^{12}(1|2) \right\} \left[\frac{(\mathcal{D}_2^+)^4 (\mathcal{D}_3^+)^4}{\hat{\Box}_2 \hat{\Box}_3} \delta^{12}(2|3) (\mathcal{D}_3^{--})^2 \right. \\
&\left. \times \delta^{(-2,2)}(u_2, u_3) \right] \frac{(\mathcal{D}_3^+)^4 (\mathcal{D}_4^+)^4}{(u_3^+ u_4^+)^3} \left\{ \frac{1}{\hat{\Box}_3} \delta^{12}(3|4) \right\} \\
&\times \left[\frac{(\mathcal{D}_4^+)^4 (\mathcal{D}_1^+)^4}{\hat{\Box}_4 \hat{\Box}_1} \delta^{12}(4|1) (\mathcal{D}_1^{--})^2 \delta^{(-2,2)}(u_4, u_1) \right] \\
&\times q_a^+(z_1, u_1) q^{+a}(z_2, u_2) q_b^+(z_3, u_3) q^{+b}(z_4, u_4) \\
= {}& \int \frac{dz du_1 \ldots du_4}{(u_1^+ u_2^+)^3 (u_3^+ u_4^+)^3} \frac{(\mathcal{D}_1^+)^4 (\mathcal{D}_2^+)^4 (\mathcal{D}_3^+)^4 (\mathcal{D}_4^+)^4}{\hat{\Box}_1^2 \hat{\Box}_2 \hat{\Box}_3^2 \hat{\Box}_4} \\
&\times \delta^{12}(z)\Big| (\mathcal{D}_3^{--})^2 \delta^{(-2,2)}(u_2, u_3) \\
&\times (\mathcal{D}_1^{--})^2 \delta^{(-2,2)}(u_4, u_1) q_a^+(1) q^{+a}(2) q_b^+(3) q^{+b}(4).
\end{aligned} \tag{3.32}$$

Using equalities $(\mathcal{D}_3^{--})^2 \delta^{(-2,2)}(u_2, u_3) = (\mathcal{D}_2^{--})^2 \delta^{(2,-2)}(u_2, u_3)$ and $(\mathcal{D}_2^+)^4 (\mathcal{D}_2^{--})^2 (\mathcal{D}_3^+)^4 \delta^{(2,-2)}(u_2, u_3) = -2\hat{\Box}_2 (\mathcal{D}_2^+)^4$, we obtain

$$\begin{aligned}
i\Gamma_4 = {}& \int dz \frac{du_1 du_2}{(u_1^+ u_2^+)^6} \\
&\times \frac{(\mathcal{D}_1^+)^4 (\mathcal{D}_2^+)^4}{\hat{\Box}_1^2 \hat{\Box}_2^2} \delta^{12}(z)\Big| q_a^+(1) q^{+a}(2) q_b^+(2) q^{+b}(1) \\
= {}& \int d\zeta^{(-4)} \frac{du_1 du_2}{(u_1^+ u_2^+)^2} \\
&\times \frac{(\mathcal{D}_2^+)^4}{\hat{\Box}_2^2} \delta^{12}(z) D_1^{++} q_a^-(1) q^{+a}(2) q_b^+(2) q^{+b}(1) \\
= {}& \int d\zeta^{(-4)} du_1 du_2 [D_1^{--} \delta^{(2,-2)}(u_1 u_2)] \\
&\times \frac{(\mathcal{D}_2^+)^4}{\hat{\Box}_2^2} \delta^{12}(z) q_a^-(1) q^{+a}(2) q_b^+(2) q^{+b}(1).
\end{aligned}$$

Writing down the relation $D_1^{--} \delta^{(-2,2)}(u_1 u_2) = D_2^{--} \delta^{(0,0)}(u_1 u_2)$ and integrating over u_2, we obtain the final result for Γ_4:

$$i\Gamma_4 = \frac{1}{2} \int d\zeta^{(-4)} du (-4 q^- q^+)^2 \frac{(D^+)^4}{\hat{\Box}^2} \delta^{12}(z)\Big|.$$

Here, it was crucial to make use of the constraints (3.27).

General term Γ_{2n} is analyzed in a similar way. First, we transform all integrals over the analytic subspace into integrals over the full superspace at each point using factors $(D_k^+)^4 (D_{k+1}^+)^4$ from the hypermultiplet propagators. Next, the integration over sets of Grassmannian coordinates and space-time coordinates is performed with the corresponding delta functions under the integral. The following expression is obtained as a result:

$$\begin{aligned}
&\int d^{12} z du_1 \ldots du_{2n} \\
&\times \frac{\delta^{(-2,2)}(u_2, u_3) \delta^{(-2,2)}(u_4, u_5) \ldots \delta^{(-2,2)}(u_{2n}, u_1)}{(u_1^+ u_2^+)^3 (u_3^+ u_4^+)^3 \ldots (u_{2n-1}^+ u_{2n}^+)^3} \\
&\times \frac{(\mathcal{D}_2^+)^4 (\mathcal{D}_4^+)^4 \ldots (\mathcal{D}_{2n}^+)^4}{\hat{\Box}_1 \hat{\Box}_2 \ldots \hat{\Box}_{2n}} \\
&\times \left\{ \delta^{12}(z - z')\Big| q_a^+(u_1) q^{+a}(u_2) q_b^+(u_3) \ldots q_c^+(u_{2n-1}) q^{+c}(u_{2n}) \right\}.
\end{aligned} \tag{3.33}$$

Integrating over $u_2, u_4, \ldots, u_{2n}$ with the use of the harmonic delta function, we then obtain

$$\begin{aligned}
&\int \frac{du_1 du_3 \ldots du_{2n-1}}{(u_1^+ u_3^+)^3 (u_3^+ u_5^+)^3 \ldots (u_{2n-1}^+ u_1^+)^3} \frac{(\mathcal{D}_1^+)^4 (\mathcal{D}_3^+)^4 \ldots (\mathcal{D}_{2n-1}^+)^4}{\hat{\Box}_1^2 \hat{\Box}_2^2 \ldots \hat{\Box}_{2n-1}^2} \\
&\times \left\{ \delta^{12}(z - z')\Big| q_a^+(u_1) q^{+a}(u_3) \right. \\
&\left. \times q_b^+(u_3) q^{+b}(u_5) \ldots q_c^+(u_{2n-1}) q^{+c}(u_1) \right\}.
\end{aligned} \tag{3.34}$$

Relabeling indices as $c \to a$, $a \to b, \ldots$; $3 \to 2, \ldots$, $(2n-1) \to n$, one arrives at the following expression:

$$i\Gamma_{2n} = \frac{4(-1)^n 2^n}{n}\int d^{12}z du_1\ldots du_n$$
$$\times \frac{(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4\ldots(\mathcal{D}_n^+)^4}{(u_1^+u_2^+)^3(u_2^+u_3^+)^3\ldots(u_n^+u_1^+)^3}\frac{1}{\hat{\Box}_1^2\hat{\Box}_3^2\ldots\hat{\Box}_{2n-1}^2} \tag{3.35}$$
$$\times\left\{\delta^{12}(z-z')\big|_{z=z'}q^{+a}(u_1)q_b^+(u_1)q^{+b}(u_2)q_c^+(u_2)\ldots q_a^+(u_n)\right\}.$$

In order to simplify it, we represent $q_b^+(u_1)$ as $q_b^+(u_1) = D_1^{++}q_b^-(u_1)$ (since q^+ is on its mass shell). Note that a structure akin to $(D^+)^5 = 0$ is obtained when D^{++} acts on $\hat{\Box}$. We then integrate by parts, throw the harmonic derivative on harmonic distributions

$$-D_1^{++}\frac{1}{(u_1^+u_2^+)^3(u_n^+u_1^+)^3}$$
$$= \frac{1}{2}\left\{(D_1^{--})^2\delta^{(3,-3)}(u_1,u_2)\frac{1}{(u_1^+u_n^+)^3} + (2\leftrightarrow n)\right\}, \tag{3.36}$$

use equality [51, 52]

$$(D_1^{--})^2\delta^{(3,-3)}(1|2) = (D_2^{--})^2\delta^{(-1,1)}(1|2), \tag{3.37}$$

and remove operator factor $(D_2^{--})^2$ from the harmonic delta function. It is easily seen that this operator can yield a nonzero result only when it acts on $(\mathcal{D}^+(u_2))^4$. We obtain

$$\int du_1\ldots\left(-\frac{1}{2}\right)\delta^{(-1,1)}(1|2)$$
$$\times\frac{(\mathcal{D}_1^+)^4(D_2^{--})^2(\mathcal{D}_2^+)^4\ldots(\mathcal{D}_n^+)^4}{(u_2^+u_3^+)^3\ldots(u_n^+u_1^+)^3}\frac{1}{\hat{\Box}_1^2\hat{\Box}_2^2\ldots\hat{\Box}_n^2} \tag{3.38}$$
$$\times\left\{\delta^{12}(z)\big|q^{+a}(u_1)q_b^-(u_1)q^{+b}(u_2)\ldots q_a^+(u_n)\right\} + (2\leftrightarrow n).$$

The second term becomes identical to the first one after the $u_2\leftrightarrow u_n$ substitution. Integrating over u_1, we obtain $(-1)(-2\hat{\Box}_2)\frac{1}{\hat{\Box}_2^4}$. At the second stage, the above operations are repeated for $q_c^+(u_2)$ (i.e., the latter is presented as $q_c^+(u_2) = D_2^{++}q_c^-$ and we integrate by parts with respect to $\mathcal{D}_2^{++}$). Following the procedure that was detailed above, we obtain factor $(-1)^2(-2\hat{\Box}_3)^2\frac{1}{\hat{\Box}_3^6}$.

After $n-4$ similar steps (i.e., writing down $q_d^+(u_3) = D_3^{++}q_d^-(u_3)$, integrating by parts, etc.), the harmonic integral is reduced to expression (3.35) with three sets of harmonics:

$$\frac{(\mathcal{D}_u^+)^4(\mathcal{D}_{u_{n-1}}^+)^4(\mathcal{D}_{u_n}^+)^4}{\hat{\Box}_u^{2n-4}\hat{\Box}_{u_{n-1}}^2\hat{\Box}_{u_n}^2}$$
$$\times\frac{(2\hat{\Box})^{n-3}}{(u^+u_{n-1}^+)^3(u_{n-1}^+u_n^+)^3(u_n^+u^+)^3} \tag{3.39}$$
$$\times\left\{\delta^{12}(z-z')\big|q^{+a}(u)\ldots q_c^+(u)q^{+c}(u_{n-1})q_d^+(u_{n-1})q^{+d}(u_n)q_a^+(u_n)\right\}.$$

At the final step, we write down $q_c^+(u) = D_u^{++}q_c^-$ and pull D_u^{++} over to the harmonic factor. Repeating the same operations, we perform u_{n-1}-integration and obtain the following resulting expression:

$$i\Gamma_{2n} = -\frac{(-2)^n 2^n}{n}\int d^{12}z du du_1\frac{(D_u^+)^4(D_{u_1}^+)^4}{\hat{\Box}_u^n\hat{\Box}_{u_1}^2}$$
$$\times\frac{1}{(u^+u_1^+)^6}\left\{\delta^{12}(z-z')\big|q^{+a}(u)\right. \tag{3.40}$$
$$\left.\times q_b^-(u)q^{+b}(u)\ldots q_c^+(u)q^{+c}(u_1)q_a^+(u_1)\right\}.$$

Let us now switch back to the analytic subspace by applying equalities (3.29):

$$\int d\zeta^{(-4)}\frac{du du_1}{(u^+u_1^+)^2}\frac{(\mathcal{D}_u^+)^4}{\hat{\Box}_u^n}$$
$$\times\delta^{12}(z-z')\big|q^{+a}(u)q_b^-(u)q^{+b}(u)\ldots q_c^+(u)q^{+c}(u_1)q_a^+(u_1).$$

We then use equality $q^+(u_1) = (u_1^+u^-)q^+(u) - (u_1^+u^+)q^-$, $q_a^-q^{-a} = 0$, which yields factor $(u_1^+u_2^+)^2$. The final result is as follows:

$$i\Gamma_{2n} = \frac{1}{n}\int d\zeta^{(-4)}du\frac{(\mathcal{D}^+)^4}{\hat{\Box}^n}\delta^{12}(z-z')\big|(-4q^-q^+)^n. \tag{3.41}$$

Constraints (3.27) are again taken into account here.

Let us now sum up all contributions Γ_{2n} (3.41). The result can be expressed through the functional determinant of a special differential operator:

$$i\Gamma = -\int d\zeta^{(-4)}du\ln\left(1+\frac{4q_a^-q^{+a}}{\hat{\Box}}\right)(\mathcal{D}^+)^4\delta^{12}(z-z')\big|_{z'=z}$$
$$= -\int d\zeta^{(-4)}du[\ln(\hat{\Box}+4q_a^-q^{+a}) \tag{3.42}$$
$$+\ln\hat{\Box}](D^+)^4\delta^{12}(z-z')\big|_{z'=z}.$$

Note that expression (3.42) represents only the hypermultiplet-dependent part of full effective action (3.24). The full effective action contains also a hypermultiplet-independent part $\Gamma(V^{++})$.

It should be emphasized that, although q_a^- is technically present in (3.42), the expression under the integral in (3.42) is an analytic superfield within the low-energy approximation (see constraints (3.27) in the hypermultiplet sector). Indeed, combination $q_a^-q^{+a}$ on the mass shell is harmonically independent [146] and proportional to $q^{ia}q_{ia}$, where q^{ia} is a (constrained) superfield given on the general $\mathcal{N}=2$ superspace. This quantity should be treated as independent of space-time coordinates if we consider only the leading low-energy approximation for the hypermultiplet dependence (see (3.27)). This agrees completely with the general scenario of calculation of the effective action as a series in derivatives of the background field. It is sufficient to set to zero all the possible derivatives of background fields in the effective action in order to isolate the terms without derivatives. Therefore, Eq. (3.42) which was formulated in this section from the very beginning is completely correct in the considered approximation.[12]

Thus, proceeding from the formulation of $\mathcal{N}=4$ supersymmetric Yang–Mills theory in $\mathcal{N}=2$ harmonic superspace and using the harmonic supergraph technique, we have obtained a superfield expression for one-loop effective action in the hypermultiplet sector (3.42). This expression is free from harmonic singularities and, as we will see, allows for direct calculations using the $\mathcal{N}=2$ superfield generalization of the heat kernel method. It will be demonstrated in the next section that general expression (3.42) allows one to obtain an exact proper time representation for the effective action and its expansion in powers of covariant spinor derivatives of the space-time constant Abelian $\mathcal{N}=2$ superfield strength $\mathcal{W}, \bar{\mathcal{W}}$ subject to the conditions

$$\begin{gathered}\mathcal{W}|_{\theta=0} = \text{const},\\ D^{\pm}_{\alpha}\mathcal{W}\big|_{\theta=0} = \text{const}, \quad \bar{D}^{\pm}_{\dot\alpha}\bar{\mathcal{W}}\big|_{\theta=0} = \text{const},\\ D^{-}_{\alpha}D^{+}_{\beta}\mathcal{W}\big|_{\theta=0} = \text{const}, \quad \bar{D}^{-}_{\dot\alpha}\bar{D}^{+}_{\dot\beta}\bar{\mathcal{W}}\big|_{\theta=0} = \text{const},\end{gathered} \tag{3.43}$$

as well as for the background hypermultiplet on the mass shell $q^{\pm a} = q^{ia}u^{\pm}_{i}$, where q^{ia} is independent of harmonics and is regarded as a superfield that is constant in space-time.

One of the basic statements regarding the hypermultiplet dependence of one-loop effective action follows directly from expression (3.42). Namely, the hypermultiplet superfield appears in the effective action in combination $\bar{\mathcal{W}}\mathcal{W} + 2q^{-}_{a}q^{+a}$ that is invariant with respect to full R-symmetry of $\mathcal{N}=4$ superalgebra under the condition that superfield strength $\mathcal{W}$ and the hypermultiplet superfield are placed on the mass shell and are assumed to be independent of space-time coordinates [146]. In order to verify the correctness of this statement, we use representation (3.23) for operator $\hat{\square}$ in the $\hat{\square} + 4q^{-}_{a}q^{+a}$ expression and obtain the following: $\hat{\square} + 4q^{-}_{a}q^{+a} = \frac{1}{2}\mathcal{D}^{\alpha\dot\alpha}\mathcal{D}_{\alpha\dot\alpha} - \frac{i}{\sqrt{2}}((\mathcal{D}^{+\alpha}\mathcal{W})\mathcal{D}^{-}_{\alpha} + (\bar{D}^{+}_{\dot\alpha}\bar{\mathcal{W}})\bar{\mathcal{D}}^{-\dot\alpha}) + 2\mathcal{W}\bar{\mathcal{W}} + 4q^{-}_{a}q^{+a}$. This expression features the combination of superfields mentioned above (two last terms without derivatives), and exactly this combination is the argument of hypermultiplet-dependent effective action in the low-energy approximation.[13]

3.5. Proper Time Representation for the Effective Action

Relation (3.42) implies several remarkable corollaries. First, we have started with the model of two interacting superfields V^{++} and q^{+}, and then summed the supergraphs in such a way that the effective action was expressed in terms of a differential operator that acts only in the vector multiplet sector. This operator contained the complete dependence on hypermultiplets. Second, relation (3.42) has the $\mathrm{Tr}\ln\hat{A}$ form with operator $\hat{A} = \hat{\square} + 4q^{-}_{a}q^{+a}$ that acts on analytic superfields. It should be emphasized that the obtained simple form of one-loop effective action is not immediately obvious; it was the result of summing an infinite series of one-loop harmonic supergraphs with an arbitrary number of external hypermultiplet lines.

Effective action (3.42) is the basis for application of the proper time representation:

$$\begin{aligned}\Gamma &= i\int d\zeta^{(-4)}du\int_0^\infty \frac{ds}{s}e^{-s(\hat{\square}+4q^{-}_{a}q^{+a})}(\mathcal{D}^{+})^{4}\delta^{12}(z-z')\big|_{z'=z}\\ &= i\int_0^\infty \frac{ds}{s}\mathrm{Tr}\left\{K(s)e^{-s(4q^{-}_{a}q^{+a})}\right\}.\end{aligned} \tag{3.44}$$

Here $K(s)$ is the superfield heat kernel, and operation Tr means taking a functional trace in the analytic subspace of the harmonic superspace, $\mathrm{Tr}K(s) = \mathrm{tr}\int d\zeta^{(-4)}K(\zeta,\zeta|s)$, where tr denotes a trace over discrete indices. The problem of finding the one-loop effective action is then reduced to calculating the kernel $K(s) = e^{-s\hat{\square}}$ By definition, operator $\hat{\square}$ has the form[14]

$$\begin{aligned}\hat{\square} &= \frac{1}{2}\mathcal{D}^{\dot\alpha\alpha}\mathcal{D}_{\alpha\dot\alpha} + \frac{i}{2}(D^{+\alpha}\mathcal{W})\mathcal{D}^{-}_{\alpha}\\ &+ \frac{i}{2}(\bar{D}^{+}_{\dot\alpha}\bar{\mathcal{W}})\bar{\mathcal{D}}^{-\dot\alpha} + \mathcal{W}\bar{\mathcal{W}}.\end{aligned} \tag{3.45}$$

In order to calculate effective action (3.44), we use the technique of symbols of operators in the analytic subspace of the full $\mathcal{N}=2$ harmonic superspace.[15] Let

[12]It should be emphasized once more that we consider only the leading (independent of derivatives) contribution of the hypermultiplet to the low-energy effective action. Naturally, this approximation breaks invariance with respect to hidden $\mathcal{N}=2$ supersymmetry. It would be exceptionally interesting and instructive to develop a mathematical procedure for the derivation of an expansion in derivatives of effective action in the hypermultiplet sector, but this problem is outside the scope of the present review.

[13]It is worth noting that this approximation breaks R-symmetries of the initial classical model. The problem of construction of a general expansion of effective action in derivatives that incorporates spinor derivatives of the hypermultiplet and retains all R-symmetries remains unresolved.

[14]Note that substitution $\mathcal{W} \to -\frac{\mathcal{W}}{\sqrt{2}}$, $\bar{\mathcal{W}} \to -\frac{\bar{\mathcal{W}}}{\sqrt{2}}$ transforms expression (3.45) into form (3.23).

[15]The technique of symbols of operators was used for the calculation of effective action in $\mathcal{N}=1$ superspace in [211, 213].

us start with the Fourier transform of the delta function in $\mathcal{N} = 2$ superspace

$$\delta^{12}(z-z') = \int \frac{d^4p}{(2\pi)^4} \int d^4\psi^+ d^4\psi^- e^{ip_m(x-x')^m} \times e^{(\theta-\theta')^{+\alpha}\psi^-_\alpha + \bar\psi^-_{\dot\alpha}(\bar\theta-\bar\theta')^{+\dot\alpha}} e^{(\theta-\theta')^{-\alpha}\psi^+_\alpha + \bar\psi^+_{\dot\alpha}(\bar\theta-\bar\theta')^{-\dot\alpha}}. \tag{3.46}$$

Here $p_M = \{p_m, \psi^\pm_\alpha, \bar\psi^\pm_{\dot\alpha}\}$ is the cotangent supervector at superspace point z, and $d^4\psi^+ = \frac{1}{16} d^2\psi^+ d^2\bar\psi^+$. As a result, the heat kernel at coincident points takes the following form:

$$K(\zeta,\zeta|s) = \int \frac{d^4p}{(2\pi)^4} d^8\psi \exp\Big\{-s\Big(\frac{1}{2}(\mathcal{D}+ip)^{\dot\alpha\alpha}(\mathcal{D}+ip)_{\alpha\dot\alpha} + \frac{i}{2}(\mathcal{D}^{+\alpha}W)(\mathcal{D}+\psi)^-_\alpha + \frac{i}{2}(\bar D^+_{\dot\alpha}\bar W) \times (\bar{\mathcal{D}}+\bar\psi)^{-\dot\alpha} + \mathcal{W}\bar{\mathcal{W}}\Big)\Big\}(\mathcal{D}^+ + \psi^+)^4 \times \mathbf{1}. \tag{3.47}$$

At the next step, we should act by operators in the exponential on unity standing on the right. After accomplishing this, all differential operators will act only on $\mathcal{W}, \bar{\mathcal{W}}$. Thus, the final result will be expressed in terms of strengths $\mathcal{W}, \bar{\mathcal{W}}$ and their spinor derivatives. This procedure is performed in the following way. First, we single out in (3.47) the exponential of main symbol $\frac{1}{2}p^{\alpha\dot\alpha}p_{\alpha\dot\alpha}$ of operator (3.45) and expand the remaining exponential in a power series in covariant derivatives. We then pull out all derivatives to the right, commuting or anticommuting them with the coefficients of operator (3.45) in the course. At the last stage, the derivatives act on unity and thus yield zero. The final step of the considered procedure involves Gaussian integration over momenta p and trivial integration over odd variables ψ. A series in covariant derivatives of strengths is then obtained. However, time-consuming and tedious calculations are needed in order to obtain the final result in a manifestly covariant form in the way described above.

The method yielding a manifestly supersymmetric asymptotic expansion of the heat kernel was developed within the $\mathcal{N} = 1$ superspace in [96, 211, 213]. We will generalize this heat kernel method to the $\mathcal{N} = 2$ harmonic superspace. Let us introduce, at each point of the superspace, a tangent space formed by a system of normal coordinates, with a fibre obtained via a parallel translation from the reference space point. Pseudodifferential operators can be rewritten in this local vector bundle representation. We will analyze the heat kernel using these operators and construct an algorithm for its asymptotic expansion.

Let us introduce the following notations:

$$\begin{gathered} A^{+\alpha} = \frac{i}{2}[D^{+\alpha}, \mathcal{W}], \quad \bar A^{+\dot\alpha} = -\frac{i}{2}[\bar D^{+\dot\alpha}, \bar{\mathcal{W}}], \\ A^{-\alpha} = [D^{-\alpha}, \mathcal{W}], \quad \bar A^{-\dot\alpha} = [\bar D^{-\dot\alpha}, \bar{\mathcal{W}}], \\ \{D^-_\alpha, A^+_\beta\} = N_{\alpha\beta} = N_{\beta\alpha} = \frac{i}{2} D^-_\alpha D^+_\beta \mathcal{W}, \\ \{\bar D^-_{\dot\alpha}, \bar A^+_{\dot\beta}\} = \bar N_{\dot\alpha\dot\beta} = \bar N_{\dot\beta\dot\alpha} = -\frac{i}{2}\bar D^-_{\dot\alpha}\bar D^+_{\dot\beta}\bar{\mathcal{W}}. \end{gathered} \tag{3.48}$$

The algebra of covariant derivatives (3.10) acquires the following form in this notation:

$$\begin{gathered} [\mathcal{D}_{\alpha\dot\alpha}, x^{\beta\dot\beta}] = 2\delta^{\beta\dot\beta}_{\alpha\dot\alpha}, \\ \{\mathcal{D}^-_\alpha, \theta^{+\beta}\} = \delta^\beta_\alpha, \quad \{\bar{\mathcal{D}}^-_{\dot\alpha}, \bar\theta^{+\dot\beta}\} = \delta^{\dot\beta}_{\dot\alpha}, \end{gathered} \tag{3.49}$$

$$\begin{gathered} [\mathcal{D}^{\dot\alpha\alpha}, \mathcal{D}_{\beta\dot\beta}] = -(\delta^{\dot\alpha}_{\dot\beta} N^\alpha_\beta + \delta^\alpha_\beta \bar N^{\dot\alpha}_{\dot\beta}) = -F^{\alpha\dot\alpha}_{\beta\dot\beta}, \\ [\mathcal{D}^-_\alpha, \mathcal{D}_{\beta\dot\beta}] = \varepsilon_{\alpha\beta}\bar A^-_{\dot\beta}, \quad [\bar{\mathcal{D}}^-_{\dot\alpha}, \mathcal{D}_{\beta\dot\beta}] = \varepsilon_{\dot\alpha\dot\beta}A^-_\beta. \\ \mathcal{D}_m A^\pm_{(\alpha,\dot\alpha)} = \mathcal{D}^\pm_{(\delta,\dot\delta)} N_{\alpha\beta} \\ = \mathcal{D}^\pm_{(\delta,\dot\delta)}\bar N_{\dot\alpha\dot\beta} = \mathcal{D}_m N_{\alpha\beta} = \mathcal{D}_m \bar N_{\dot\alpha\dot\beta} = 0. \end{gathered} \tag{3.50}$$

Thus, the set of covariant derivatives together with the on-shell background superfields $\mathcal{W}, \bar{\mathcal{W}}$ corresponding to the space-time constant configurations (3.43) and their lower-order derivatives form a finite-dimensional superalgebra (3.48)–(3.50).

At the next step, we lift the action of shifted operators $X_m = \mathcal{D}_m + ip_m$, $X^\pm_{(\alpha,\dot\alpha)} = \mathcal{D}^\pm_{(\alpha,\dot\alpha)} + \psi^\pm_{(\alpha,\dot\alpha)}$, which satisfy the same algebra (3.48)–(3.50), to the tangent space through "exponential mapping" $X(p_M, \partial/\partial p_M) = U^{-1}X_M U$ with[16]

$$U = e^{-\partial^{-\alpha}_\psi \mathcal{D}^+_\alpha - \bar{\mathcal{D}}^+_{\dot\alpha}\bar\partial^{-\dot\alpha}_\psi} \times e^{2\theta^{-\alpha}p_{\alpha\dot\alpha}\bar\partial^{+\dot\alpha}_\psi - 2\partial^{+\alpha}_\psi p_{\alpha\dot\alpha}\bar\theta^{-\dot\alpha}} \cdot e^{\partial^{+\alpha}_\psi \mathcal{D}^-_\alpha + \bar{\mathcal{D}}^-_{\dot\alpha}\bar\partial^{+\dot\alpha}_\psi} \cdot e^{-\frac{i}{2}\partial^{\dot\alpha\alpha}_p \mathcal{D}_{\alpha\dot\alpha}}.$$

Here, the set of derivatives

$$\partial^{\mp\alpha}_\psi \equiv \frac{\partial}{\partial\psi^\pm_\alpha}, \quad \bar\partial^{\mp\dot\alpha}_\psi \equiv \frac{\partial}{\partial\bar\psi^\pm_{\dot\alpha}},$$
$$\partial^{\alpha\dot\alpha}_p \equiv \frac{\partial}{\partial p_{\alpha\dot\alpha}} \quad [\partial^M, p_N\} = \delta^M_N,$$

serves as tangent vectors that form the basis of the normal coordinate system.

[16] These transformations are basically aimed at eliminating operators $\mathcal{D}_m, \mathcal{D}^+_\alpha, \mathcal{D}^-_{\dot\alpha}$ from X_M. In what follows, the same notation X_M are equally used for the transformed and the initial quantities.

The action of operator U on shifted operators X_M is given by

$$\begin{gathered}U^{-1}X^{+}_{\alpha}U=\psi^{+}_{\alpha},\quad U^{-1}\bar{X}^{+}_{\dot\alpha}U=-\bar{\psi}^{+}_{\dot\alpha},\\ U^{-1}X^{-}_{\alpha}U=-\psi^{-}_{\alpha}+2\bar{\partial}^{-\dot\alpha}_{\psi}p_{\alpha\dot\alpha}+\mathbb{O}(\partial^{-}_{\psi},\partial_p),\\ U^{-1}\bar{X}^{-}_{\dot\alpha}U=\bar{\psi}^{-}_{\dot\alpha}+2\partial^{-\alpha}_{\psi}p_{\alpha\dot\alpha}+\mathbb{O}(\partial^{-}_{\psi},\partial_p),\\ \{X^{+}_{\alpha},\bar{X}^{-}_{\dot\alpha}\}=2X_{\alpha\dot\alpha}-\{X^{-}_{\alpha},\bar{X}^{+}_{\dot\alpha}\},\\ X_{\alpha\dot\alpha}=ip_{\alpha\dot\alpha}+\partial^{+}_{\psi\alpha}(\bar{D}^{-}_{\dot\alpha}\bar{\mathcal{W}})-\bar{\partial}^{+}_{\psi\dot\alpha}(D^{-}_{\alpha}\mathcal{W})\\ -\frac{1}{8}\left\{\partial^{\dot\beta}_{p\alpha}(\bar{D}^{+}_{\dot\beta}\bar{D}^{-}_{\dot\alpha}\bar{\mathcal{W}})+\partial^{\beta}_{p\dot\alpha}(D^{-}_{\beta}D^{+}_{\alpha}\mathcal{W}\right\}+\mathbb{O}(\partial^{-}_{\psi},\partial_p).\end{gathered}\tag{3.51}$$

The mapping of superspace functions at point z onto the tangent superspace is set by transformations

$$\begin{gathered}\mathcal{W}\to\mathcal{W}-\partial^{+\alpha}_{\psi}(D^{-}_{\alpha}\mathcal{W})+\mathbb{O}(\partial^{-}_{\psi},\partial_p),\\ \bar{\mathcal{W}}\to\bar{\mathcal{W}}+\bar{\partial}^{+\dot\alpha}_{\psi}(\bar{D}^{-}_{\dot\alpha}\bar{\mathcal{W}})+\mathbb{O}(\partial^{-}_{\psi},\partial_p),\\ D^{+}_{\alpha}\mathcal{W}\to D^{+}_{\alpha}\mathcal{W}-\partial^{+\beta}_{\psi}(D^{-}_{\beta}D^{+}_{\alpha}\mathcal{W})+\mathbb{O}(\partial^{-}_{\psi},\partial_p),\\ \bar{D}^{+}_{\dot\alpha}\bar{\mathcal{W}}\to\bar{D}^{+}_{\dot\alpha}\bar{\mathcal{W}}+\bar{\partial}^{+\dot\beta}_{\psi}(\bar{D}^{-}_{\dot\beta}\bar{D}^{+}_{\dot\alpha}\bar{\mathcal{W}})+\mathbb{O}(\partial^{-}_{\psi},\partial_p),\\ D^{-}_{\alpha}\mathcal{W}\to D^{-}_{\alpha}\mathcal{W}+\mathbb{O}(\partial^{-}_{\psi},\partial_p),\\ \bar{D}^{-}_{\dot\alpha}\bar{\mathcal{W}}\to D^{-}_{\dot\alpha}\bar{\mathcal{W}}+\mathbb{O}(\partial^{-}_{\psi},\partial_p).\end{gathered}\tag{3.52}$$

Note that in this representation the operators X, together with superfields and their derivatives, satisfy the same algebra (3.49), (3.50). Actually, in the case of arbitrary background fields, all the quantities defined above are represented by infinite series in ∂_p and finite series in powers of Grassmannian derivatives ∂_ψ with coefficients given at fixed point z^A. However, representation (3.51), (3.52) is exact for the considered background.

The actual calculation of effective action (3.44) with kernel (3.47) is based on the following observation. The operator exponential $e^{-s\hat{\square}_U}$, where operator $\hat{\square}_U$ is expressed in terms of shifted variables X (3.51), can be regarded as an operator of evolution in a quantum Bose–Fermi system with Hamiltonian $\hat{H}=\hat{\square}_U$. Equations (3.45) and (3.51) demonstrate that Hamiltonian $\hat{H}$ is a quadratic form of operators $p,\partial_p,\psi,\partial_\psi$ with constant coefficients (owing to the constraints imposed on the considered background fields). Therefore, the calculation of $\mathrm{Tr}K(s)=\int d\zeta^{-4}K(\zeta,\zeta|s)$ with kernel $K(\zeta,\zeta|s)$, which is defined by (3.47), is an exactly solvable problem.

Let us return to Eq. (3.47) in which all operators and fields (together with their derivatives) are written in representation (3.51). This is equivalent to a statement that the heat kernel is extended to the tangent bundle at superspace point z, where supercoordinate z is regarded as a constant parameter. According to (3.47), the evolution operator should finally act on unity. It is evident that, in order to determine the result of such action, one should pull over all derivatives with respect to p and ψ to the right and discard them at the last step (when they act on unity). This procedure can be performed, based on the Baker–Campbell–Hausdorff formula corresponding to algebra (3.49), (3.50). As a result, we obtain the so-called evolution operator symbol. This symbol should be integrated over bosonic variables p and fermionic variables ψ, which would lead to the trace of heat kernel (3.47). It is also worth noting that the exponential factor of the evolution operator contains only the operators $\mathcal{D}^-\sim\psi^-$. All $\mathcal{D}^+\sim\psi^+$ are concentrated in the $(\mathcal{D}^{+4})$ pre-exponential factor and saturate the integral with respect to $d^4\psi^+$. Therefore, we should omit all terms $\mathbb{O}\left(\partial^-_\psi\right)$ in operator $\hat{H}$ so as to obtain a less complicated expression for (3.51), (3.52). In addition, in order to perform Berezin integration in the sector of variables ψ^-, we should take out "projector" $(\psi^-)^4$ from the considered exponential.

This algorithm can be implemented efficiently if we present exponential $K(s)=e^{-s\hat{\square}}$ as a product of several operator exponentials. This construction helps circumvent certain difficulties that arose in previous attempts to calculate the effective action of $\mathcal{N}=2$ gauge theory directly in $\mathcal{N}=2$ superspace. Let us write operator $K(s)=e^{-s\hat{\square}}$ in the form

$$\begin{aligned}K(s)&=\exp\left(-s\left\{A^{+}\mathcal{D}^{-}+\bar{A}^{+}\bar{\mathcal{D}}^{-}+\frac{1}{2}\mathcal{D}^{\dot\alpha\alpha}\mathcal{D}_{\alpha\dot\alpha}+\mathcal{W}\bar{\mathcal{W}}\right\}\right)\\ &=\exp\{-f_{\alpha\dot\alpha}(s)\mathcal{D}^{\dot\alpha\alpha}\}\exp\left\{-s\frac{1}{2}\mathcal{D}^{\dot\alpha\alpha}\mathcal{D}_{\dot\alpha\alpha}\right\}\\ &\quad\times\exp\{-\Omega(s)\}\exp\{-s(A^{+\alpha}\mathcal{D}^{-}_{\alpha}+\bar{A}^{+\dot\alpha}\bar{\mathcal{D}}^{-}_{\dot\alpha})\}\end{aligned}\tag{3.53}$$

with certain unknown coefficients in the right-hand side. These coefficients can be determined directly (i.e., based on the Baker–Campbell–Hausdorff theorem; see the representation of the Baker–Campbell–Hausdorff formula below) or as a solution of a system of differential equations for coefficients. These methods yield the same results.

In order to construct the system of equations just mentioned, one should replace K in $\left(\frac{d}{ds}K\right)K^{-1}$ by the first and the second lines of (3.53). The following equations for functions $f^{\dot\alpha\alpha}(s)$ are thus produced:

$$\begin{aligned}\dot{f}_{\alpha\dot\alpha}(s)&=-f_{\beta\dot\beta}F^{\dot\beta\beta}_{\dot\alpha\alpha}-A^{+\beta}(D^{-}_{\beta}f_{\alpha\dot\alpha})-\bar{A}^{+\dot\beta}(\bar{D}^{-}_{\dot\beta}f_{\alpha\dot\alpha})\\ &+A^{+}_{\beta}\bar{A}^{-}_{\dot\beta}\left(\int_0^s d\tau e^{\tau F}\right)^{\dot\beta\beta}_{\dot\alpha\alpha}+\bar{A}^{+}_{\dot\beta}A^{-}_{\beta}\left(\int_0^s d\tau e^{\tau F}\right)^{\dot\beta\beta}_{\dot\alpha\alpha}.\end{aligned}\tag{3.54}$$

Likewise, the equation for function Ω is

$$\dot{\Omega}(s) - \mathcal{W}\bar{\mathcal{W}} = -A^{+\alpha}(D^-_\alpha\Omega) - \bar{A}^{+\dot\alpha}(\bar{D}^-_{\dot\alpha}\Omega) + A^+_\alpha f^{\alpha\dot\alpha}\bar{A}^-_{\dot\alpha} + \bar{A}^+_{\dot\alpha} f^{\dot\alpha\alpha}A^-_\alpha - \frac{1}{2}A^+_{\dot\beta}\bar{A}^-_{\dot\beta} \tag{3.55}$$
$$\times\left(\int_0^s d\tau e^{\tau F}\right)^{\dot\beta\beta}_{\dot\alpha\alpha} F^{\dot\alpha\alpha}_{\dot\rho\rho}f^{\dot\rho\rho} - \frac{1}{2}\bar{A}^+_{\dot\beta}A^-_\beta\left(\int_0^s d\tau e^{\tau F}\right)^{\dot\beta\beta}_{\dot\alpha\alpha} F^{\dot\alpha\alpha}_{\dot\rho\rho}f^{\dot\rho\rho}.$$

It is easy to demonstrate that the solution of Eq. (3.54) can be written as

$$f_{\alpha\dot\alpha} = -A^+_\delta\mathcal{N}^{\delta\dot\delta}_{\alpha\dot\alpha}(s)\bar{A}^-_{\dot\delta} - \bar{A}^+_{\dot\delta}\bar{\mathcal{N}}^{\dot\delta\delta}_{\dot\alpha\alpha}(s)A^-_\delta, \tag{3.56}$$

where functions $\mathcal{N}(N,\bar{N})$, $\bar{\mathcal{N}}(N,\bar{N})$ are given below. The solution of Eq. (3.55) is as follows:

$$\Omega(s) = s\mathcal{W}\bar{\mathcal{W}} + A^{+\alpha}\Omega^-_\alpha(s) + \bar{A}^{+\dot\alpha}\bar{\Omega}^-_{\dot\alpha}(s) + (A^+)^2\Psi^{(-2)}(s) + (\bar{A}^+)^2\bar{\Psi}^{(-2)}(s) + A^{+\alpha}\bar{A}^+_{\dot\alpha}\Psi^{\dot\alpha(-2)}_\alpha(s). \tag{3.57}$$

Note that this solution is a polynomial of a finite order in powers of Grassmannian elements $A^\pm$ and $\bar{A}^\pm$ (3.48). Coefficients $\Omega^-_\alpha(s)$, $\bar{\Omega}^-_{\dot\alpha}(s)$, $\Psi^{(-2)}(s)$, $\bar{\Psi}^{(-2)}(s)$, and $\Psi^{\dot\alpha(-2)}_\alpha(s)$ are given by (3.78)–(3.84).

It is instructive to present the last exponential in (3.53) in the form

$$e^{-s(A^+\mathcal{D}^- + \bar{A}^+\bar{\mathcal{D}}^-)} = 1 + a^{+\alpha}\mathcal{D}^-_\alpha + \bar{a}^{+\dot\alpha}\bar{\mathcal{D}}^-_{\dot\alpha} + f^{+2}(\mathcal{D}^-)^2 + \bar{f}^{+2}(\bar{\mathcal{D}}^-)^2 + f^{+2\dot\alpha\alpha}\mathcal{D}^-_\alpha\bar{\mathcal{D}}^-_{\dot\alpha} + \Xi^{+3\dot\alpha}\bar{\mathcal{D}}^-_{\dot\alpha}(\mathcal{D}^-)^2 + \Xi^{+3\alpha}\mathcal{D}^-_\alpha(\bar{\mathcal{D}}^-)^2 + \Omega^{+4}(\mathcal{D}^-)^2(\bar{\mathcal{D}}^-)^2. \tag{3.58}$$

The coefficients in this expansion can be determined exactly. Note that $\Omega^{+4}(s) \sim (A^+)^4$. The expression under the integral for kernel (3.47) can be presented as a product of a Schwinger-type kernel $e^{-s\frac{1}{2}\mathcal{D}^{\alpha\dot\alpha}\mathcal{D}_{\alpha\dot\alpha}}$ and terms of the expansion of (3.58) in powers of $\mathcal{D}^-$, as well as the expansion of $e^{-f_{\alpha\dot\alpha}\mathcal{D}^{\alpha\dot\alpha}}$, $e^{-\Omega(s)}$ in powers of Grassmannian quantities $A^+, \bar{A}^+$. This leads to the following representation:

$$K(s) = \int\frac{d^4p}{(2\pi)^4}d^8\psi e^{-s\frac{1}{2}X^{\alpha\dot\alpha}X_{\alpha\dot\alpha}} \times\left\{1 + \frac{1}{2}f_{\alpha\dot\alpha}(s)X^{\alpha\dot\alpha}f_{\beta\dot\beta}(s)X^{\beta\dot\beta} + \ldots\right\} \times e^{-\Omega(s)}\{1 + \ldots + \Omega^{+4}(s)(\psi^-)^4\}(\psi^+)^4\times\mathbf{1}. \tag{3.59}$$

One can considerably simplify this intricate expression by making use of the remarkable properties of the Berezin integral. Only the last term in the last bracket gives a nontrivial contribution. Since $\Omega^{+4} \sim (A^+)^4$ and all $f_{\alpha\dot\alpha}$, Ω are constructed from elements $A^+, \bar{A}^+$, we can omit all these functions (except for $e^{-s\mathcal{W}\bar{\mathcal{W}}}$) in the expression for $K(s)$. The following expression is thus obtained:

$$K(s) = \int\frac{d^4p}{(2\pi)^4}K_{Sch}(s)e^{-s\mathcal{W}\bar{\mathcal{W}}}\Omega^{+4}(s). \tag{3.60}$$

The last step involves calculating the Schwinger kernel for operator $\square_{cov}(X_m)$ $K_{Sch}(s) \equiv \int\frac{d^4p}{(2\pi)^4}\cdot e^{-s\square_{cov}(X_m)}$, where operators $X_{\alpha\dot\alpha}$ are defined in (3.51). Such calculations have now become common (see, for example, [96, 107, 144]).[17] Therefore, we quote only the final result

$$K_{Sch}(s) = \frac{i}{(4\pi s)^2}\frac{s^2(N^2 - \bar{N}^2)}{\cosh(sN) - \cosh(s\bar{N})}. \tag{3.61}$$

Here $N = \sqrt{-\frac{1}{2}D^4\mathcal{W}^2}$. This quantity can be expressed through two invariants of the Abelian vector field, $\mathcal{F} = \frac{1}{4}F^{mn}F_{mn}$ and $\mathcal{G} = \frac{1}{4}{}^{\star}F^{mn}F_{mn}$, as $N = \sqrt{2(\mathcal{F} + i\mathcal{G})}$. In the context of $\mathcal{N} = 4$ supergauge theory, kernel (3.61) was found in [144] with the use of various indirect approaches. Here we have obtained a complete expression for kernel (3.61) in terms of $\mathcal{N} = 2$ harmonic superfields.

3.6. Effective Action and Its Expansion in Covariant Spinor Derivatives

The proper time method in harmonic superspace was developed in the previous subsection. Let us apply this technique for the construction of effective action.

The effective action is written in form (3.44), where the heat kernel at coincident points is defined by (3.47). We apply expansion (3.60), (3.61) for $K_{Sch}(s)$ (3.90), $\Omega^{(+4)}$ and take the following facts into account: (i) equation (3.47) already contains $(\psi^+)^4$, thus one can use $\int d^4\psi^+(\psi^+)^4 = 1$ directly; (ii) it is sufficient to retain only the last term $\Omega^{+4}(\psi^-)^4$ in Eq. (3.58) in order to saturate the integration with respect to ψ^-; (iii) since $\Omega^{+4} \sim (A^+)^4$ (see (3.90)), all terms depending on A^+ should be omitted in (3.53).

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[17] Note that it is trivial to perform the *tr* operation over Lorentz indices, due to the identity

$$N^{\ \beta}_{\alpha}N^{\ \delta}_{\beta} = -\frac{1}{4}D^-_\alpha D^{+\beta}\mathcal{W}D^-_\beta D^{+\delta}\mathcal{W} = -\frac{1}{8}D^-_\alpha D^{+\beta}D^-_\beta D^{+\delta}\mathcal{W}^2 = -\frac{1}{2}\delta^\delta_\alpha(D)^4\mathcal{W}^2 = \delta^\delta_\alpha N^2.$$

The following resulting form of effective action is thus obtained:

$$\Gamma = \frac{1}{(4\pi)^2}\int d\zeta^{(-4)}du\int_0^\infty \frac{ds}{s^3}e^{-s(2\mathcal{W}\bar{\mathcal{W}}+4q_a^- q^{+a})} \times \frac{s^2(N^2-\bar{N}^2)}{\cosh(sN)-\cosh(s\bar{N})}\frac{1}{16}(D^{+}\mathcal{W})^2 \times (\bar{D}^{+}\bar{\mathcal{W}})^2\frac{\cosh(sN)-1}{N^2}\frac{\cosh(s\bar{N})-1}{\bar{N}^2}. \quad (3.62)$$

One can show that the integrand in (3.62) may be expanded in a power series in s^2N^2, $s^2\bar{N}^2$. After changing the proper time parameter s by $s' = 2s\mathcal{W}\bar{\mathcal{W}}$, we obtain an expansion in powers of $s'^2\frac{4N^2}{4(\mathcal{W}\bar{\mathcal{W}})^2}$ and their conjugates. In addition, since the expression under the integral sign in (3.62) is already proportional to $(A^+)^4$, we can replace the $N^2, \bar{N}^2$ quantities in each term of expansion with superconformal invariants Ψ^2 and $\bar{\Psi}^2$ [144] that are given by

$$\bar{\Psi}^2 = \frac{1}{\bar{\mathcal{W}}^2}D^4\ln\mathcal{W} = \frac{1}{2\bar{\mathcal{W}}^2} \times \left\{\frac{N_\alpha^\beta N_\beta^\alpha}{\mathcal{W}^2} + 4\frac{A^{+\alpha}N_\alpha^\beta A_\beta^-}{\mathcal{W}^3} + 3\frac{(A^+)^2(A^-)^2}{\mathcal{W}^4}\right\} \quad (3.63)$$

and its conjugate. Having performed all these operations, one can show that each term of expansion is written as an integral over a general $\mathcal{N} = 2$ superspace.

Expression (3.62) without hypermultiplets was obtained in [144, 146] by other methods. The hypermultiplet dependence of effective action was obtained in [215] in terms of $\mathcal{N} = 1$ superfields; its transformation into an $\mathcal{N} = 2$ harmonic form was performed in [215] based on a heuristic reasoning for the restoration of a manifestly $\mathcal{N} = 2$ supersymmetric form of effective action that is written initially in terms of $\mathcal{N} = 1$ superfields. The complete hypermultiplet dependence of effective action (3.62) was obtained above completely in terms of harmonic superfields, and this dependence coincides with the one given in [215]. Thus, we have confirmed the correctness of the consideration used in [215].

Let us present the first few terms of expansion of effective action (3.62) in a power series of $N^2, \bar{N}^2$ and compare them to the results of $\mathcal{N} = 1$ superfield calculations based on the above-mentioned special consideration for the restoration of a manifestly $\mathcal{N} = 2$ supersymmetric form of effective action in the formalism of $\mathcal{N} = 1$ superfields. Our aim is to show that the expansion in spinor derivatives in (3.62) actually reproduces the expansion in [215]. Each term of the expansion of effective action contains a certain power of Abelian strength F_{mn}. This expansion allows one to isolate an explicit dependence on q^-q^+. Since $N^2, \bar{N}^2$ include the spinor covariant derivatives of superfield strengths (see (3.48)), the expansion in these quantities is exactly the expansion in spinor covariant derivatives of strengths $\mathcal{W}, \bar{\mathcal{W}}$.

Using expansions $K_{Sch}(s)$ and $\Omega^{+4}(s)$, we arrive at the following expression:

$$\Gamma = \frac{1}{(8\pi)^2}\int d\zeta^{(-4)}du\int_0^\infty ds\cdot s e^{-s(2\mathcal{W}\bar{\mathcal{W}}+4q^-q^+)} \times D^{+}\mathcal{W}D^{+}\mathcal{W}\bar{D}^{+}\bar{\mathcal{W}}\bar{D}^{+}\bar{\mathcal{W}} \times\left\{1+\frac{s^4}{2\cdot 5!}D^4\mathcal{W}^2\bar{D}^4\bar{\mathcal{W}}^2+\ldots\right\}.$$

The leading low-energy correction corresponds to term F^4. Integrating over s, we obtain

$$\Gamma_{F^4} = \frac{1}{(4\pi)^2}\int d\zeta^{(-4)}du\frac{1}{16}\frac{D^{+}\mathcal{W}D^{+}\mathcal{W}}{\mathcal{W}^2}\frac{\bar{D}^{+}\bar{\mathcal{W}}\bar{D}^{+}\bar{\mathcal{W}}}{\bar{\mathcal{W}}^2} \times\frac{1}{\left(1-\left(-\frac{2q^-q^+}{\mathcal{W}\bar{\mathcal{W}}}\right)\right)^2} = \sum_{k=0}^\infty\frac{1}{(4\pi)^2}\int d\zeta^{(-4)}du \times\frac{1}{16}\frac{D^{+}\mathcal{W}\ldots\bar{D}^{+}\bar{\mathcal{W}}}{(\mathcal{W}\bar{\mathcal{W}})^{k+2}}(k+1)(-2q^-q^+)^k \quad (3.64)$$

$$\times\frac{1}{(4\pi)^2}\int d\zeta^{(-4)}du\frac{1}{16}\left\{D^{+2}\ln\mathcal{W}\bar{D}^{+2}\ln\bar{\mathcal{W}} + \sum_{k=1}^\infty\frac{1}{k^2(k+1)}D^{+2}\frac{1}{\mathcal{W}^k}\bar{D}^{+2}\frac{1}{\bar{\mathcal{W}}^k}(-2q^-q^+)^k\right\},$$

which yields

$$\Gamma_{F^4} = \frac{1}{(4\pi)^2}\int d^{12}z \times\left\{\ln\mathcal{W}\ln\bar{\mathcal{W}} + \sum_{k=1}^\infty\frac{1}{k^2(k+1)}\left(\frac{-2q^{ai}q_{ai}}{\mathcal{W}\bar{\mathcal{W}}}\right)^k\right\}. \quad (3.65)$$

This expression coincides with the results of [147, 215]:[18]

$$\Gamma_{F^4} = \frac{1}{(4\pi)^2}\int d^{12}z \times\left\{\ln\mathcal{W}\ln\bar{\mathcal{W}} + \mathrm{Li}_2(X) + \ln(1-X) - \frac{1}{X}\ln(1-X)\right\},$$

---

[18] We may use the relation $\int d\zeta^{(-4)}(D^+)^4 = \int d^{12}z$ and $\int du = 1$, since the hypermultiplet superfields on the mass shell are presented as $q^{\pm a} = q^{ia}u_i^\pm$ (and superstrengths $\mathcal{W}, \bar{\mathcal{W}}$ do not depend on harmonics) in the central basis of $\mathcal{N} = 2$ harmonic superspace.

where $X = \frac{-2q^{ai}q_{ai}}{\mathcal{W}\bar{\mathcal{W}}}$, and $\mathrm{Li}_2(X)$ is the Euler dilogarithm.

Next-to-leading correction to F^8-term[19] has the form:

$$\Gamma_{F^8} = \sum_{k=0}^{\infty}\frac{1}{2(4\pi)^2}\int d\zeta^{(-4)}du$$
$$\times\frac{1}{16}\frac{D^{+}\mathcal{W}\ldots\bar{D}^{+}\bar{\mathcal{W}}}{(\mathcal{W}\bar{\mathcal{W}})^6}D^4\mathcal{W}^2\bar{D}^4\bar{\mathcal{W}}^4 \tag{3.66}$$
$$\times\frac{1}{5!}(k+1)(k+2)(k+3)(k+4)(k+5)\left(\frac{-2q^{-}q^{+}}{\mathcal{W}\bar{\mathcal{W}}}\right)^k.$$

Using extended expressions

$$D^4\mathcal{W}^2 4D^{+}_{\alpha}D^{-\beta}\mathcal{W}D^{+\alpha}D^{-}_{\beta}\mathcal{W},$$

and

$$D^{+\alpha}D^{+}_{\alpha}(D^{+\delta}\mathcal{W}D^{+}_{\delta}\mathcal{W}D^{-\beta}\mathcal{W}D^{-}_{\beta}\mathcal{W})$$
$$= 2D^{+\delta}\mathcal{W}D^{+}_{\delta}\mathcal{W}D^{+}_{\alpha}D^{-\beta}\mathcal{W}D^{+\alpha}D^{-}_{\beta}\mathcal{W},$$

we obtain a sequence of identities

$$\frac{1}{(\mathcal{W})^{k+6}}D^{+}\mathcal{W}D^{+}\mathcal{W}D^4\mathcal{W}$$
$$= 2D^{+}D^{+}\left\{\frac{D^{+}\mathcal{W}D^{+}\mathcal{W}}{\mathcal{W}^{k+4}}\frac{D^{-}\mathcal{W}D^{-}\mathcal{W}}{\mathcal{W}^2}\right\}$$
$$= -2D^{+}D^{+}\left\{\frac{1}{(k+2)(k+3)}D^{+}D^{+}\frac{1}{\mathcal{W}^{k+2}}D^{-}D^{-}\ln\mathcal{W}\right\}$$
$$= -2D^{+}D^{+}\left\{\frac{1}{(k+2)(k+3)}\frac{1}{\mathcal{W}^{k+2}}D^4\ln\mathcal{W}\right\}.$$

Similar operations are also performed for the complex conjugate term. Using these identities and restoring the full measure $d^{12}z = d\zeta^{(-4)}(D^{+})^4$, one obtains the following factor:

$$\frac{1}{(k+2)^2(k+3)^2}\frac{1}{\bar{\mathcal{W}}^2}D^4\ln\mathcal{W}\frac{1}{\mathcal{W}^2}\bar{D}^4\ln\bar{\mathcal{W}}\left(\frac{-2q^{-}_{a}q^{+a}}{\mathcal{W}\bar{\mathcal{W}}}\right)^k.$$

This allows one to present Γ_{F^8} in the form:

$$\Gamma_{F^8} = \frac{1}{2(4\pi)^2 5!}\int d^{12}z$$
$$\times\sum_{k=0}^{\infty}\frac{(k+1)(k+4)(k+5)}{(k+2)(k+3)}\Psi^2\bar{\Psi}^2\left(\frac{-2q^{-}_{a}q^{+a}}{\mathcal{W}\bar{\mathcal{W}}}\right)^k$$
$$= \frac{1}{2(4\pi)^2 5!}\int d^{12}z\Psi^2\bar{\Psi}^2\left\{\frac{1}{(1-X)^2}+\frac{4}{(1-X)}\right. \tag{3.67}$$
$$\left.+\frac{6X-4}{X^3}\ln(1-X)+4\frac{X-1}{X^2}\right\}.$$

[19] The F^6-correction does not emerge in the one-loop effective action for $\mathcal{N} = 4$ Yang–Mills theory [90, 144].

This result coincides with the one obtained in $\mathcal{N} = 1$ approach using the consideration from [215].

Analogous analysis allows us to obtain, in principle, any term $\Gamma_{F^{2n}}$ of the expansion of effective action (3.62) in derivatives. We pay attention to the fact that in each term of expansion the integrals over the analytic subspace can be transformed into integrals over the full $\mathcal{N} = 2$ superspace.

3.7. Useful Representation of the Baker–Campbell–Hausdorff Formula

The heat kernel associated with operator $\mathbb{O}$ is defined as a matrix element of operator $e^{\mathbb{O}}$. In many cases the operator $\mathbb{O}$ is a linear combination of basis operators that form a certain (super)algebra. Then the heat kernel calculation is simplified greatly due to possibility to use the Baker–Campbell–Hausdorff (BCH) formula. According to this formula, the exponent, which is a product of two exponents with noncommuting operators A and B, can be presented as a series in powers of commutators $[A, B]$, $[A, [A, B]]$, $[B, [A, B]]$, etc. We will derive another convenient representation for the BCH formula:

$$e^{A+B} = e^{C_1+C_2+C_3+\ldots}e^{A}, \tag{3.68}$$

where the operators C_k are expressed through commutators of operators A and B. We will show that operators C_k can be defined in such a way that each C_k contains the kth power of operator B and all powers of operator A. This representation of the BCH formula turns out to be useful in the cases when operators A and B are associated with a certain (super)algebra that allows one to sum operator series $C_1 + C_2 + C_3 + \ldots$ to explicit expressions.

Let us introduce variable t in (3.68):

$$e^{A+tB} = e^{tC_1+t^2C_2+t^3C_3+\ldots}e^{A}. \tag{3.69}$$

We then define function

$$\mathcal{F} = e^{A+tB}e^{-A} = e^{\sum_{k=1}^{\infty}t^kC_k(A,B)} \tag{3.70}$$

and determine the appropriate operators C_k in (3.69). It is evident that at $t = 0$ (3.69) will be the identity and at $t = 1$ we obtain the initial relation (3.68). Let us calculate the logarithmic derivative of function (3.69) with respect to t. On the one hand,

$$\dot{\mathcal{F}}\mathcal{F}^{-1} = \int_0^1 d\tau e^{\tau A+\tau tB}Be^{-\tau A-\tau tB}. \tag{3.71}$$

On the other hand,

$$\mathcal{F} = e^{tC_1 + t^2 C_2 + \dots}, \quad \dot{\mathcal{F}}\mathcal{F}^{-1} = C_1 + 2tC_2 + \dots. \tag{3.72}$$

If we set $t = 0$, one obtains

$$C_1 = \int_0^1 d\tau e^{\tau A} B e^{-\tau A} = \int_0^1 d\tau \mathcal{B}(\tau). \tag{3.73}$$

In order to determine C_2, we should calculate the first-order derivative of the logarithmic derivative with respect to t:

$$\begin{aligned}\frac{d}{dt}(\dot{\mathcal{F}}\mathcal{F}^{-1}) &= \int_0^1\int_0^1 d\tau' d\tau \\ &\times \{e^{\tau'\tau A + \tau'\tau t B}(\tau B)e^{-\tau'\tau A - \tau'\tau t B}\}\{e^{\tau A + \tau t B} B e^{-\tau A - \tau t B}\} \\ &+ \int_0^1\int_0^1 d\tau' d\tau \{e^{\tau A + \tau t B} B e^{-\tau A - \tau t B}\} \\ &\times \{e^{\tau'\tau A + \tau'\tau t B}(-\tau B)e^{-\tau'\tau A - \tau'\tau t B}\}.\end{aligned} \tag{3.74}$$

At $t = 0$, we obtain the expression for C_2:

$$2C_2 = \int_0^1\int_0^1 d\tau d\tau' \cdot \tau \cdot [\mathcal{B}(\tau'\tau), \mathcal{B}(\tau)]. \tag{3.75}$$

Following the same procedure, one can find all operators C_k. For example, operator C_3 is constructed by calculating the second-order derivative of the logarithmic derivative determined above. This results in the following relation:

$$\begin{aligned}6C_3 + [C_1, C_2] &= \int_0^1 d\tau'' d\tau' d\tau \\ &\times \tau^2\{[\mathcal{B}(\tau'\tau), [\mathcal{B}(\tau''\tau), \mathcal{B}(\tau)]] \\ &- \tau'[\mathcal{B}(\tau), [\mathcal{B}(\tau''\tau'\tau), \mathcal{B}(\tau'\tau)]]\},\end{aligned} \tag{3.76}$$

where C_1 and C_2 have been determined earlier. Operator C_4 is derived from the relation

$$\begin{aligned}24C_4 + 6[C_1, C_3] + [C_1, [C_1, C_2]] &= \int_0^1 d\tau''' d\tau'' d\tau' d\tau \cdot \tau^2\{\tau'\tau[[\mathcal{B}(\tau'''\tau'\tau), \mathcal{B}(\tau'\tau], [\mathcal{B}(\tau''\tau), \mathcal{B}(\tau)]] \\ &- \tau''\tau[\mathcal{B}(\tau'\tau), [\mathcal{B}(\tau), [\mathcal{B}(\tau'''\tau''\tau), \mathcal{B}(\tau''\tau)]]] + \tau[\mathcal{B}(\tau'\tau), [\mathcal{B}(\tau''\tau), [\mathcal{B}(\tau'''\tau), \mathcal{B}(\tau)]]]\} \\ &\times \tau^2\tau'\{\tau''\tau'\tau[[[\mathcal{B}(\tau'''\tau''\tau'), \mathcal{B}(\tau''\tau'\tau)], \mathcal{B}(\tau'\tau)], \mathcal{B}(\tau)] \\ &+ \tau'\tau[[[\mathcal{B}(\tau''\tau'\tau), [\mathcal{B}(\tau'''\tau'\tau), \mathcal{B}(\tau'\tau)], \mathcal{B}(\tau)] + \tau[[\mathcal{B}(\tau''\tau'\tau), \mathcal{B}(\tau'\tau)], [\mathcal{B}(\tau'''\tau), \mathcal{B}(\tau)]]\},\end{aligned} \tag{3.77}$$

and all the other terms of the BCH series in formula (3.68) can be found in a similar way.

The problem in the present case consists in rewriting exponential (3.53) of a sum of operators that satisfy algebraic relations (3.48), (3.50) in the form of a product of exponentials of separate operators. This problem is solved in two steps.

First, we take as A in (3.68) the operator $A^+D^- + \bar{A}^+\bar{D}^-$ and present C_1 as a linear combination of operators $\frac{1}{2}\mathcal{D}^{\alpha\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}}$, $f^{\alpha\dot{\alpha}}(A^\pm, \bar{A}^\pm, N, \bar{N})\mathcal{D}_{\alpha\dot{\alpha}}$ with certain coefficients[20] plus certain functions of $\mathcal{W}, \bar{\mathcal{W}}, A^\pm, \bar{A}^\pm, N, \bar{N}$ as central elements. This means that all other operators $C_2, C_3, \dots$ should be proportional to operator $\mathcal{D}_{\alpha\dot{\alpha}}$ with certain functions as a coefficient. Therefore, series $C_1 + C_2 + C_3 + \dots.$ is reduced to summing these coefficient functions (this summing can be performed explicitly). The final result is $-s\frac{1}{2}\mathcal{D}^{\alpha\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}}$ plus $f^{\alpha\dot{\alpha}}(A^\pm, \bar{A}^\pm, N, \bar{N})\mathcal{D}_{\alpha\dot{\alpha}}$, plus a certain central element.

Second, we apply Eq. (3.68) once more to expression $\exp\left(-s\frac{1}{2}\mathcal{D}^{\alpha\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}} + f^{\alpha\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}}\right)$ and use operator $-s\frac{1}{2}\mathcal{D}^{\alpha\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}}$ as A. All operators C_k are again proportional to one operator $\mathcal{D}_{\alpha\dot{\alpha}}$, and the set of coefficient functions can be summed to explicit expressions. As a result, we obtain the right part of expression (3.53). All coefficient functions are given in Subsection 3.8 (see relations (3.78)–(3.84)). These functions can also be derived from differential equations (3.54), (3.55). The results obtained based on the BCH formula and the solutions of the above-mentioned differential equations agree with each other.

3.8. Coefficient Functions in the Heat Kernel Expansion

The solutions of linear differential equations (3.54) and (3.55) for $f_{\alpha\dot{\alpha}}(s)$ and $\Omega(s)$ can be found exactly

[20] For example, $\left[A^+\mathcal{D}^-, \frac{1}{2}(\mathcal{D}^{\alpha\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}})\right] = -A^{+\alpha}\bar{A}^{-\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}}$.

and are of the form (3.56), (3.57). The coefficients of the expansion $f_{\alpha\dot{\alpha}}(s)$ over the basis of Grassmannian elements A^+_α, $\bar{A}^+_{\dot{\alpha}}$ are as follows:

$$\mathcal{N}^{\delta\dot{\delta}}_{\alpha\dot{\alpha}} = \int_0^s d\tau\left(\frac{e^{-\tau N}-e^{\tau\bar{N}}}{N+\bar{N}}\cdot e^{-s\bar{N}}\right)^{\delta\dot{\delta}}_{\alpha\dot{\alpha}} = -\frac{e^{-sF}-1}{NF}+\frac{e^{-s\bar{N}}-1}{N\bar{N}}, \tag{3.78}$$

$$\bar{\mathcal{N}}^{\dot{\delta}\delta}_{\dot{\alpha}\alpha} = \int_0^s d\tau\left(\frac{e^{-\tau\bar{N}}-e^{\tau N}}{N+\bar{N}}\cdot e^{-sN}\right)^{\dot{\delta}\delta}_{\dot{\alpha}\alpha} = -\frac{e^{-sF}-1}{\bar{N}F}+\frac{e^{-sN}-1}{N\bar{N}}. \tag{3.79}$$

The coefficients of power expansion of $\Omega(s)$ in the same basis A^+_α, $\bar{A}^+_{\dot{\alpha}}$ are given by

$$\Omega^-_\alpha = -\bar{\mathcal{W}}\left\{\frac{e^{-sN}+sN-1}{N^2}\right\}^\beta_\alpha \bar{A}_\beta, \tag{3.80}$$

$$\bar{\Omega}^-_{\dot{\alpha}} = -\mathcal{W}\bar{A}^-_{\dot{\beta}}\left\{\frac{e^{-s\bar{N}}+s\bar{N}-1}{\bar{N}^2}\right\}^{\dot{\beta}}_{\dot{\alpha}}, \tag{3.81}$$

$$\begin{aligned}\Psi^{(-2)} &= \frac{1}{8}(\bar{A}^-)^2\operatorname{tr}\sum_{n=0}^{\infty}\sum_{p=1}^{n}\frac{s^{n+2}}{(n+2)!}C^p_n(-F)^{n-p} \\ &\quad\times\{N^{p-1}-(-1)^n\bar{N}^{p-1}\} \\ &= (\bar{A}^-)^2\left\{\frac{s^3}{6}+\frac{s^5}{5!}(N^2+\bar{N}^2)+\ldots\right\},\end{aligned} \tag{3.82}$$

$$\begin{aligned}\bar{\Psi}^{(-2)} &= \frac{1}{8}(A^-)^2\operatorname{tr}\sum_{n=0}^{\infty}\sum_{p=1}^{n}\frac{s^{n+2}}{(n+2)!}C^p_n(-F)^{n-p} \\ &\quad\times\{\bar{N}^{p-1}-(-1)^n N^{p-1}\} \\ &= (A^-)^2\left\{\frac{s^3}{6}+\frac{s^5}{5!}(N^2+\bar{N}^2)+\ldots\right\},\end{aligned} \tag{3.83}$$

$$\Psi^{\dot{\alpha}(-2)}_\alpha = \Psi^{\dot{\alpha}\delta}_{\alpha\dot{\delta}}A^-_\delta\bar{A}^{-\dot{\delta}},$$

$$\begin{aligned}\Psi^{\dot{\alpha}\delta}_{\alpha\dot{\delta}} &= \frac{1}{N\bar{N}(N-\bar{N})}+\frac{1}{N\bar{N}}\left\{\frac{e^{-sN}}{\bar{N}}-\frac{e^{s\bar{N}}}{N}\right\} \\ &+\frac{N^2+\bar{N}^2}{2N^2\bar{N}^2(\bar{N}-N)}e^{s(\bar{N}-N)}+\frac{\bar{N}^2e^{sF}-N^2e^{-sF}}{2N^2\bar{N}^2(N+\bar{N})} \\ &= \frac{s^3}{3}+\frac{s^4}{8}(\bar{N}-N)+\frac{7s^5}{5!}(N^2+\bar{N}^2)+\ldots.\end{aligned} \tag{3.84}$$

The coefficients of expansion in derivatives of operator $\exp\{-s(A^+\mathcal{D}^-+\bar{A}^+\bar{\mathcal{D}}^-)\}$, which are determined in (3.58), are as follows:

$$a^{+\alpha} = A^{+\beta}\left(\frac{e^{-sN}-1}{N}\right)^\alpha_\beta, \quad \bar{a}^{+\dot{\alpha}} = \bar{A}^{+\dot{\beta}}\left(\frac{e^{-s\bar{N}}-1}{\bar{N}}\right)^{\dot{\alpha}}_{\dot{\beta}}, \tag{3.85}$$

$$\begin{aligned}f^{+2} &= -\frac{1}{4}(A^+)^2\operatorname{tr}\left(\frac{\cosh(sN)-1}{N^2}\right), \\ \bar{f}^{+2} &= -\frac{1}{4}(\bar{A}^+)^2\operatorname{tr}\left(\frac{\cosh(s\bar{N})-1}{\bar{N}^2}\right),\end{aligned} \tag{3.86}$$

$$f^{+2\dot{\alpha}\alpha} = -A^{+\beta}\bar{A}^{+\dot{\beta}}\left(\frac{e^{-sN}-1}{N}\right)^\alpha_\beta\left(\frac{e^{s\bar{N}}-1}{\bar{N}}\right)^{\dot{\alpha}}_{\dot{\beta}}, \tag{3.87}$$

$$\begin{aligned}\Xi^{+3\dot{\alpha}} &= -\frac{1}{4}(A^+)^2\bar{A}^{+\dot{\beta}}\left(\frac{e^{s\bar{N}}-1}{\bar{N}}\right)^{\dot{\alpha}}_{\dot{\beta}} \\ &\quad\times\operatorname{tr}\left(\frac{\cosh(sN)-1}{N^2}\right),\end{aligned} \tag{3.88}$$

$$\begin{aligned}\Xi^{+3\alpha} &= -\frac{1}{4}(\bar{A}^+)^2A^{+\beta}\left(\frac{e^{-sN}-1}{N}\right)^\alpha_\beta \\ &\quad\times\operatorname{tr}\left(\frac{\cosh(s\bar{N})-1}{\bar{N}^2}\right),\end{aligned} \tag{3.89}$$

$$\begin{aligned}\Omega^{+4} &= -\frac{1}{16}(A^+)^2(\bar{A}^+)^2 \\ &\times\operatorname{tr}\left(\frac{\cosh(sN)-1}{N^2}\right)\operatorname{tr}\left(\frac{\cosh(s\bar{N})-1}{\bar{N}^2}\right).\end{aligned} \tag{3.90}$$

3.9. Summary

The problem of construction of one-loop low-energy effective action in $\mathcal{N}=4$, $SU(2)$ gauge theory was studied. The theory is formulated in $\mathcal{N}=2$ harmonic superspace and features manifest $\mathcal{N}=2$ supersymmetry and additional hidden $\mathcal{N}=2$ supersymmetry that close on $\mathcal{N}=4$ supersymmetry on the mass shell. A new approach to the construction of effective action that depends on all fields of the $\mathcal{N}=4$ vector multiplet was presented. The main advantage of this approach consists in the fact that manifest $\mathcal{N}=2$ supersymmetry is preserved at all stages of calculations.

In the context of $\mathcal{N}=2$ supersymmetry, the effective action under consideration is a functional of superfields of the $\mathcal{N}=2$ vector multiplet and the hypermultiplet. The theory is quantized within the $\mathcal{N}=2$ background field method, thus allowing one to obtain a manifestly gauge-invariant form of effective action. The effective action is calculated in the low-energy approximation under the assumption that on-shell $\mathcal{N} = 2$ superfield strengths $\mathcal{W}, \overline{\mathcal{W}}$ and the on-shell superfields of hypermultiplets q^{ia} are space-time constants. The effective action is given by an analytic harmonic superspace integral of a function, which depends on superstrengths $\mathcal{W}, \overline{\mathcal{W}}$, their spinor covariant derivatives, and the hypermultiplet superfields. This dependence was exactly found in the low-energy approximation. The results of [215] given in the previous section were also confirmed and justified. Such a one-loop effective action in the hypermultiplet sector was calculated in [215] in terms of $\mathcal{N} = 1$ superfields based on special gauge fixing and several heuristic algorithms for restoring a manifestly $\mathcal{N} = 2$ supersymmetric form of effective action. This form is reproduced automatically by the method considered in the present section.

The general method for finding the one-loop low-energy effective action for the theory under consideration in terms of $\mathcal{N} = 2$ harmonic superfields was developed. An infinite series of covariant harmonic supergraphs with an arbitrary number of the hypermultiplet external lines is the basis of our analysis. Each of these supergraphs is written as an integral over the analytic subspace, and all contributions are summed up. The result is given by (3.42). This expression is analyzed using the proper time and the operator symbols methods. As a result we obtain the expression (3.62) for effective action.

At least two questions concerning the hypermultiplet dependence of effective action in $\mathcal{N} = 4$ gauge theory still remain open. The first problem consists in constructing the effective action with nonvanishing spinor covariant derivatives of the hypermultiplet superfields. The low-energy effective action in this case can be written as an expansion in spinor derivatives of superstrengths and hypermultiplet superfields. Knowing this expansion, one could determine whether the effective action is invariant under the hidden quantum $\mathcal{N} = 2$ supersymmetries analogous to the hidden $\mathcal{N} = 2$ supersymmetry of classical action. The second problem consists in obtaining the hypermultiplet dependence of higher-loop contributions to the effective action. We hope that the methods described above will be of use for studying these problems.

4. HYPERMULTIPLET DEPENDENCE OF ONE-LOOP EFFECTIVE ACTION IN $\mathcal{N} = 2$ SUPERCONFORMAL THEORIES

4.1. Introduction

$\mathcal{N} = 2$ supersymmetric gauge theories of general position in four dimensions are basically the theories of the $\mathcal{N} = 2$ vector gauge multiplet that interacts with massless hypermultiplets belonging to certain representations R of gauge group G. The $\mathcal{N} = 4$ supersymmetric gauge theory discussed above is a special case corresponding to the hypermultiplet in the adjoint representation. All these gauge models possess only one-loop divergences [41, 188] and can be finite and, consequently, superconformal (under certain restrictions imposed on the field contents of hypermultiplet matter). The finiteness (and superconformality) condition in a model with n_σ hypermultiplets in representation R_σ of gauge group G can be written in a simple universal form [188]

$$C(G) = \sum_\sigma n_\sigma T(R_\sigma), \tag{4.1}$$

where $C(G)$ is the quadratic Casimir operator for the adjoint representation, and $T(R_\sigma)$ is the quadratic Casimir operator for representation R_σ. The simplest solution of Eq. (4.1) corresponds to $\mathcal{N} = 4$ gauge theory with $n_\sigma = 1$ and all fields in the adjoint representation. It is evident that other solutions can also be found; for example, in the case of group $\mathrm{SU}(N)$ with hypermultiplets in the fundamental representation, we obtain $T(R) = 1/2$, $C(G) = N$, and $n_\sigma = 2N$. A large class of $\mathcal{N} = 2$ superconformal models was constructed exploiting the *AdS/CFT* correspondence hypothesis (see, for example, [189] and references therein; the examples of such models and the structure of vacuum states were discussed in detail in [115]). In the present section, the structure of low-energy one-loop effective action for general $\mathcal{N} = 2$ superconformal theories is analyzed.

The effective action of $\mathcal{N} = 4$ gauge theory and $\mathcal{N} = 2$ superconformal models in the $\mathcal{N} = 2$ vector multiplet sector was studied by various methods [41, 82, 86, 107, 115, 144]. However, the problem of hypermultiplet dependence of effective action for these theories remained open for the long time.

It was noted already that the low-energy effective action in $\mathcal{N} = 4$ theory containing both the $\mathcal{N} = 2$ vector multiplet background fields and the hypermultiplet background fields was first constructed in [146] and was later studied in detail in [216]. In the present section, the hypermultiplet dependence of effective action for $\mathcal{N} = 2$ superconformal models is considered. Such models are finite like $\mathcal{N} = 4$ SYM theory, therefore it is reasonable to expect that the hypermultiplet dependence of the corresponding effective actions is similar to the dependence found in $\mathcal{N} = 4$ theory. However, it is not immediately obvious. Indeed, the $\mathcal{N} = 4$ gauge theory is a special case of $\mathcal{N} = 2$ superconformal models that is distinguished by the presence of an additional hidden $\mathcal{N} = 2$ supersymmetry. It was noted in [146] that this additional $\mathcal{N} = 2$

supersymmetry is key to finding the explicit dependence of effective action of $\mathcal{N} = 4$ gauge theory on hypermultiplet fields. In the general case, such hidden $\mathcal{N} = 2$ supersymmetries are lacking; therefore, the derivation of effective action for general $\mathcal{N} = 2$ superconformal models in the hypermultiplet sector is an intriguing problem on its own.

In this section we will find the full $\mathcal{N} = 2$ supersymmetric one-loop effective action depending on both the background vector multiplet fields and the hypermultiplet background fields in the mixed phase, where the scalar fields of both the vector multiplet and the hypermultiplet have nonzero vacuum expectation values.[21]

Similar to the $\mathcal{N} = 4$ gauge theory considered in the previous section, $\mathcal{N} = 2$ supersymmetric models dissused here are formulated in harmonic superspace [51, 56]. We develop a systematic approach to finding the terms with a fixed number of field derivatives in the one-loop effective action with the use of the heat kernel for certain differential operators defined on a harmonic superspace. The heat kernel trace containg the dependence on the $\mathcal{N} = 2$ vector multiplet and background hypermultiplet superfields is calculated. The component structure of the leading quantum corrections is studied both for on-shell and for off-shell background superfields. It is found that these quantum corrections contain, among other terms, interactions of the Chern–Simons type. The fact that such explicitly scale invariant (but *P*-odd) terms containing both scalar and vector fields need to be present in the effective action of $\mathcal{N} = 4$ gauge theory was noted in [193]. The hypothesis that terms with higher derivatives are present in the effective action of $\mathcal{N} = 2$ models in a harmonic superspace was put forward in [195]. We will show that the terms in effective action proposed in above paper can be reproduced as a result of direct calculations in supersymmetric quantum field theory.

The section is organized as follows. Subsection 4.2 contains a formulation of $\mathcal{N} = 2$ supersymmetric models in harmonic superspace and a description of the corresponding vacua structure. The basic elements of the $\mathcal{N} = 2$ supersymmetric background field method is also given there. The structure of a superspace differential operator associated with the hypermultiplet dependence of one-loop effective action constructed on this vacuum is discussed in Subsection 4.3. Subsection 4.4 is focused on direct calculations of one-loop low-energy effective action for the background field subject to the mass shell conditions (see Eq. (4.6)). Besides, the bosonic component effective action containing the terms with four space-time derivatives of scalar component fields of the hypermultiplet is derived there. Similar Chern–Simons like terms were discussed in [195]. The possible contributions to the effective action from the background off-shell hypermultiplet (4.6) are analyzed in Subsection 4.5. It is demonstrated that the corresponding contribution in the bosonic sector contains terms of the Chern–Simons type (similar to the ones proposed in [195]) with three space-time derivatives. The results are summarized in Subsection 4.6.

4.2. Model and Background–Quantum Splitting

As it was already noted in the previous section the harmonic superspace approach provides a manifestly covariant description of $\mathcal{N} = 2$ supersymmetric theories on classical and quantum levels. Its key advantage is that the $\mathcal{N} = 2$ vector multiplet and the hypermultiplet are described within this approach by unconstrained superfields that "live" on an analytic subspace with coordinates $\zeta^M = (x^m_A, \theta^{+\alpha}, \bar{\theta}^+_{\dot\alpha}, u^\pm_i)$.

The $\mathcal{N} = 2$ vector multiplet is described by a real analytic superfield (gauge potential) $V^{++} = V^{++I}(\zeta)T_I$ taking the values in the Lie algebra of the gauge group. Prepotential V^{++} satisfies the reality condition $\widetilde{V^{++}} = V^{++}$ with respect to generalized conjugation (the product of complex conjugation and antipodal mapping on harmonic sphere S^2). The gauge group acts on V^{++} as $\delta V^{++} = -\mathcal{D}^{++}\lambda$, where λ is an arbitrary real analytic superfunction. In the Wess–Zumino gauge, the superfield V^{++} has a finite number of component fields $\phi, \bar{\phi}, A_m, \lambda_\alpha, \bar{\lambda}_{\dot\alpha}, D^{(ij)}$, which correspond to the field content of $\mathcal{N} = 2$ vector multiplet.

The hypermultiplet transforming in representation *R* of the gauge group is described by analytic superfield $q^+(\zeta)$ and its conjugate $\tilde{q}^+(\zeta)$ (see the definition of conjugation in [51]) that takes the values in representation *R*. Scalar component fields $f^i(x_A)$ of the hypermultiplet together with complex-conjugate fields $\bar{f}^i = (f_i)^\dagger$ form an $SU(2)$ doublet. Together with spinor physical fields, they emerge as the lower components of the $\theta^+, \bar{\theta}^+, u^\pm_i$ expansion of superfields $q^+, \tilde{q}^+$.

The classical action of $\mathcal{N} = 2$ gauge theory interacting with matter hypermultiplets consists of two parts: the action of "pure" $\mathcal{N} = 2$ gauge theory and the action of N_c hypermultiplets $q^{+a}, a = 1, \dots, 2N_c$, in the fundamental, the adjoint, or any other representation of the gauge group. The full action has the following general form in harmonic superspace [51]:

$$S = \frac{1}{2g^2}\mathrm{tr}\int d^8z\mathcal{W}^2 + \frac{1}{2}\int d\zeta^{(-4)} q^{+f}_a(D^{++} + igV^{++})q^{+a}_f. \tag{4.2}$$

[21] Examples of effective action in the Higgs branch for the $\mathcal{N} = 2$ gauge theory were given in [79].

Here index f characterizes the "flavor" group representation with respect to which the q_a^{+f} hypermultiplet forms an N_f-dimensional vector. We use a symplectic-covariant formulation $\widetilde{q_a^+} \equiv q^{+a} = \Omega^{ab} q_b^+$, where $\Omega^{ab} = \Omega^{ba}$ is the invariant tensor of symplectic group $USp(2N_c)$[22], for each index f. Thus, the superfield q_f^{+b} is a matrix of size $2N_c \times N_f$. The covariant derivative acts on hypermultiplet superfields as $\mathcal{D}^{++} q_f^{+a} = D^{++} q_f^{+a} + i\mathbf{V}_b^{++a} q_f^{+b}$, where $\mathbf{V}_b^{++a} = V^{++I}(\mathbf{T}_I)_b^a$ and $\mathbf{T}_I = \begin{pmatrix} -T_I & 0 \\ 0 & T_I \end{pmatrix}$. All the other notations are the same as in the previous section. Action (4.2) is manifestly $\mathcal{N} = 2$ supersymmetric by construction. For the sake of simplicity, we assume at the intermediate calculation stages that coupling constant g equals unity. An explicit dependence on g is easily restored in the resulting expressions for effective action.

It is worth reminding that the strength $\mathcal{W}$ superfield is expressed through nonanalytic harmonic connection V^{--}

$$\mathcal{W} = -\frac{1}{4}(\bar{D}^+)^2 V^{--}, \quad \overline{\mathcal{W}} = -\frac{1}{4}(D^+)^2 V^{--}. \qquad (4.3)$$

It follows from this representation that the strengths $\mathcal{W}, \overline{\mathcal{W}}$ are gauge-covariant, u-independent ($\mathcal{D}^{\pm\pm}\mathcal{W} = 0$), and covariant-chiral (antichiral) ($\bar{\mathcal{D}}^{\pm}_{\dot\alpha}\mathcal{W} = 0$, $\mathcal{D}^{\pm}_{\alpha}\overline{\mathcal{W}} = 0$) superfields that satisfy Bianchi equalities $(\mathcal{D}^{\pm})^2\mathcal{W} = (\bar{\mathcal{D}}^{\pm})^2\overline{\mathcal{W}}$.

Action (4.2) possesses the superconformal symmetry $SU(2,2|2)$, the explicit realization of which in harmonic superspace was given in [51]. The low-energy effective action of $\mathcal{N} = 2$ gauge theory constructed on general vacuum involves the dependence only on the massless $U(1)$ vector multiplet and massless neutral hypermultiplets, since charged vector multiplets and hypermultiplets acquire mass via the Higgs mechanism and do not produce a contribution in the low-energy approximation. The vacuum moduli space for the theories of the considered type is given by the relations [74]

$$[\bar\phi, \phi] = 0, \quad \phi f_i = 0, \; \bar f^i \bar\phi = 0, \; \bar f^{(i} T_I f^{i)} = 0. \; (4.4)$$

[22] A global subgroup of the gauge group should be a subgroup in $USp(2N_c)$ at constant N_c. The existence of isomorphisms $USp(2) \sim SU(2)$, $USp(4) \sim SO(5)$ should be noted.

Here $\phi, \bar\phi$ are scalar mutually conjugate components of the $\mathcal{N} = 2$ vector multiplet, and f_i are complex scalar hypermultiplet components.

The vacuum structure is characterized by the solutions of Eqs. (4.4). These solutions can be classified by phases (or branches) of the considered gauge model [74]. In the purely Coulomb branch at $f_i = 0$ and $\phi \neq 0$, Cartan subgroup $\mathrm{U}(1)^{\mathrm{rank(G)}}$ is the unbroken gauge group. In the purely Higgs branch (i.e., at $f_i \neq 0$), gauge symmetries are broken completely; therefore, massless gauge bosons are absent in this phase. It is well known that the conditions of F- and D-flatness characterizing the Higgs branch can be associated with ADHM constraints that define the instanton moduli space. In the mixed phase (i.e., at the direct product of Coulomb and Higgs branches with certain expectation vacuum values of fields $\phi, \bar\phi$ and f_i being nonzero), the gauge group is broken to $\tilde G \times K$, where K is a certain Abelian subgroup, and $\mathrm{rank}(\tilde G) < \mathrm{rank}(G)$.

Following [115], we impose the specific constraints on the background fields of the $\mathcal{N} = 2$ vector multiplet and the hypermultiplet. They are chosen in such a way that only their projections onto a fixed direction in the vacua moduli space remain nonzero. In particular, their scalar fields should be the solutions of Eqs. (4.4):

$$V^{++} = \mathbf{V}^{++}(\zeta)H, \quad q^+ = \mathbf{q}^+(\zeta)\Upsilon. \qquad (4.5)$$

Here H is the fixed generator of Abelian Cartan subalgebra corresponding to subgroup K, and Υ is the fixed vector in the R-space of the gauge group representation by which the hypermultiplet is transformed. This vector is chosen so that $H\Upsilon = 0$ and $\bar\Upsilon \mathbf{T}_I \Upsilon = 0$. Equations (4.5) distinguish a single U(1) vector multiplet and a single hypermultiplet that is neutral with respect to the U(1)-gauge subgroup with generator H. The choice of H and Υ can be constrained by the requirement of invariance of field configuration (4.5) with respect to the maximal unbroken gauge subgroup.

At tree level and energies below the symmetry breaking scale, theory describes the dynamics of free fields of the massless $\mathcal{N} = 2$ vector multiplet and the hypermultiplet with their fields being directed along a certain preferred vector in the vacuum moduli space. Low-energy propagating fields of massless neutral hypermultiplets and U(1) multiplets, which form together the complete set of on-shell superfields, have the following properties:

$$\begin{gathered} (D^{\pm})^2\mathcal{W} = (\bar D^{\pm})^2\overline{\mathcal{W}} = 0, \\ D^{++}q^{+a} = (D^{--})^2 q^{+a} = D^{--}q^{-a} = 0, \\ q^{-a} = D^{--}q^{+a}, \quad D^-_{(\alpha,\dot\alpha)} = 0. \end{gathered} \qquad (4.6)$$

All notations corresponds to the book [51]. Equations (4.6) eliminate the auxiliary fields and put physical fields on shell. At the quantum level, the exchange by virtual particles produces the corrections to the action of massless fields.

Manifestly $\mathcal{N} = 2$ supersymmetric Feynman rules in harmonic superspace were constructed in [56] (see also [41, 52]). We will quantize $\mathcal{N} = 2$ supergauge theories using the $\mathcal{N} = 2$ supersymmetric background field method [41, 52] by decomposing fields V^{++}, q^{+a} into a sum of background fields V^{++}, q^{+a} parameterized in accordance with (4.5) and quantum fields v^{++}, Q^{+a}. The Lagrangian will be presented as a power series in quantum fields. This procedure allows one to determine quantum effective action for an arbitrary $\mathcal{N} = 2$ supersymmetric gauge model in the form that preserves manifest $\mathcal{N} = 2$ supersymmetry and classical gauge invariance. The initial infinitesimal gauge transformations are realized in two ways: first, as background transformations

$$\delta V^{++} = -\mathcal{D}^{++}\lambda, \quad \delta v^{++} = i[\lambda, v^{++}], \tag{4.7}$$

and second, as quantum transformations

$$\delta V^{++} = 0, \quad \delta v^{++} = -\mathcal{D}^{++}\lambda - i[v^{++}, \lambda]. \tag{4.8}$$

In the framework of the background–quantum splitting, the classical action of the "pure" $\mathcal{N} = 2$ gauge theory is written as

$$\begin{aligned} S_{SYM}[V^{++} + v^{++}] &= S_{SYM}[V^{++}] + \frac{1}{4}\int d\zeta^{(-4)} du\, v^{++} \\ &\times (D^+)^2 \mathcal{W}_\lambda - \mathrm{tr}\int d^{12}z \sum_{n=2}^{\infty} \frac{(-ig)^{n-2}}{n} \\ &\times \int du_1 \ldots du_n \frac{v_\tau^{++}(z, u_1) \ldots v_\tau^{++}(z, u_n)}{(u_1^+ u_2^+) \ldots (u_n^+ u_1^+)}. \end{aligned} \tag{4.9}$$

Here $\mathcal{W}_\lambda$ and v_τ^{++} denote $\mathcal{W}$ and v^{++} in the λ- and τ-bases, respectively. They are related to each other via "bridge" b, $\mathcal{W}_\lambda = e^{ib}\mathcal{W}e^{-ib}$, $v_\tau^{++} = e^{-ib}v^{++}e^{ib}$. The quantum part of action depends on V^{++} through the dependence of v_τ^{++} on the bridge that is a complex function of V^{++}. The hypermultiplet action is split in accordance with the following representation:

$$\begin{aligned} S_H(q+Q) &= S_H[q] + \int d\zeta^{(-4)} du Q_a^+ \mathcal{D}^{++} q^{+a} \\ &+ \frac{1}{2}\int d\zeta^{(-4)} du q_a^+ i v^{++} q^{+a} + \frac{1}{2}\int d\zeta^{(-4)} du \\ &\times \{Q_a^+ \mathcal{D}^{++} Q^{+a} + Q_a^+ i v^{++} q^{+a} \\ &+ q_a^+ i v^{++} Q^{+a} + Q_a^+ i v^{++} Q^{+a}\}. \end{aligned} \tag{4.10}$$

Terms linear in v^{++} and q^+ in (4.9) and (4.10) define the equations of motion. These terms should be omitted when one considers effective action on the mass shell.

We follow the Faddeev–Popov procedure in constructing the effective action. Within the background field method, we should fix the gauge only with respect to quantum transformations (4.8). In accordance with the analysis in [52], we introduce the gauge fixing function

$$\mathcal{F}_\tau^{(4)} = D^{++} v_\tau^{++} = e^{-ib}(\mathcal{D}^{++} v^{++})e^{ib} = e^{-ib}\mathcal{F}^{(4)}e^{ib},$$

with quantum transformations (4.8) acting on it in the following way:

$$\delta\mathcal{F}_\tau^{(4)} = e^{-ib}\{\mathcal{D}^{++}(\mathcal{D}^{++}\lambda + i[v^{++}, \lambda])\}e^{ib}. \tag{4.11}$$

The Faddeev–Popov determinant is obtained using Eq. (4.11):

$$\Delta_{FP}[v^{++}, V^{++}] = \mathrm{Det}[\mathcal{D}^{++}(\mathcal{D}^{++} + i v^{++})].$$

In order to present $\Delta_{FP}[v^{++}, V^{++}]$ as a functional integral, we introduce two real analytic fermionic ghosts $\mathbf{b}$ and $\mathbf{c}$ in the adjoint representation of the gauge group and determine the corresponding ghost action

$$\begin{aligned} &S_{FP}[\mathbf{b}, \mathbf{c}, v^{++}, V^{++}] \\ &= \mathrm{tr}\int d\zeta^{(-4)} du \mathbf{b} \mathcal{D}^{++}(\mathcal{D}^{++}\mathbf{c} + i[v^{++}, \mathbf{c}]). \end{aligned} \tag{4.12}$$

As a result, effective action $\Gamma[V^{++}, q^+]$ is obtained in the following form:

$$e^{i\Gamma[V^{++}, q^+]} = e^{iS_{cl}[V^{++}, q^+]}\int \mathcal{D}v^{++}\mathcal{D}Q^+\mathcal{D}\mathbf{b}\mathcal{D}\mathbf{c}\, e^{i(\Delta S_{SYM}[v^{++}, V^{++}] + \Delta S_H[v^{++}, V^{++}, Q^+, q^+] + S_{FP}[\mathbf{b}, \mathbf{c}, v^{++}, V^{++}])}\delta[\mathcal{F}^{(4)} - f^{(4)}], \tag{4.13}$$

where $f^{(4)}(\zeta, u)$ is an external V^{++}-independent analytic superfield taking the values in the Lie algebra, and $\delta[\mathcal{F}^{(4)}]$ is a definite analytic functional delta function. In order to transform the integral representation of $\Gamma[V^{++}, q^+]$ into a more convenient form, we average the right hand part of Eq. (4.13) with weight

$$\Delta[V^{++}]\exp\left\{\frac{i}{2\alpha}\mathrm{tr}\int d^{12}z du_1 du_2 \times f_\tau^{(4)}(z,u_1)\frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3}f_\tau^{(4)}(z,u_2)\right\}. \quad (4.14)$$

Here α is an arbitrary gauge parameter. Functional $\Delta[V^{++}]$ is derived from relation

$$1 = \Delta[V^{++}]\int \mathcal{D}f^{(4)}\exp\left\{\frac{i}{2\alpha}\mathrm{tr}\int d^{12}z du_1 du_2 \times f_\tau^{(4)}(z,u_1)\frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3}f_\tau^{(4)}(z,u_2)\right\}. \quad (4.15)$$

Using standard equality $\int d\zeta^{(-4)}(D^+)^4 L(z,u) = \int d^{12}z L(z,u)$ in expression

$$\Delta^{-1}[V^{++}] = \int \mathcal{D}f^{(4)}\exp\left\{\frac{i}{2\alpha}\mathrm{tr}\int d\zeta_1^{(-4)}d\zeta_2^{(-4)}z du_1 du_2 \times f^{(4)}(\zeta_1,u_1)A(1,2)f^{(4)}(\zeta_2,u_2)\right\},$$

we express $\Delta[V^{++}]$ through a special background-dependent operator $A = \frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3}(D_1^+)^4(D_2^+)^4 \times \delta^{12}(z_1 - z_2)$, operating on the space of analytic superfields with values in the Lie algebra of the gauge group:

$$\Delta[V^{++}] = \mathrm{Det}^{1/2}A. \quad (4.16)$$

In order to calculate Det$A$, we express it as the functional integral over analytic superfields

$$\mathrm{Det}^{-1}A = \int \mathcal{D}\chi^{(4)}\mathcal{D}\rho^{(4)}\exp\left\{i\mathrm{tr}\int d\zeta_1^{(-4)} \times du_1 d\zeta_2^{(-4)}du_2\chi^{(4)}(1)A(1,2)\rho^{(4)}(2)\right\}, \quad (4.17)$$

and perform a change of functional integration variables

$$\rho^{(4)} = (\mathcal{D}^{++})^2\sigma, \quad \mathrm{Det}\frac{\delta\rho^{(4)}}{\delta\sigma} = \mathrm{Det}(\mathcal{D}^{++})^2.$$

We obtain[23]

$$\begin{aligned}&\mathrm{tr}\int d\zeta_1^{(-4)}du_1 d\zeta_2^{(-4)}du_2\chi^{(4)}(1)A(1,2)\rho^{(4)}(2)\\ &= \mathrm{tr}\int d^{12}z du_1 du_2\chi_\tau^{(4)}(1)\frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3}(D_2^{++})^2\sigma_\tau(2)\\ &= \frac{1}{2}\mathrm{tr}\int d^{12}z du\chi_\tau^{(4)}(D^{--})^2\sigma_\tau\\ &= -\mathrm{tr}\int d\zeta^{(-4)}du\chi^{(4)}\widehat{\Box}\sigma,\end{aligned} \quad (4.18)$$

where

$$\widehat{\Box} = -\frac{1}{2}(\mathcal{D}^+)^4(\mathcal{D}^{--})^2. \quad (4.19)$$

Equations (4.16)–(4.18) yield

$$\Delta[V^{++}] = \mathrm{Det}^{1/2}\widehat{\Box}_{(4,0)}\mathrm{Det}^{1/2}(\mathcal{D}^{++})^2. \quad (4.20)$$

Functional $\Delta[V^{++}]$ can now be presented as the following functional integral:

$$\Delta[V^{++}] = \mathrm{Det}^{1/2}\widehat{\Box}_{(4,0)} \times \int \mathcal{D}\varphi e^{-\frac{i}{2}\mathrm{tr}\int d\zeta^{(-4)}du\mathcal{D}^{++}\varphi\mathcal{D}^{++}\varphi}. \quad (4.21)$$

Here integration variable φ is a bosonic real analytic superfield that takes the values in the Lie algebra of the gauge group. Superfield φ is basically a Nielsen–Kallosh ghost for the theory under consideration. Therefore, the quantum $\mathcal{N} = 2$ gauge theory within the background field approach is defined by a set of three ghosts: two fermionic ones (**b** and **c**) and bosonic ghost φ. The actions of ghosts S_{FP} and S_{NK} are defined by expressions (4.12) and (4.21) and coincide with the known action of the ω-hypermultiplet.

After averaging the effective action with weight (4.14), we obtain the following representation for the functional integral:

$$e^{i\Gamma[V^{++},q^+]} = e^{iS_{cl}[V^{++},q^+]}\mathrm{Det}^{1/2}\widehat{\Box}_{(4,0)} \times \int \mathcal{D}v^{++}\mathcal{D}Q^+\mathcal{D}\mathbf{b}\mathcal{D}\mathbf{c}\mathcal{D}\varphi e^{iS_q[v^{++},Q^+,\mathbf{b},\mathbf{c},\varphi,V^{++},q^+]}, \quad (4.22)$$

where

$$\begin{aligned}S_q[v^{++},Q^+,\mathbf{b},\mathbf{c},\varphi,V^{++},q^+] &= \Delta S_{SYM}[v^{++},V^{++}]\\ &+ S_{GF}[v^{++},V^{++}] + \Delta S_H[v^{++},V^{++},Q^+,q^+]\\ &+ S_{FP}[\mathbf{b},\mathbf{c},v^{++},V^{++}] + S_{NK}[\varphi,V^{++}].\end{aligned}$$

---

[23] Equality $(D_2^{++})^2\frac{u_1^- u_2^-}{(u_1^+ u_2^+)^3} = (u_2^+ u_1^-)(D_2^{--})^2\delta^{(3,-3)}(u_2,u_1)$ was used.

Here $S_{GF}[v^{++}, V^{++}]$ is the contribution of the gauge fixing term to the quantum action:

$$
\begin{aligned}
S_{GF}[v^{++}, V^{++}] &= \frac{1}{2\alpha}\mathrm{tr}\int d^{12}z du_1 du_2 \frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3} \\
&\times [D_1^{++} v_\tau^{++}(1) D_2^{++} v^{++}(2)] = \frac{1}{2\alpha} \\
&\times \mathrm{tr}\int d^{12}z du_1 du_2 \frac{v_\tau^{++}(1) v_\tau^{++}(2)}{(u_1^+ u_2^+)^2} - \frac{1}{4\alpha} \\
&\times \mathrm{tr}\int d^{12}z du\, v_\tau^{++}(D^{--})^2 v_\tau^{++}.
\end{aligned} \tag{4.23}
$$

Let us consider the sum of parts of ΔS_{SYM} (4.9) and S_{GF} (4.23) that are quadratic in v^{++}. This sum has the following form:

$$
\begin{aligned}
&\frac{1}{2}\left(1 + \frac{1}{\alpha}\right)\mathrm{tr}\int d^{12}z du_1 du_2 \frac{v_\tau^{++}(1) v_\tau^{++}(2)}{(u_1^+ u_2^+)^2} \\
&+ \frac{1}{2\alpha}\mathrm{tr}\int d^{12}z du\, v^{++} \widehat{\Box}\, v^{++},
\end{aligned}
$$

where definition (3.18) was used. In order to simplify calculations further, we choose the Fermi–Feynman gauge with $\alpha = -1$. Now we can write the final result for effective action $\Gamma[V^{++}, q^+]$ in the form (4.22) with quantum action S_q being expressed as

$$
\begin{aligned}
&S_q[v^{++}, Q^+, \mathbf{b}, \mathbf{c}, \varphi, V^{++}, q^+] \\
&= S_2[v^{++}, Q^+, \mathbf{b}, \mathbf{c}, \varphi, V^{++}, q^+] \\
&+ S_{int}[v^{++}, Q^+, \mathbf{b}, \mathbf{c}, V^{++}, q^+],
\end{aligned}
$$

where

$$
\begin{aligned}
S_2 &= -\frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)} du\, v^{++}\widehat{\Box}\, v^{++} + \mathrm{tr}\int d\zeta^{(-4)} du\, \mathbf{b}(\mathcal{D}^{++})^2 \mathbf{c} \\
&+ \frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)} du\, \varphi(\mathcal{D}^{++})^2 \varphi + \frac{1}{2}\int d\zeta^{(-4)} du \\
&\times \{Q_a^+ \mathcal{D}^{++} Q^{+a} + Q_a^+ i v^{++} q^{+a} + q_a^+ i v^{++} Q^{+a}\},
\end{aligned} \tag{4.24}
$$

$$
\begin{aligned}
S_{int} &= -\mathrm{tr}\int d\zeta^{(-4)} du_1 \ldots du_n \sum_{n=3}^{\infty} \frac{(-i)^{n-2}}{n} \\
&\times \frac{v_\tau^{++}(z, u_1)\ldots v_\tau^{++}(z, u_n)}{(u_1^+ u_2^+)\ldots(u_n^+ u_1^+)} - i\mathrm{tr}\int d\zeta^{(-4)} du \\
&\times \mathcal{D}^{++}\mathbf{b}[v^{++}, \mathbf{c}] + \frac{1}{2}\int d\zeta^{(-4)} du\, Q_a^+ i v^{++} Q^{+a}.
\end{aligned} \tag{4.25}
$$

Actions S_2 and S_{int} define completely, in a manifestly supersymmetric and gauge-invariant form, the structure of perturbative expansion of effective action $\Gamma[V^{++}, q^+]$ in the $\mathcal{N} = 2$ gauge theory that interacts with hypermultiplets.

Note that not all hidden symmetries of classical action are preserved in the quantum case in the Faddeev–Popov quantization scheme. According to the results of analysis undertaken in [112], the problem of preservation of a certain manifest global symmetry at the quantum level is basically equivalent to finding the gauge conditions that are covariant under these symmetries. Such conditions do not exist in the case of conformal symmetry, and any special conformal transformation should be accompanied by field-dependent nonlocal gauge transformations in order to restore the gauge orbit [112]. The invariance of functional integral under combined conformal and gauge transformations leads to a modification of the Ward conformal identities for effective action.

Action S_2 defines the propagators depending on background fields [52]. Three types of covariant propagators for material and gauge fields are needed in $\mathcal{N} = 2$ harmonic superspace within the background field method. Green's function $G^{(2,2)}(z, z')$ that is associated with $\widehat{\Box}$ is subject to the Feynman boundary condition and satisfies equation $G^{(2,2)}(1|2) = -\mathbf{1}\delta^{(2,2)}(1|2)$, where $\delta^{2,2}(\zeta_1, \zeta_2)$ is an analytic delta function.[24] So, it is defined by

$$
\begin{aligned}
i\langle v^{++}(z, u) v^{++}(z', u')\rangle &= G^{(2,2)}(z, u, z', u') \\
&= -\frac{1}{\widehat{\Box}}(\mathcal{D}^+)^4\{\mathbf{1}\delta^{12}(z - z')\delta^{(-2,2)}(u, u')\}.
\end{aligned}
$$

It sometimes is instructive to rewrite $G^{(2,2)}$, following [51], in a form that is explicitly analytic with respect to both arguments

$$
\begin{aligned}
G^{(2,2)}(1, 2) &= -\frac{1}{2\widehat{\Box}_1\widehat{\Box}_2}(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4 \\
&\times \{\mathbf{1}\delta^{12}(z_1 - z_2)(D_2^{--})^2\delta^{(-2,2)}(u_1, u_2)\}.
\end{aligned} \tag{4.26}
$$

This representation can be used to calculate those superdiagrams that contain the products of harmonic distributions.

Propagator of Q^+ hypermultiplet is defined by action (4.24) and has the form:

$$
\begin{aligned}
i\langle Q^+(\zeta_1, u_1, \zeta_2, u_2)\rangle &= G_b^{a(1.1)}(1|2) \\
&= -\delta_b^a \frac{(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4}{(u_1^+ u_2^+)^3}\frac{1}{\widehat{\Box}_1}\delta^{12}(z_1 - z_2).
\end{aligned} \tag{4.27}
$$

---

[24] $\delta^{(q,4-q)}(\zeta_1, u_1|\zeta_2, u_2) = (D_1^+)^4\delta^{12}(z_1 - z_2)\delta^{(q-4,4-q)}(u_1, u_2)$
$= (D_2^+)^4\delta^{12}(z_1 - z_2)\delta^{(q,-q)}(u_1, u_2)$.

It is not too difficult to see that this manifestly analytic expression is a solution of equation $\mathcal{D}_1^{++}G^{(1,1)} = \delta_A^{(3,1)}(1|2)$.

The equation for the Green's function of the second type hypermultiplet described by neutral real analytic superfield $\omega(\zeta, u)$ is as follows:

$$(\mathcal{D}_1^{++})^2 G^{(0,0)}(1|2) = \delta_A^{(4,0)}(1|2).$$

The corresponding expression for $G^{(0,0)}$ is written as

$$i\langle\omega(1), \omega^T(2)\rangle = G^{(0,0)}(1|2)$$
$$= -\frac{1}{\widehat{\Box}_1}(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4\left\{\mathbf{1}\delta^{12}(z_1-z_2)\frac{u_1^-u_2^-}{(u_1^+u_2^+)^3}\right\}. \quad (4.28)$$

The operator $\widehat{\Box} = -\frac{1}{2}(\mathcal{D}^+)^4(\mathcal{D}^{--})^2$ present in the Green's functions transforms each covariant-analytic superfield into a covariant-analytic one. Using algebra (3.10), one can rewrite it as a second-order differential operator on the space of such superfields [52]:

$$\widehat{\Box} = \frac{1}{2}\mathcal{D}^{\alpha\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}} + \frac{i}{2}(\mathcal{D}^{+\alpha}\mathcal{W})\mathcal{D}_\alpha^- + \frac{i}{2}(\overline{\mathcal{D}}_{\dot{\alpha}}^+\overline{\mathcal{W}})\overline{\mathcal{D}}^{-\dot{\alpha}}$$
$$+\frac{1}{2}\{\mathcal{W}, \overline{\mathcal{W}}\} - \frac{i}{4}(\overline{\mathcal{D}}^+\overline{\mathcal{D}}^+\overline{\mathcal{W}})\mathcal{D}^{--} + \frac{i}{8}[\mathcal{D}^+, \mathcal{D}^-]\mathcal{W}. \quad (4.29)$$

Among the important properties of this operator one notes the preservation of analyticity, $[(\mathcal{D}^+)^4, \widehat{\Box}] = 0$. The coefficients of this operator depend on background superfields $\mathcal{W}, \overline{\mathcal{W}}$.

In the case of a background belonging to the Abelian subgroup of the gauge group and satisfying the mass shell conditions, an additional restriction takes place: $\mathcal{D}^{\pm}\mathcal{W} = D^{\pm}\mathcal{W}$ (and a similar condition for $\overline{\mathcal{W}}$ with $\overline{D}^{\pm}$). Thus, spinor derivatives become background-independent in this case. In addition, we should omit the two last terms in (4.29), since they vanish on the mass shell.

The part of the action that is quadratic in quantum gauge superfields can be simplified easily by expanding these matrix superfields over some basis. We choose quantum superfields in unique correspondence with roots of the Lie algebra of gauge group G: $v = \sum_\alpha v^\alpha E_\alpha + \sum_i v^i H_i$. Here E_α is the generator corresponding to root α and normalized as $\mathrm{tr}(E_\alpha E_{-\beta}) = \delta_{\alpha,-\beta}$, and H_i are the generators of Cartan subalgebra of rank (G) that satisfy commutation relations $[H_i, E_\alpha] = \alpha(H_i)E_\alpha$. Using this notation, one can rewrite action (4.24) in terms of coefficients of expansion over v in this basis. This form of effective action is convenient for calculations and is used in Section 4 in various cases.

After the gauge background superfield is removed, Green's functions (4.26), (4.27), and (4.28) go over to free propagators constructed in [56, 51].

4.3. Structure of One-Loop Effective Action

Let us consider the loop expansion of effective action in the framework of the background field method. The effective action is defined by vacuum diagrams (i.e., diagrams without external lines) with background field dependent propagators and vertices. The formal expression for one-loop action $\Gamma[V^{++}, q^{++}]$ is written in the theory under consideration as a functional integral (4.22), and the full quadratic action of quantum fields is defined in (4.24).

Expressions (4.22) and (4.24) define completely the manifestly supersymmetric and gauge-invariant structure of perturbation theory for the calculation of effective action of $\mathcal{N} = 2$ gauge theory with hypermultiplets. We use expressions (4.26) and (4.27) for propagators of the quantum vector multiplet v^{++} and hypermultiplets Q^{+a}, respectively. Vertices can be determined directly from expression (4.25). It is easy to see that ghosts are not associated with the background hypermultiplet and, consequently, do not contribute to the hypermultiplet-dependent part of one-loop effective action. In the vector sector of $\mathcal{N} = 2$ gauge theory, when the matter hypermultiplet is integrated out, one-loop effective action $\Gamma[V^{++}]$ takes the following form:

$$\Gamma[V^{++}] = \frac{i}{2}\mathrm{Tr}_{(2,2)}\ln\widehat{\Box}$$
$$-\frac{i}{2}\mathrm{Tr}_{(4,0)}\ln\widehat{\Box} - \frac{i}{2}\mathrm{Tr}_{ad}\ln(\mathcal{D}^{++})^2$$
$$+ i\mathrm{Tr}_{R_q}\ln\mathcal{D}^{++} + \frac{i}{2}\mathrm{Tr}_{R_\omega}\ln(\mathcal{D}^{++})^2.$$

At present, the holomorphic and nonholomorphic parts of low-energy effective action of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric gauge theories in the Coulomb branch (including the Euler–Heisenberg action in the presence of a covariantly constant vector multiplet) are known exactly (see, for example, [86, 82]). The general structure of low-energy effective action in such theories is given by [107, 144]

$$\Gamma = S_{cl} + \int d^{12}z = \{c\ln\mathcal{W}\ln\overline{\mathcal{W}}$$
$$+\int d^{12}z\ln\mathcal{W}\Lambda\left(\frac{D^4\ln\mathcal{W}}{\overline{\mathcal{W}}^2}\right) + c.c.$$
$$+\int d^{12}z\Upsilon\left(\frac{\overline{D}^4\ln\overline{\mathcal{W}}}{\mathcal{W}^2}, \frac{D^4\ln\mathcal{W}}{\overline{\mathcal{W}}^2}\right)\right\} + \ldots,$$

where Λ and Υ are the holomorphic and real analytic functions of (anti)chiral superconformal invariants. It is known that the c-term at the component level produces quantum corrections with four derivatives including the F^4-term (see, for example, [82]). The hypermultiplet-dependent part of effective action in $\mathcal{N} = 4$ gauge theory in the leading order is also known [146, 215, 216].

For further analysis of effective action it is convenient to diagonalize the action of quantum fields $S^{(2)}$. To do this one shifts the hypermultiplet variables in the functional integral in the following way:

$$\begin{aligned} Q^{+a} &= \xi^{+a} + i\int d\zeta_2^{(-4)} q^{+b}(2) v^{++}(2) G_b^{a(1.1)}(1|2), \\ Q_a^{+} &= \xi_a^{+} - i\int d\zeta_2^{(-4)} G_a^{b(1.1)}(1|2) v^{++}(2) q_b^{+}(2), \end{aligned} \tag{4.30}$$

where ξ^{+a}, ξ_a^{+} are new independent variables of functional integration. It is evident that the Jacobian of the change of variables (4.30) equals unity. Here $G_b^{a(1.1)}(1|2)$ is the background-dependent propagator (4.27) for superfields Q^{+a}, Q_b^{+}. The following expression in terms of the new set of quantum fields is obtained for the hypermultiplet-dependent part of quadratic action:

$$\begin{aligned} S_H^{(2)} &= -\frac{1}{2}\int d\zeta^{(-4)} \xi^{a+} \mathcal{D}^{++} \xi_a^{+} - \frac{1}{2}\int d\zeta_1^{(-4)} d\zeta_2^{(-4)} \\ &\times q^{+a}(1) v^{++}(1) G_a^{b(1.1)}(1|2) v^{++}(2) q_b^{+}(2). \end{aligned}$$

Then, the quadratic part of action in the vector multiplet sector is extended by

$$\begin{aligned} S_v^{(2)} &= -\frac{1}{2}\mathrm{tr}\int d\zeta_1^{(-4)} v_1^{++} \int d\zeta_2^{(-4)} \\ &\times [\widehat{\Box}\delta_A^{(2.2)}(1|2) + q^{+a}(1) G_a^{b(1.1)}(1|2) q_b^{+}(2)] v_2^{++}. \end{aligned} \tag{4.31}$$

Expression (4.31) written as an analytic nonlocal superfunctional will serve as a starting point for our calculations of one-loop effective action in the hypermultiplet sector. Our aim in this and next subsections is to find the leading low-energy contributions to the effective action for slowly varying hypermultiplet superfields (i.e., when all derivatives of the background hypermultiplet can be neglected). We will show that in this case such nonlocal interactions can be localized at a point.

Using relation $v_2^{++} = \int d\zeta_3^{(-4)} \delta_A^{(2.2)}(2|3) v_3^{++}$, we can rewrite the expansion for $S_v^{(2)}$ (4.31) in the following form:

$$\begin{aligned} S_v^{(2)} &= -\frac{1}{2}\mathrm{tr}\int d\zeta_1^{(-4)} v_1^{++} \int d\zeta_2^{(-4)} \Big[\widehat{\Box}\delta_A^{(2.2)}(1|2) \\ &+ \int d\zeta_3^{(-4)} q^{+a}(1) G_a^{b(1.1)}(1|3) q_b^{+}(3) \delta_A^{(2.2)}(3|2)\Big] v_2^{++}. \end{aligned} \tag{4.32}$$

We then use the explicit expression for Green's function (4.27) and a relation that allows one to present $(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4$ as a polynomial in powers of $(u_1^+u_2^+)$ [107]

$$\begin{aligned} (\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4 &= (\mathcal{D}_1^+)^4\Big[(\mathcal{D}_1^-)^4(u_1^+u_2^+)^4 \\ &- \frac{i}{2}\Delta_1^{--}(u_1^+u_2^+)^3(u_1^-u_2^+) - \widehat{\Box}_1(u_1^+u_2^+)^2(u_1^-u_2^+)^2\Big], \end{aligned} \tag{4.33}$$

where

$$\begin{aligned} \Delta^{--} &= \mathcal{D}^{\alpha\dot{\alpha}}\mathcal{D}_\alpha^-\overline{\mathcal{D}}_{\dot{\alpha}}^- + \frac{1}{2}\mathcal{W}(\mathcal{D}^-)^2 \\ &+ \frac{1}{2}\overline{\mathcal{W}}(\overline{\mathcal{D}}^-)^2 + (\mathcal{D}^-\mathcal{W})\mathcal{D}^- + (\overline{\mathcal{D}}^-\overline{\mathcal{W}})\overline{\mathcal{D}}^-. \end{aligned} \tag{4.34}$$

Since $G^{(1.1)}(1, 2) = -G^{(1.1)}(2, 1)$, the nonlocal term in (4.32) takes the form

$$\begin{aligned} &\int d\zeta_3^{(-4)} q^{+a}(1)(\mathcal{D}_3^+)^4\Big[(\mathcal{D}_3^-)^4(u_3^+u_1^+)\frac{1}{\widehat{\Box}_3} \\ &- \frac{i}{2}\Delta_3^{--}(u_3^-u_1^+)\frac{1}{\widehat{\Box}_3} - \frac{(u_3^-u_1^+)^2}{u_3^+u_1^+}\Big]\delta^{12}(1|3) q_a^+(3)\delta_A^{(2.2)}(3|2). \end{aligned}$$

There are three terms within square brackets. It is easy to see that the first two terms include derivatives that produce derivatives of hypermultiplet superfields in the effective action. Since we are interested only in contributions with no derivatives, these terms can be omitted. Therefore, only the third term in square brackets requires consideration.

Now we apply the relation $\int d\zeta_3^{(-4)}(\mathcal{D}_3^+)^4 = \int d^{12}z_3 du_3$, which allows to integrate over z_3, and obtain

$$-\int du_3 q^{+a}(1)\frac{(u_3^-u_1^+)^2}{(u_3^+u_1^+)} q_a^+(u_3, z_1)\delta_A^{(2.2)}(u_3, z_1|2).$$

Next, we pick up the explicit harmonic dependence of the hypermultiplet on the mass shell $q^{+a}(3) = u_{3i}^+ q^{ia}$ and go to the limit of coincident harmonic arguments $u_1 = u_3$ using the harmonic part of delta function $\delta_A^{(2.2)}(u_3, z_1|2)$. After that we obtain: $\int du_3 \frac{u_{3i}^+}{u_3^+u_1^+} = -u_{1i}^-$. As a result, the term under consideration takes the form

$$q^{+a}(1) q_a^-(1)\delta_A^{(2.2)}(1|2).$$

Expression $q^{+a}(1)q_a^-(1) = q^{ia}q_{ia}$ will be regarded hereafter as a slowly changing superfield, and all its derivatives will be neglected. This is exactly the superfield argument that was obtained in [216] after summing up the harmonic superdiagrams.

Thus, the second term in (4.32) becomes local in the leading low-energy approximation. As a result, the operator of quadratic fluctuations for quantum superfield v^{++} (we expand superfield v^{++} over the generator basis as $v^{++} = v_I^{++}T_I$ and consider only the v_I^{++} superfield components) takes the form:

$$[\widehat{\square}_{IJ} + q^{+a}(z_1, u_1)\{T_I, T_J\}q_a^-(z_1, u_1)]\delta_A^{(2.2)}(1|2), \quad (4.35)$$

where

$$\widehat{\square}_{IJ} = \mathrm{tr}\left\{T_{(I}\square T_{J)} + \frac{i}{2}T_{(I}[\mathcal{D}^{+\alpha}\mathcal{W}, T_{J)}]\mathcal{D}_\alpha^- \right.$$
$$\left. + \frac{i}{2}T_{(I}[\overline{\mathcal{D}}^+_{\dot\alpha}\overline{\mathcal{W}}, T_{J)}]\overline{\mathcal{D}}^{-\dot\alpha} + T_{(I}[\mathcal{W}, [\overline{\mathcal{W}}, T_{J)}]]\right\}.$$

Here $\square = \frac{1}{2}\mathcal{D}^{\alpha\dot\alpha}\mathcal{D}_{\alpha\dot\alpha}$ is a covariant d'Alembertian.

Thus, using the harmonic formulation of $\mathcal{N} = 2$ gauge theory interacting with hypermultiplets and the technique of nonlocal shifts, we have demonstrated that the complete dependence on the background hypermultiplet was concentrated in the quantum vector multiplet sector after the action was modified. Therefore, the one-loop effective action is given by

$$\Gamma^{(1)}[V^{++}, q^+] = \Gamma_v^{(1)}[V^{++}, q^+] + \tilde\Gamma^{(1)}[V^{++}]. \quad (4.36)$$

The first term in (4.36) is produced by quantum vector multiplet v_I^{++}:

$$\Gamma_v^{(1)}[V^{++}, q^+] = \frac{i}{2}\mathrm{Tr}\ln(\widehat{\square}_{IJ} + q^{+a}\{T_I, T_J\}q_a^-). \quad (4.37)$$

The second term in (4.36) is stipulated by contribution of ghosts and quantum hypermultiplet ξ_a^+ and does not depend on the background hypermultiplet. Therefore, the complete dependence of one-loop effective action on the background hypermultiplet superfield is contained within operator

$$\widehat{\square}_{IJ} + q^{+a}\{T_I, T_J\}q_a^-, \quad (4.38)$$

acting on v_I^{++}. If q^+ belongs to the fundamental representation, this dependence has the form of a mass matrix of the vector multiplet

$$(\mathcal{M}_v^2)_{IJ} = \mathrm{tr}\{[T_I, \mathcal{W}][\overline{\mathcal{W}}, T_J] + (I \leftrightarrow J)\} + q^{+a}\{T_I, T_J\}q_a^-. \quad (4.39)$$

If q^+ transforms in arbitrary matrix representation, the dependence is of the form of matrix

$$(\mathcal{M}_v^2)_{IJ} = \mathrm{tr}\{[T_I, \mathcal{W}][\overline{\mathcal{W}}, T_J] + [q^{+a}, T_I][T_J, q_a^-]\} + (I \leftrightarrow J). \quad (4.40)$$

Thus, the dependence on the hypermultiplet is transferred completely to the quantum superfields v^{++} sector and is concentrated in background-covariant operator (4.38). Representations (4.36) and (4.37) serve as the starting point for the calculation of one-loop effective action. Note that we have imposed no restrictions on the space-time dependence of the hypermultiplet superfield (apart from the properties implied by mass shell conditions (4.6)).

Note that the structure of the gauge group realized on superfields $\mathcal{W}$, q_a^+ has so far been completely arbitrary in our analysis. Now we choose the background superfields along a given direction in the vacuum moduli space in such a way that their scalar fields will be the solutions of Eqs. (4.4). Let the background vector multiplet and the hypermultiplet be of the form (4.5), where H is the fixed generator of Cartan subalgebra. This agrees with the assumption that gauge group G is broken to subgroup $\tilde G \times K$, where K is the Abelian Cartan subgroup with an algebra to which generator H belongs. In this case, a unique vacuum combination $\mathcal{W}\overline{\mathcal{W}}$ exists for the $\mathcal{N} = 2$ background vector multiplet, and a unique vacuum combination $q^{+a}q_a^-$ is present for the background hypermultiplet.[25] The operator acting on quantum superfields of the vector multiplet, which are defined in (4.38), then assumes universal form

$$\square + \frac{i}{2}\alpha(H)(\mathcal{D}^+\mathcal{W}\mathcal{D}^- + \overline{\mathcal{D}}^+\overline{\mathcal{W}}\overline{\mathcal{D}}^-) + \alpha^2(H)\mathcal{W}\overline{\mathcal{W}} + q^{+a}q_a^-Z. \quad (4.41)$$

Here $\square$ is a covariant d'Alembertian, combination $q^{+a}q_a^-$ ($a = 1, 2$) has no matrix indices (since a fixed direction in the moduli space was chosen), and matrix Z features indices I, J coming from the expression $\{T_I, T_J\}$ after fixing the background hypermultiplet as in (4.5). All matrices in (4.41) with $\mathcal{W}$, $\overline{\mathcal{W}}$ are diagonal in indices of generators of the subgroup $\tilde G$.

We are interested only in hypermultiplet-dependent terms in one-loop effective action (4.37). Let us explain how such terms can emerge in (4.37). The

[25] If background fields correspond to several Cartan generators H_i, the effective action is a sum of contributions over index i with the structure of each contribution corresponding to the above case. Therefore, without losing generality, we may restrict ourselves to the case of a single fixed generator H.

mass matrix has the following structure: $(\mathcal{M}_v^2) = \alpha^2(H)\mathcal{W}\overline{\mathcal{W}} \cdot Y + q^{+a}q_a^- \cdot Z$. Matrix Y has $n(H)$ eigenvectors corresponding to an eigenvalue of 1. The matrix in brackets in (4.41) has the same eigenvectors. As to Z, two options are available:

(i) Matrix Z has $n(\Upsilon)$ eigenvectors shared with matrix Y ($n(\Upsilon) \le n(H)$) with eigenvalues $r(\Upsilon)$. The effective action is then the sum over various $r(\Upsilon)$ values. Therefore, without losing generality, we can assume that a sole eigenvalue $r(\Upsilon)$ with $n(\Upsilon)$ eigenvectors, which are also the eigenvectors of matrix Y, exists. Therefore, the hypermultiplet dependence of the effective action is in this case given by

$$\Gamma_v^{(1)}[V^{++}, q^+] = \frac{i}{2}n(\Upsilon)\mathrm{Tr}\ln\Big[\Box + \frac{i}{2}\alpha(H)(\mathcal{D}^+\mathcal{W}\mathcal{D}^- + \overline{\mathcal{D}}^+\overline{\mathcal{W}}\overline{\mathcal{D}}^-) + \alpha^2(H)\mathcal{W}\overline{\mathcal{W}} + r(\Upsilon)q^{+a}q_a^-\Big]. \tag{4.42}$$

Here Tr denotes the functional trace of operators acting on analytic superfields with the corresponding $U(1)$ charge.[26] Those eigenvectors of matrix Y that do not match the eigenvectors of matrix Z produce no contribution to the hypermultiplet dependence of effective action.

(ii) Matrices Y and Z have no any common eigenvectors. If this is the case, the hypermultiplet-dependent part of effective action vanishes.

Thus, the hypermultiplet-dependent effective action is defined by expression (4.42). The actual calculation of effective action is discussed in the next subsection.

4.4. Calculation of One-Loop Effective Action

Expression (4.42) is the basis for the analysis of the hypermultiplet dependence of effective action. This expression will be rewritten in a form that allows a calculation in the framework of the superfield generalization of the Fock–Schwinger proper time method [2] ($\mathcal{N} = 1$ superfield proper time method was developed in [39, 40, 211, 213]). We will follow the general procedure outlined in [216] where the proper time method was used to analyze the hypermultiplet dependence of effective action of $\mathcal{N} = 4$ Yang–Mills theory.

[26] Specifically, if $\mathcal{A}^{(p,\,4-p)}(\zeta_1,\zeta_2)$ is the kernel of an operator acting on the space of covariant-analytic superfields with charge p,

$$\mathrm{Tr}\mathcal{A}^{(p,4-p)} = \mathrm{tr}\int d\zeta^{(-4)}du\mathcal{A}^{(p,4-p)}(\zeta,\zeta),$$

where "tr" denotes the trace over group indices.

In the framework of the Fock–Schwinger proper time representation, the effective action (4.42) is written as

$$\begin{aligned}\Gamma_v^{(1)}[V^{++}, q^+] &= \frac{i}{2}n(\Upsilon)\int d\zeta^{(-4)}du \\ &\times \int_0^\infty \frac{ds}{s}e^{-s\left(\Box + \frac{i}{2}\alpha(H)(\mathcal{D}^+\mathcal{W}\mathcal{D}^- + \overline{\mathcal{D}}^+\overline{\mathcal{W}}\overline{\mathcal{D}}^-) + \mathcal{M}_v^2\right)} \\ &\times (\mathcal{D}^+)^4[\delta^{12}(z-z')\delta^{(-2,2)}(u,u')]\big|_{z=z',\,u=u'} \\ &= \int_0^\infty \frac{ds}{s}\mathrm{Tr}K(s),\end{aligned} \tag{4.43}$$

where $\mathcal{M}_v^2 = \alpha^2(H)\mathcal{W}\overline{\mathcal{W}} + r(\Upsilon)q^{+a}q_a^-$, and $K(s)$ is the superfield heat kernel of the operator. Symbol Tr denotes the functional trace in the analytic subspace of harmonic superspace, $\mathrm{Tr}K(s) = \mathrm{tr}\int d\zeta^{(-4)}K(\zeta,\zeta|s)$, where tr is the trace over discrete indices. The representation of effective action in the form (4.43) allows us to calculate it directly as an expansion in powers of covariant spinor derivatives of Abelian strength superfields $\mathcal{W}, \overline{\mathcal{W}}$. The leading low-energy terms in this expansion correspond to background gauge superfields that are constant in space-time, $D_\alpha^- D_\beta^+\mathcal{W} =$ const and $\overline{D}_{\dot\alpha}^-\overline{D}_{\dot\beta}^+\overline{\mathcal{W}} =$ const, and to the background hypermultiplet on the mass shell. In addition, we assume that the hypermultiplet is a slowly varying function on a superspace and neglect all hypermultiplet derivatives in the process of calculation of superfield effective action. This, however, should not result in the lack of space-time derivatives needed in the component effective Lagrangian. The Grassmannian measure in the integral over harmonic superspace $d^4\theta^+d^4\theta^-$ produces four space-time derivatives in the component expansion of the superfield Lagrangian. Therefore, the assumptions made above provide obtaining the component effective Lagrangian with four space-time derivatives of the scalar hypermultiplet component. The possible off-shell contributions to the hypermultiplet dependence of effective action are discussed in the next subsection.

The process of construction of effective action (4.43) is based on the calculation of the superfield heat kernel $K(s)$. Even considering the properties of a nonanalytic expression under the integral, the basic idea is to stay within the analytic subspace at all calculation steps and avoid the artificial transformation of an analytic integral into an integral over the full superspace, in which case integrands often include ill-defined products of harmonic distributions. The integration with analytic measure can be regarded as a "projector" that removes all harmonic singularities automatically. Besides, since the covariant d'Alembertian does not contain an effectively acting $\mathcal{D}^+$, while working out the heat kernel we will never obtain derivative $\mathcal{D}^+q^-$; only the derivative $\mathcal{D}^-q^-$, can be obtained, and it vanishes on the mass shell. When this calculation method is used, quantum corrections with higher derivatives should emerge in the form

$$\int d\zeta^{(-4)}(\mathcal{D}^+)^4\mathcal{H}(\mathcal{W},\overline{\mathcal{W}},q^+,q^-). \qquad (4.44)$$

In the case of covariantly constant hypermultiplet ($\mathcal{D}_m q^+ = 0$) and vector multiplet ($\mathcal{D}_m\mathcal{W} = \mathcal{D}_m\overline{\mathcal{W}} = 0$), the heat kernel can be calculated exactly. This can be done more conveniently separating the contributions of the "diamagnetic" and "paramagnetic" parts of operator $\widehat{\square}$. We follow here the general calculation scheme [216] and take into account only those aspects that are essential for the theory under consideration. First, we apply the Baker–Campbell–Hausdorff formula in order to represent $K(s)$ as a product of several operator exponentials,[27]

$$\begin{aligned}K(s) &= \exp\left(-s\left\{A^+\mathcal{D}^- + \overline{A}^+\overline{\mathcal{D}}^- + \frac{1}{2}\mathcal{D}^{\alpha\dot\alpha}\mathcal{D}_{\alpha\dot\alpha} + \mathcal{M}_v^2\right\}\right)\\ &= \exp\{-f_{\alpha\dot\alpha}(s)\mathcal{D}^{\alpha\dot\alpha}\}\exp\left\{-s-\frac{1}{2}\mathcal{D}^{\alpha\dot\alpha}\mathcal{D}_{\alpha\dot\alpha}\right\}\\ &\quad\times\exp\{-\Omega(s)\}\exp\{-s(A^+\mathcal{D}^- + \overline{A}^+\overline{\mathcal{D}}^-)\},\end{aligned} \qquad (4.45)$$

with certain unknown coefficients in the right-hand side. A system of differential equations can be derived to determine these coefficients. The equation for function $f^{\dot\alpha\alpha}(s)$ has the form:

$$\begin{aligned}\frac{d}{ds}f_{\alpha\dot\alpha}(s) &= -f_{\beta\dot\beta}F^{\dot\beta\beta}_{\dot\alpha\alpha} - A^{+\beta}(D^-_\beta f_{\alpha\dot\alpha}) - \overline{A}^{+\dot\beta}(\overline{D}^-_{\dot\beta}f_{\alpha\dot\alpha})\\ &+ A^+_\beta\overline{A}^-_{\dot\beta}\left(\int_0^s d\tau e^{\tau F}\right)^{\dot\beta\beta}_{\dot\alpha\alpha} + \overline{A}^+_{\dot\beta}A^-_\beta\left(\int_0^s d\tau e^{\tau F}\right)^{\dot\beta\beta}_{\dot\alpha\alpha}.\end{aligned}$$

One can show that the solution to this equation is given by

$$f_{\alpha\dot\alpha} = -A^+_\delta\mathcal{F}^{\delta\dot\delta}_{\alpha\dot\alpha}\overline{A}^-_{\dot\delta} - \overline{A}^+_{\dot\delta}\overline{\mathcal{F}}^{\delta\dot\delta}_{\alpha\dot\alpha}A^-_\delta, \qquad (4.46)$$

[27]We use the following notation:

$$A^{+\alpha} = \frac{i}{2}\alpha(H)(\mathcal{D}^{+\alpha}\mathcal{W}),\quad \overline{A}^{+\dot\alpha} = -\frac{i}{2}\alpha(H)(\overline{\mathcal{D}}^{+\dot\alpha}\overline{\mathcal{W}}),$$

$$\mathcal{N}^{\;\beta}_\alpha = D^-_\alpha A^{+\beta},\quad \overline{\mathcal{N}}^{\;\dot\beta}_{\dot\alpha} = \overline{D}^-_{\dot\alpha}\overline{A}^{+\dot\beta}.$$

where the expressions for functions $\mathcal{F}(\mathcal{N},\overline{\mathcal{N}},s)$, $\overline{\mathcal{F}}(\mathcal{N},\overline{\mathcal{N}},s)$ can be found in [216]. A similar equation for function Ω is written as

$$\begin{aligned}\frac{d}{ds}\Omega(s) - \mathcal{M}_v^2 &= -A^{+\alpha}(D^-_\alpha\Omega) - \overline{A}^{+\dot\alpha}(\overline{D}^-_{\dot\alpha}\Omega)\\ &+ A^+_\alpha f^{\alpha\dot\alpha}\overline{A}^-_{\dot\alpha} + \overline{A}^+_{\dot\alpha}f^{\dot\alpha\alpha}A^-_\alpha - \frac{1}{2}A^+_\beta\overline{A}^-_{\dot\beta}\left(\int_0^s d\tau e^{\tau F}\right)^{\dot\beta\beta}_{\dot\alpha\alpha}\\ &\times F^{\dot\alpha\alpha}_{\dot\rho\rho}f^{\dot\rho\rho} - \frac{1}{2}\overline{A}^+_{\dot\beta}A^-_\beta\left(\int_0^s d\tau e^{-\tau F}\right)^{\dot\beta\beta}_{\dot\alpha\alpha}F^{\dot\alpha\alpha}_{\dot\rho\rho}f^{\dot\rho\rho}.\end{aligned}$$

The solution of this equation is as follows:

$$\begin{aligned}\Omega(s) &= s\mathcal{M}_v^2 + A^{+\alpha}\Omega^-_\alpha(s) + \overline{A}^{+\dot\alpha}\overline{\Omega}^-_{\dot\alpha}(s)\\ &+ (A^+)^2\Psi^{(-2)}(s) + (\overline{A}^+)^2\overline{\Psi}^{(-2)}(s) + A^{+\alpha}\overline{A}^+_{\dot\alpha}\Psi^{\dot\alpha(-2)}_\alpha(s).\end{aligned} \qquad (4.47)$$

Note that this solution is a polynomial of a finite order in powers of Grassmannian quantities $A^\pm, \overline{A}^\pm$. All coefficients are given in [216]. The last exponential in (4.45) should then be transformed into the form

$$\begin{aligned}\exp\{-s(A^+\mathcal{D}^- + \overline{A}^+\overline{\mathcal{D}}^-)\} &= 1 + a^{+\alpha}\mathcal{D}^-_\alpha + \overline{a}^{+\dot\alpha}\mathcal{D}^-_{\dot\alpha}\\ &+ f^{+2}(\mathcal{D}^-)^2 + \overline{f}^{+2}(\overline{\mathcal{D}}^-)^2 + f^{+2\dot\alpha\alpha}\mathcal{D}_\alpha\overline{\mathcal{D}}_{\dot\alpha}\\ &+ \Xi^{+3\dot\alpha}\overline{\mathcal{D}}^-_{\dot\alpha}(\mathcal{D}^-)^2 + \Xi^{+3\alpha}\mathcal{D}^-_\alpha(\overline{\mathcal{D}}^-)^2 + \Omega^{+4}(\mathcal{D}^-)^2(\overline{\mathcal{D}}^-)^2.\end{aligned} \qquad (4.48)$$

The coefficients of this expansion were determined exactly and are given in [216].

For further analysis it is essential to note that

$$\begin{aligned}\Omega^{+4} &= -\frac{1}{16}(A^+)^2(\overline{A}^+)^2\\ &\times\mathrm{tr}\left(\frac{\cosh(s\mathcal{N})-1}{\mathcal{N}^2}\right)\mathrm{tr}\left(\frac{\cosh(s\overline{\mathcal{N}})-1}{\overline{\mathcal{N}}^2}\right).\end{aligned} \qquad (4.49)$$

One can show that only this last term in the expansion of the exponential survives in the limit of coincident arguments $\theta^+ = \theta^{+'}$, to which one should pass in (4.43) with the relation $(D^-)^4(D^+)^4\,\delta^8(\theta-\theta')\big|_{\theta=\theta'} = 1$ taken into account. All the other terms with less than four derivatives (D^-) vanish in the above limit. Thus, an expression containing the maximum admissible power of Grassmannian-odd quantities $A^+, \overline{A}^+$ is obtained as a coefficient. All the other dependences on $A^+, \overline{A}^+$ in operator exponential (4.45) make no contribution, and we obtain the following expression for effective action:

$$\Gamma^{(1)}_{V}[V^{++}, q^+] = \frac{i}{2}n(\Upsilon)\int d\zeta^{(-4)}e^{-s\mathcal{M}_V^2}K_{Sch}(s)$$
$$\times (A^+)^2(\bar{A}^+)^2\mathrm{tr}\left(\frac{\cosh(s\mathcal{N})-1}{\mathcal{N}^2}\right)\mathrm{tr}\left(\frac{\cosh(s\bar{\mathcal{N}})-1}{\bar{\mathcal{N}}^2}\right), \tag{4.50}$$

where $K_{Sch}(s)$ is the superfield generalization of the Schwinger kernel [2, 144]. The latter is defined in the following way:

$$K_{Sch}(x, x', s) = e^{-\frac{s}{2}\mathcal{D}^{\dot{\alpha}\alpha}\mathcal{D}_{\dot{\alpha}\alpha}}\{\mathbf{1}\delta^4(x-x')\}.$$

The calculation of this heat kernel and its functional trace is performed in a standard way (for a more detailed description, see [107, 211, 213]). The final result of this calculation is

$$K_{Sch}(s) = \frac{i}{(4\pi s)^2}\frac{s^2(\mathcal{N}^2-\bar{\mathcal{N}}^2)}{\cosh(s\mathcal{N})-\cosh(s\bar{\mathcal{N}})}.$$

Here $\mathcal{N}$ is expressed as $\mathcal{N} = \sqrt{-\frac{1}{2}D^4\mathcal{W}^2}$ and, finally, in terms of two invariants of the Abelian vector field $\mathcal{F} = \frac{1}{4}F^{mn}F_{mn}$ and $\mathcal{G} = \frac{1}{4}\overset{\star}{F}{}^{mn}F_{mn}$: $\mathcal{N} = \sqrt{2(\mathcal{F}+i\mathcal{G})}$.

Expression (4.50) is the final result for the hypermultiplet-dependent low-energy one-loop effective action of the Euler–Heisenberg type. It is worth pointing out that the total dependence on the background hypermultiplet is contained in $\mathcal{M}_V^2$. The explicit expression for effective action is as follows:

$$\Gamma^{(1)}[V^{++}, q^+] = \frac{1}{(4\pi)^2}n(\Upsilon)\int d\zeta^{(-4)}du$$
$$\times\int_0^\infty\frac{ds}{s^3}e^{-s(\alpha^2(H)\mathcal{W}\bar{\mathcal{W}}+r(\Upsilon)q^{+a}q_a^-)}\frac{\alpha^4(H)}{16}$$
$$\times(D^+\mathcal{W})^2(\bar{D}^+\bar{\mathcal{W}})^2\frac{s^2(\mathcal{N}^2-\bar{\mathcal{N}}^2)}{\cosh(s\mathcal{N})-\cosh(s\bar{\mathcal{N}})} \tag{4.51}$$
$$\times\frac{\cosh(s\mathcal{N})-1}{\mathcal{N}^2}\cdot\frac{\cosh(s\bar{\mathcal{N}})-1}{\bar{\mathcal{N}}^2}.$$

The integrand in (4.51) can be expanded in the series of powers of $s^2\mathcal{N}^2$, $s^2\bar{\mathcal{N}}^2$. After changing the proper time variable s to $s'\mathcal{W}\bar{\mathcal{W}}$, we obtain an expansion in powers of $s'^2\frac{\mathcal{N}^2}{(\mathcal{W}\bar{\mathcal{W}})^2}$ and their conjugates. Since the expression under the integral in (4.51) contains $(D^+\mathcal{W})^2(\bar{D}^+\bar{\mathcal{W}})^2$ as a factor, we can substitute superconformal invariants Ψ^2 and $\bar{\Psi}^2$ in each term of the expansion of $\mathcal{N}^2$ and $\bar{\mathcal{N}}^2$ in these invariants [144] with $\bar{\Psi}^2 = \frac{1}{\bar{\mathcal{W}}^2}D^4\ln\mathcal{W} = \frac{1}{2\bar{\mathcal{W}}^2}\left\{\frac{\mathcal{N}_\alpha^\beta\mathcal{N}_\beta^\alpha}{\mathcal{W}^2}+\mathcal{O}(D^+\mathcal{W})\right\}$ and its conjugate. It can then be demonstrated that each term in the expansion is written as an integral over full $\mathcal{N} = 2$ superspace.

It is interesting and instructive to calculate the leading part of effective action (4.51). The analysis of expression (4.51) (see [213, 215, 216] for details) yields

$$\Gamma^{(1)}_{\mathrm{lead}} = \frac{1}{(4\pi)^2}n(\Upsilon)\int d\zeta^{(-4)}du\frac{1}{16}$$
$$\times\frac{D^+\mathcal{W}D^+\mathcal{W}}{\mathcal{W}^2}\frac{\bar{D}^+\bar{\mathcal{W}}\bar{D}^+\bar{\mathcal{W}}}{\bar{\mathcal{W}}^2}\frac{1}{(1-X)^2},$$

where

$$X = \frac{-q^{+a}q_a^-}{\mathcal{W}\bar{\mathcal{W}}}\frac{r(\Upsilon)}{\alpha^2(H)}. \tag{4.52}$$

At the next step, we rewrite this expression in the following form:

$$\frac{1}{(4\pi)^2}\int d\zeta^{(-4)}du\frac{1}{16}\left\{D^{+2}\ln\mathcal{W}\bar{D}^{+2}\ln\bar{\mathcal{W}}+\sum_{k=1}^\infty\frac{1}{k^2(k+1)}\right.$$
$$\left.\times D^{+2}\frac{1}{\mathcal{W}^k}\bar{D}^{+2}\frac{1}{\bar{\mathcal{W}}^k}\left(-\frac{r(\Upsilon)q^{+a}q_a^-}{\alpha^2(H)}\right)^k\right\} = \frac{1}{(4\pi)^2} \tag{4.53}$$
$$\times\int d^{12}zdu\left\{\ln\mathcal{W}\ln\bar{\mathcal{W}}+\sum_{k=1}^\infty\frac{1}{k^2(k+1)}X^k\right\}.$$

The obtained result agrees, except the group factor Υ, with the expressions found earlier in [146, 216]:

$$\Gamma^{(1)}_{\mathrm{lead}} = \frac{n(\Upsilon)}{(4\pi)^2}\int dud^{12}z\left[\ln\mathcal{W}\ln\bar{\mathcal{W}}+\mathrm{Li}_2(X)\right. \tag{4.54}$$
$$\left.+\ln(1-X)-\frac{1}{X}\ln(1-X)\right].$$

Here $\mathrm{Li}_2(X)$ is the Euler dilogarithm. Next-to-leading corrections to (4.54) can also be calculated. The remarkable property of the low-energy effective action (4.54) is that the factor $r(\Upsilon)/\alpha(H)$ emerges in argument of X. This factor is stipulated by the vacuum structure of the model under consideration and depends on the specific features of symmetry breaking. The hypermultiplet dependence of effective action similar to (4.54) was first found in [146] for $\mathcal{N}$ = 4 Yang–Mills theory and was then studied by various methods in [147, 216].

Both the $\mathcal{N}$ = 2 vector multiplet and the hypermultiplet in $\mathcal{N}$ = 4 Yang–Mills theory belong to the adjoint representation of the gauge group; therefore, the above coefficient in X equals unity. As a result, we arrive at the conclusion that the hypermultiplet-dependent low-energy effective action has a general form (4.54) in *all* $\mathcal{N}$ = 2 superconformal models; the models differ only in the coefficient $r(\Upsilon)/\alpha(H)$ in X (4.52). The same holds true for the general expression (4.50). The different models also differ in coefficients $n(\Upsilon)$ at integrals (4.50) and (4.54). This universality of effective action of $\mathcal{N}$ = 2 superconformal gauge theories is a noteworthy result that demonstrates the actual potential of the harmonic approach in the quantum region.

Let us now discuss some selected component terms in effective action (4.54). The component structure of (4.54) was studied in [146] in the bosonic sector for constant background fields F_{mn}, ϕ, $\bar{\phi}$, f^i, $\bar{f}_i$. However, it was already noted that superfield effective action (4.54) allows one to calculate the terms in effective action up to space-time derivatives of the fourth order on component fields. Our aim is to find these terms in the sector of scalar fields of the hypermultiplet. To this end, we omit all components of the background superfield except the scalars $\phi, \bar{\phi}$ in the $\mathcal{N}$ = 2 vector multiplet and scalars $f, \bar{f}$ in the hypermultiplet and integrate over $d^4\theta^+ d^4\theta^- = (D^-)^4(D^+)^4$. We act by these derivatives on the series under the integral in (4.53). In order to obtain the higher space-time derivatives of scalar components of the hypermultiplet, one should throw exactly two spinor derivatives on each hypermultiplet superfield. After some algebra, the following terms with four space-time derivatives on $q^{\pm}$ in the component expansion of effective action (4.53) are obtained:

$$\Gamma^{(1)}_{\text{lead}} = \int d^4x du \frac{n(\Upsilon)}{(4\pi)^2} \sum_{k=2}^{\infty} \frac{1}{16} \frac{k-1}{k(k+1)} \frac{X^{k-2}}{(\mathcal{W}\bar{\mathcal{W}})^2}$$
$$\times \Big\{ -\bar{D}^{+\dot{\alpha}} D^{+\alpha} q^-_b \bar{D}^+_{\dot{\alpha}} D^-_{\beta} q^{+(b} \bar{D}^{-\dot{\beta}} D^{-\beta} q^{+a)} \bar{D}^-_{\dot{\beta}} D^+_{\alpha} q^-_a$$
$$+ \frac{1}{2} \bar{D}^{+\dot{\alpha}} D^{+\alpha} q^-_b \bar{D}^{-\dot{\beta}} D^{-\beta} q^{+b} \bar{D}^-_{\dot{\beta}} D^-_{\beta} q^{+a} \bar{D}^+_{\dot{\alpha}} D^+_{\alpha} q^-_a$$
$$+ \frac{1}{2} \bar{D}^{-\dot{\beta}} D^{+\alpha} q^-_b \bar{D}^{+\dot{\alpha}} D^{-\beta} q^{+b} \bar{D}^+_{\dot{\alpha}} D^-_{\beta} q^{+a} \bar{D}^-_{\dot{\beta}} D^+_{\alpha} q^-_a \Big\}\Big|_{\theta=0}.$$

The direct calculation[28] of each term of this expression shows that among many terms with four derivatives, an intriguing term of the following special form is present:

$$\Gamma^{(1)}_{\text{lead}} = \frac{-1}{8\pi^2} n(\Upsilon) \left(\frac{r(\Upsilon)}{\alpha(H)}\right)^2 \times \left[\frac{X_0 - 2}{X_0^3} \ln(1 - X_0) - \frac{2}{X_0^2}\right] \times \int du d^4x \frac{1}{(\phi\bar{\phi})^2} i\varepsilon^{\mu\nu\lambda\rho} \partial_\mu \tilde{q}^+ \partial_\nu q^+ \partial_\lambda \tilde{q}^- \partial_\rho q^-. \tag{4.55}$$

The first term of expansion of this expression in powers of $X_0 = \frac{r(\Upsilon)\bar{f}^i f_i}{\alpha^2 \bar{\phi}\phi}$ is given by

$$\Gamma^{(1)}_{\text{lead}} = -\frac{n(\Upsilon)}{48\pi^2} \left(\frac{r(\Upsilon)}{\alpha(H)}\right)^2 \int d^4x \frac{1}{(\phi\bar{\phi})^2} i\varepsilon^{\mu\nu\lambda\rho} \times (\partial_\mu \bar{f}^i \partial_\nu f_i \partial_\lambda \bar{f}^j \partial_\rho f_j - \partial_\mu \bar{f}^i \partial_\nu \bar{f}_i \partial_\lambda f^j \partial_\rho f_j). \tag{4.56}$$

We have omitted all terms with expressions of the $\partial^\mu f \partial_\mu f$ type. Expressions (4.55) and (4.56) are similar to the Chern–Cimons action for a multicomponent complex scalar field. The possibility that such terms can arise in the effective action was discussed in [193, 195] in the context of $\mathcal{N}$ = 4, 2 supersymmetric gauge models and for $d = 6$, $\mathcal{N} = (2, 0)$ superconformal models. In the present case, expression (4.55) emerged as a result of direct calculations in supersymmetric quantum field theory.

As an example, we present the values of $\alpha(H)$, $r(\Upsilon)$, and $n(\Upsilon)$ for models in [115].

(i) The $\mathcal{N}$ = 4 Yang–Mills theory with gauge groups SU(N), Sp($2N$), and SO(N). The hypermultiplet sector here includes one hypermultiplet in the adjoint representation of the gauge group. The background is chosen in such a way that the gauge groups be broken according to the options $SU(N) \to SU(N-1) \times U(1)$, $Sp(2N) \to Sp(2N-2) \times U(1)$, and $SO(N) \to SO(N-2) \times U(1)$. All background fields are directed along element H = U(1) of the Cartan subalgebra (with $\Upsilon = H$). The mass matrix is expressed as

$$(\mathcal{M}^2_v)_{IJ} = (\mathcal{W}\bar{\mathcal{W}} + \mathbf{q}^{+a}\mathbf{q}^-_a)(\alpha(H))^2 \delta_{I,J},$$

[28] The following relations are used here: $\bar{D}^+ D^+ q^- = \bar{D}^+ D^+ D^{--} q^+ = -\bar{D}^+ D^- q^+ = -2i\sigma^\mu \partial_\mu q^+$, $\bar{D}^- D^- q^+ = 2i\sigma^\mu \partial_\mu q^-$, $\bar{D}^+ D^- q^+ = 2i\sigma^\mu \partial_\mu q^+$, and $\bar{D}^- D^+ q^- = -2i\sigma^\mu \partial_\mu q^-$. Rule $\int du u^+_i u^+_j u^-_k u^-_l = \frac{1}{6}(\epsilon_{ik}\epsilon_{jl} + \epsilon_{il}\epsilon_{jk})$ is also applied.

and the traces in Eq. (4.36) lead to a coefficient $n(\Upsilon)$ that is equal to the number of roots with $\alpha(H) \neq 0$ (i.e., the number of unbroken generators):

$$n(\Upsilon) = \begin{cases} 2(N-1) \text{ for } \mathrm{SU}(N), \\ 4N-2 \text{ for } \mathrm{Sp}(2N) \text{ and } \mathrm{SO}(2N+1), \\ 4N-1 \text{ for } \mathrm{SO}(2N). \end{cases}$$

It follows from the expression for the mass matrix that $r(\Upsilon) = \alpha(H)$ in this case.

(ii) This model was proposed in [189]. The group $\mathrm{USp}(2N) = \mathrm{Sp}(2N, \mathbb{C}) \cap \mathrm{U}(2N)$ was taken as the gauge one. The model contains four hypermultiplets q_F^+ in the fundamental representation and one hypermultiplet q_A^+ in the antisymmetric traceless representation of group $\mathrm{USp}(2N)$. Background fields $\mathcal{W}$, q_F^+, and q_A^+ are chosen as the solutions of Eq. (4.4) with the maximal unbroken gauge subgroup $\mathrm{USp}(2N-2) \times \mathrm{U}(1)$:

$$\mathcal{W} = \frac{W}{\sqrt{2}}\mathrm{diag}(1, \underbrace{0, \ldots, 0}_{N-1}, -1, \underbrace{0, \ldots, 0}_{N-1}), \quad q_F^+ = 0,$$

$$(q_A^+)_\alpha^\beta = \frac{\mathbf{q}^+}{\sqrt{2N(N-1)}}$$

$$\times\, \mathrm{diag}(N-1, \underbrace{-1, \ldots, -1}_{N-1}, N-1, \underbrace{-1, \ldots, -1}_{N-1}).$$

Mass matrix $(\mathcal{M}_v^2)_{IJ}$ was constructed in [115]. It has $n(\Upsilon) = 4(N-1)$ eigenvectors with eigenvalues $\mathcal{M}_v^2 = \bar{\mathcal{W}}\mathcal{W} + \frac{N}{N-1}\bar{\mathbf{q}}^j\mathbf{q}_j$.

(iii) The $\mathcal{N} = 2$ superconformal model with the simplest "quiver" of gauge theories [189]. The gauge group is the product $\mathrm{U}(N)_L \times \mathrm{SU}(N)_R$. The model contains two hypermultiplets q^+ and $\tilde{q}^+$ in bifundamental representations $(\mathbf{N}, \bar{\mathbf{N}})$ and $(\bar{\mathbf{N}}, \mathbf{N})$ of the gauge group. The solutions of Eq. (4.4) with nonvanishing hypermultiplet components that fix flat directions in massless $\mathcal{N} = 2$ gauge theories were constructed in [115]. The vacuum moduli space for these models incorporates the following configurations:

$$\mathcal{W}_L = \mathcal{W}_R = \frac{\mathcal{W}}{N\sqrt{2(N-1)}}$$

$$\times\, \mathrm{diag}(N-1, \underbrace{-1 \ldots, -1}_{N-1}),$$

$$q^+ = \tilde{q}^+ = \frac{\mathbf{q}^+}{\sqrt{2}}\mathrm{diag}(1, 0, \ldots, 0).$$

Fig. 1. One-loop supergraph.

These configurations leave unbroken gauge group $\mathrm{SU}(N-1) \times \mathrm{SU}(N-1)$ and the diagonal U(1) subgroup in $\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R$, which is related to the choice of W. On this background, the mass matrix has an eigen value of $\mathcal{M}_v^2 = \frac{1}{N-1}\bar{\mathcal{W}}\mathcal{W} + \frac{1}{N}\mathbf{q}^{+a}\mathbf{q}_a^-$; correspondingly, $n(\Upsilon) = 4(N-1)$.

4.5. Hypermultiplet-Dependent Contribution to the Effective Action off the Mass Shell

All the above analysis and study on the hypermultiplet dependence of effective action [146, 216] were based on the assumption that hypermultiplet q^+ satisfies on-shell conditions (4.6) and the restriction $q^+ = D^{++}q^-$. It was noted in [146] that these conditions are sufficient to obtain all the leading low-energy contributions to the effective action. In the present subsection, we relax these restrictions and study a few possible leading contributions with the minimum number of space-time derivatives to the component effective action.

Let us consider the superdiagram in the Fig. 1 with two external hypermultiplet legs and with all propagators depending on the background $\mathcal{N} = 2$ vector multiplet. The wavy line denotes the $\mathcal{N} = 2$ gauge superfield propagator, and solid external and internal lines correspond to the hypermutiplet background superfields and the quantum hypermultiplet propagator, respectively. We assume for simplicity that the background field is Abelian and omit all group factors. The corresponding contribution to the effective action has the form:

$$\begin{aligned} i\Gamma_2 = &\int d\zeta_1^{(-4)} d\zeta_2^{(-4)} du_1 du_2 \\ &\times \left[\frac{(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4}{(u_1^+u_2^+)^3}\frac{1}{\widehat{\square}_1}\delta^{12}(1|2)\right]\left[\frac{(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4}{\widehat{\square}_2\widehat{\square}_1}\right. \\ &\left.\times\, \delta^{12}(2|1)(\mathcal{D}_1^{--})^2\delta^{(-2,2)}(u_2, u_1)\right] \\ &\times\, \tilde{q}^+(z_1, u_1)q^+(z_2, u_2). \end{aligned} \tag{4.57}$$

As usual, we pick up factor $(D^+)^4$ from the vector multiplet propagator to restore the full $\mathcal{N}=2$ integration measure. We then shrink the loop to a point by transferring the operators $\widehat{\Box}$ and $(\mathcal{D}^+)^4$ from the first δ-function to the other one, and so remove one integration. Operator $\widehat{\Box}$ does not act on q^+ in the process, since we are interested in the contributions that produce terms with the minimum number of space-time derivatives in the component form of effective action. As a result, we obtain

$$i\Gamma_2 = \int \frac{d\zeta_1^{(-4)}du_1du_2}{(u_1^+u_2^+)^3}\frac{(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4(\mathcal{D}_1^+)^4}{\widehat{\Box}_2\widehat{\Box}_1^2}$$
$$\times\,\delta^{12}(z-z')(\mathcal{D}_1^{--})^2\delta^{(-2,2)}(u_2,u_1)\tilde{q}^+(z_1,u_1)q^+(z_1,u_2).$$

We then use twice the relation (4.33), which allows one to present $(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4$ as a polynomial in powers of $(u_1^+u_2^+)$. Multiplying operator $(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4(\mathcal{D}_1^+)^4$ by distribution $1/(u_1^+u_2^+)^3$, we obtain a polynomial containing powers of $(u_1^+u_2^+)$ from first to fifth.

First power corresponds exactly to the contribution of the type that was analyzed in the previous section, since one of the derivatives in $(D^{--})^2$ can be used to transform $(u_1^+u_2^+)$ into $(u_1^+u_2^-)\big|_{u_1=u_2}=1$ in the limit of coincident points. Another D^{--} transforms q^+ into q^- All this was done already in Subsection 2.4. Therefore, retaining only the first order in $(u_1^+u_2^+)$, we determine the contribution of combination q^+q^- without derivatives. It was noted in Subsection 2.4 that this contribution could be obtained through the analysis of a hypermultiplet satisfying mass shell conditions (4.6).

We consider here a new contribution to the effective action from term $(u_1^+u_2^+)^2$:

$$\begin{aligned}\frac{(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4(\mathcal{D}_1^+)^4}{(u_1^+u_2^+)^3} &= \ldots+(u_1^+u_2^+)^2\\ &\times(u_1^-u_2^+)(u_2^-u_1^+)(\mathcal{D}_1^+)^4\\ \times\Big[\frac{i}{2}\widehat{\Box}_1\Delta_2^{--}(u_2^+u_1^-)&-\frac{i}{2}\Delta_1^{--}\widehat{\Box}_2(u_1^+u_2^-)\Big]+\ldots.\end{aligned}\tag{4.58}$$

The ellipsis denotes terms with $(u_1^+u_2^+)$ to powers other than 2. It can be shown that such terms vanish in the limit of coincident points. Pulling $(D^{--})^2$ over to $(u_1^+u_2^+)^2$, we obtain

$$\begin{aligned}i\Gamma_2 &= i\int d\zeta^{(-4)}du(\mathcal{D}^+)^4\frac{1}{\widehat{\Box}^3}(\underbrace{\widehat{\Box}\Delta^{--}}_{\Gamma_2(1)}-\underbrace{\Delta^{--}\widehat{\Box}}_{\Gamma_2(2)})\\ &\times\delta^{12}(z-z')\big|_{z=z'}\tilde{q}^+(z,u)q^+(z,u),\end{aligned}\tag{4.59}$$

where Δ^{--} is defined (4.34).

Let us consider each of the two mentioned contributions separately. We use representation

$$\frac{1}{\widehat{\Box}^2}\Delta^{--}\delta^{12}(z-z')| = \int ds\, se^{s\widehat{\Box}}\Delta^{--}\delta^{12}(z-z')|,\tag{4.60}$$

where | denotes the limit of coincidence $z=z'$. We can now use the expansion of heat kernel in derivatives. The primary goal is to collect as many factors $\mathcal{D}^+$, $\mathcal{D}^-$, which act on $(\theta^+-\theta'^+)^4(\theta^--\theta'^-)^4$, as possible and retain the minimum order in s in the proper time integral. Higher orders in s generate higher powers of spinor derivatives in the effective action. In order to construct the necessary operator $(D^-)^4$, we expand the exponential in (4.60), which contains $D^+WD^-+c.c.$, to the second order and act on the term $\frac{1}{2}\mathcal{W}(\mathcal{D}^-)^2+c.c.$ contained in Δ^{--}. Representation (4.60) allows us to express the leading contribution to $\Gamma_2(1)$ in the following way:

$$\begin{aligned}\Gamma_2(1) &= -\int d^{12}zdu\int_0^\infty ds\cdot s\int\frac{d^4p}{(2\pi)^4}\\ &\times e^{-sp^2}e^{s(\mathcal{W}\overline{\mathcal{W}}-\varepsilon)}\frac{s^2}{32}\overline{\mathcal{W}}(D^{+\alpha}\mathcal{W}D_\alpha^+\mathcal{W})\\ &\times(D^-)^2(\bar{D}^-)^2\delta^8(\theta-\theta')\big|\tilde{q}^+q^++\text{c.c.}\end{aligned}$$

Using the trivial integration over p and s, we reduce this contribution to

$$\begin{aligned}\Gamma_2(1) &= \frac{i}{32\pi^2}\int d^{12}zdu\frac{1}{\overline{\mathcal{W}}}\frac{D^+\mathcal{W}D^+\mathcal{W}}{\mathcal{W}^2}\\ &\times\tilde{q}^+(z,u)q^+(z,u)(\mathcal{D}^-)^4\delta^8(\theta-\theta')|\\ &+\frac{i}{32\pi^2}\int d^{12}zdu\frac{1}{\mathcal{W}}\frac{\bar{D}^+\overline{\mathcal{W}}\bar{D}^+\overline{\mathcal{W}}}{\overline{\mathcal{W}}^2}\\ &\times\tilde{q}^+(z,u)q^+(z,u)(\mathcal{D}^-)^4\delta^8(\theta-\theta')|.\end{aligned}\tag{4.61}$$

We then perform the same procedures for the second underlined contribution $\Gamma_2(2)$, retaining the same order in s and D^-, $\bar{D}^-$ as in expression (4.61):

$$\Gamma_2(2) = -\int d^{12}z du \tilde{q}^+ q^+ \int_0^\infty \frac{ds s^2}{2} \int \frac{d^4p}{(2\pi)^4} e^{-sp^2 + isp\mathcal{D} + s\hat{\Box}}$$

$$\times (ip^{\alpha\dot{\alpha}} \mathcal{D}_\alpha^- \bar{\mathcal{D}}_{\dot{\alpha}}^- + \Delta^{--})\left(-\frac{1}{2}p^{\alpha\dot{\alpha}} p_{\alpha\dot{\alpha}} + ip^{\alpha\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}} + \hat{\Box}\right)\delta^8(\theta - \theta')| = -\int d^{12}z du \tilde{q}^+ q^+ \int_0^\infty \frac{ds s^2}{2}$$

$$\times \int \frac{d^4p}{(2\pi)^4} e^{-sp^2 + s\mathcal{W}\bar{\mathcal{W}} - \varepsilon s} \frac{1}{2}\bar{\mathcal{W}}(\bar{D}^-)^2 \frac{1}{4}(D^+\mathcal{W})(D^+\mathcal{W})\left(-\frac{s^2}{4}p^{\alpha\dot{\alpha}} p_{\alpha\dot{\alpha}} + s\right)\frac{1}{2}(D^-)^2\delta^8(\theta - \theta')| + (\mathcal{W} \leftrightarrow \bar{\mathcal{W}}).$$

The integration in momentum in this expression[29] yields $\frac{i}{(4\pi s)^2}\left\{-\frac{s^2}{4}\frac{4}{s} + s\right\} = 0$. It now becomes apparent that no leading term of the type (4.61) is present in $\Gamma_2(2)$. It is then easy to see that contribution (4.61) can be rewritten as (we use $\int d^2\bar{\theta}^- = \bar{D}^{+2}$)

$$-\frac{i}{32\pi^2}\int d^4x d^4\theta^+ d^2\theta^- du (\bar{D}^+)^2 (D^+)^2 (\ln\mathcal{W})\frac{1}{\bar{\mathcal{W}}}$$

$$\times \tilde{q}^+(z, u) q^+(z, u)(\mathcal{D}^-)^4 \delta^8(\theta - \theta')| + \text{c.c.}$$

A nonzero result is obtained when all D^+-factors act only on the spinor delta function. Thus, the considered contribution is written as an integral over measure $d^4x du d^4\theta^+ d^2\theta^-$ that is $3/4$ of the full $\mathcal{N} = 2$ integration measure over harmonic superspace $d^4x du d^4\theta^+ d^4\theta^-$.

Thus, the hypermultiplet-dependent part of effective action contains the following term:

$$\Gamma_2 = -\frac{i}{32\pi^2}\int d^4x du d^4\theta^+ d^2\theta^- \frac{1}{\bar{\mathcal{W}}}\ln(\mathcal{W})\tilde{q}^+ q^+\big|_{\bar{\theta}^- = 0}$$
$$- \frac{i}{32\pi^2}\int d^4x du d^4\theta^+ d^2\theta^- \frac{1}{\mathcal{W}}\ln(\bar{\mathcal{W}})\tilde{q}^+ q^+\big|_{\theta^- = 0}. \tag{4.62}$$

The presence of this term in the effective action for $\mathcal{N} = 2$ supersymmetric models was suggested in [195]. We have shown here how a term with such a structure can be obtained within supersymmetric quantum field theory.

It is instructive to determine the component form of nonstandard superfield action (4.62). We consider here only the bosonic sector in (4.62). After integration over anticommuting variables, which effectively amounts to action of supercovariant derivatives at $\theta = 0$, we obtain

$$\Gamma_2 = \frac{i}{4\pi^2}\int d^4x du \frac{1}{\mathcal{W}\bar{\mathcal{W}}}$$
$$\times D_\beta^+ \bar{D}_{\dot{\alpha}}^- \tilde{q}^+ \bar{D}^{-\dot{\alpha}} D_\alpha^- q^+ D^{+\beta} D^{-\alpha}\mathcal{W}\big|_{\theta, \bar{\theta} = 0} + \frac{i}{4\pi^2}$$
$$\times \int d^4x du \frac{1}{\mathcal{W}\bar{\mathcal{W}}}\bar{D}_{\dot{\beta}}^+ D^{-\alpha}\tilde{q}^+ D_\alpha^- \bar{D}_{\dot{\alpha}}^- q^+ \bar{D}^{+\dot{\beta}} \bar{D}^{-\dot{\alpha}}\bar{\mathcal{W}}\big|_{\theta, \bar{\theta} = 0}.$$

Since $D_\alpha^- \bar{D}_{\dot{\alpha}}^- q^+ = -2i\partial_{\alpha\dot{\alpha}} f^-$ and $\bar{D}_{\dot{\beta}}^+ D_\beta^- \tilde{q}^+ = 2i\partial_{\beta\dot{\beta}}\tilde{f}^+$, we have

$$\Gamma_2 = \frac{i}{4\pi^2}\int d^4x du \frac{1}{\phi\bar{\phi}}\partial_{\beta\dot{\alpha}}\tilde{f}^+ \partial_\alpha^{\dot{\alpha}} f^- F^{\beta\alpha}$$
$$+ \frac{i}{\pi^2}\int d^4x du \frac{1}{\phi\bar{\phi}}\partial_{\dot{\beta}}^{\alpha}\tilde{f}^+ \partial_{\alpha\dot{\alpha}} f^- \bar{F}^{\dot{\beta}\dot{\alpha}}, \tag{4.63}$$

where $F^{\beta\alpha}$, $\bar{F}^{\dot{\beta}\dot{\alpha}}$ are spinor components of Abelian strength F_{ab}. We then convert spinor indices into vector ones. As a result, we obtain a contribution to the effective action that is similar to the Chern–Simons term and contains three space-time derivatives:

$$\Gamma_2 = -\frac{1}{2\pi^2}\int d^4x \frac{1}{\phi\bar{\phi}}\varepsilon^{mnab}\partial_m \bar{f}^i \partial_n f_i F_{ab}. \tag{4.64}$$

This expression is the simplest contribution to the hypermultiplet-dependent effective action off the mass shell (4.6) for the background hypermultiplet. Naturally, other (more complicated) contributions containing hypermultiplet derivatives also exist. One can determine them using the same algorithm that led to (4.62). Here we have only demonstrated the procedure of calculation of contributions to the effective action in the form of an integral over 3/4 of the full $\mathcal{N} = 2$ harmonic superspace.

[29] $\int \frac{d^4p}{(2\pi)^4} e^{-sp^2} = \frac{i}{(4\pi s)^2}$, $\int \frac{d^4p}{(2\pi)^4} p_{\alpha\dot{\alpha}} p^{\beta\dot{\beta}} e^{-sp^2} = \frac{i}{(4\pi s)^2}\frac{1}{s}\delta_{\alpha\dot{\alpha}}^{\beta\dot{\beta}}$.

4.6. Summary

The one-loop low-energy effective action in $\mathcal{N} = 2$ superconformal models was studied. These models are formulated in harmonic superspace, and their field content satisfies the condition of ultraviolet finiteness of the theory (4.1). The effective action depends on the background Abelian $\mathcal{N} = 2$ vector multiplet superfield and the hypermultiplet background superfield that satisfy special restrictions (4.4) and (4.5) defining the vacuum structure of models. The effective action is calculated based on the $\mathcal{N} = 2$ background field method for the background hypermultiplet on the mass shell (4.6), as well as off the mass shell.

It was demonstrated that the hypermultiplet-dependent one-loop effective action in the considered theory is related to a special superfield operator (4.38) that acts only in the sector of quantum fields of the vector multiplet. The coefficients of this operator contain background superfields and depend on the specific details of gauge symmetry breaking. It was proven that one could calculate the one-loop effective action just by considering the simple case when this operator takes the general form (4.41).

The hypermultiplet-dependent one-loop low-energy effective action was presented as an integral over proper time. It was demonstrated that analyzing the vector multiplet and hypermultiplet on the mass shell is sufficient to determine the low-energy contributions to the effective action. The final result for this case is given by relation (4.50) that is the $\mathcal{N} = 2$ superfield counterpart of the Euler–Heisenberg action. The leading part of low-energy effective action (4.54) has a universal form[30] and depends on $X = \dfrac{-\mathbf{q}^{+a}\mathbf{q}^{-}_{a}}{\mathcal{W}\bar{\mathcal{W}}}\dfrac{r(\Upsilon)}{\alpha(H)}$ (4.52). The latter quantity involves the details of the vacuum structure of the model. Using superfield effective action (4.54), we construct the leading (in space-time derivatives) terms in the sector of scalar components of the hypermultiplet. These terms contain four space-time derivatives of the scalar field and are similar to Chern–Simons action (4.55).

There were found possible contributions to the effective action that can be obtained if we go beyond the mass shell conditions (4.6) for the background hypermultiplet. The harmonic superdiagram with two external hypermultiplet lines and the background-dependent propagator of the vector multiplet was calculated, and its leading low-energy contribution was found. It was demonstrated that the final result has an intriguing superfield structure and is written as an integral over 3/4 of the full $\mathcal{N} = 2$ harmonic superspace (4.62). The possible presence of such terms in the effective action of $\mathcal{N} = 2$ supersymmetric theories was discussed recently in [195]. We determine the component structure of effective action (4.62) in the bosonic sector retaining scalar components of the background hypermultiplet and vector components of the background $\mathcal{N} = 2$ gauge multiplet. The obtained expression (4.64) is similar to Chern–Simons terms and contains three space-time derivatives of component fields.

In conclusion, let us emphasize once again that we have analyzed the general structure of hypermultiplet-dependent one-loop low-energy effective action of $\mathcal{N} = 2$ superconformal models. A general expression for the effective action of the hypermultiplet on the mass shell was found. A special manifestly $\mathcal{N} = 2$ supersymmetric subleading contribution, which is written as an integral over 3/4 of the full $\mathcal{N} = 2$ harmonic superspace, was calculated for the hypermultiplet off the mass shell. It is worth pointing out that such contributions deserve a separate study.

5. ONE-LOOP EFFECTIVE ACTION IN $\mathcal{N} = 2$ SUPERSYMMETRIC MASSIVE YANG–MILLS THEORY

5.1. Introduction

The problem of quantization of the massive non-Abelian Yang–Mills theory has a long history (see, for example, [157–162]). Gauge invariance is usually associated with masslessness of the corresponding gauge fields. However, it is evident that massive degrees of freedom of vector bosons should be taken into account in order to understand the phenomenology of particles.

Several different mechanisms through which vector fields acquire mass are now known. The common feature of these mechanisms, which are compatible with gauge invariance, is an increase in the number of physical degrees of freedom (relative to that in the massless theory). Of course, the dominant and generally accepted model lying at the basis of the Standard Model is the mechanism of spontaneous symmetry breaking with additional physical degrees of freedom implied by the Higgs effect (i.e., taken from the Higgs multiplet containing scalar Higgs fields). In unitary gauge, only Higgs bosons are left from this multiplet, and a fraction of massless gauge fields become massive due to the eating of the other scalar (Goldstone) components, the number of which equals the number of massive gauge fields.

Although the existence of Higgs bosons has been proven experimentally and this aspect of the Standard Model is fully justified, other mechanisms of boson and fermion mass generation in gauge theories can also be of interest (provided that no extra degrees of freedom emerge in the spectrum of physical particles) in the context of development of quantum field theory.

[30] This form of low-energy effective action (4.54) was first found in [146] for $\mathcal{N} = 4$ gauge theory.

The model in which gauge fields acquire mass through their gauge-invariant coupling to the real pseudoscalar Stueckelberg field [163] described by the nonlinear sigma model is the most frequently used alternative to the Higgs mechanism. In unitary gauge, this field is absorbed completely by the longitudinal component of a massive vector. The difference with the Higgs mechanism consists in the fact that Stueckelberg fields are transformed by the nonlinear realization of the gauge group, which does not require the introduction of any additional Higgs fields that complete nonlinear fields to a linear multiplet, right from the outset.[31] A fairly comprehensive review of literature on this topic is found in [166]; among numerous recent publications, paper [167] deserves a mention.

In addition to models with Stueckelberg fields, one can consider non-Abelian vector-tensor gauge theories with topological interactions. All such theories are classically equivalent to non-Abelian theories of massive vector fields with Stueckelberg fields. The same degrees of freedom can be described by any of the two dual representations. In certain contexts, one formulation can turn out to be more convenient than the other one; therefore, it is useful to know both formulations in detail and understand their interrelation.[32]

It should be noted that such models arise in the low-energy limit of superstring theory and in the context of supergravity in higher dimensions. For example, the degrees of freedom of a massive antisymmetric tensor field, which are responsible for one of the possible mechanisms of spontaneous symmetry breaking (see [66]), arise naturally in the recently studied compactifications of a type II superstring on Calabi—Yau manifolds in the presence of nontrivial fluxes of the gauge three-form. This fact inspired a renewed interest in detailed studies of massive $\mathcal{N} = 1$ and $\mathcal{N} = 2$ tensor multiplets and their couplings to scalar and vector multiplets. Note also that the couplings to tensor multiplets play an important role in the mechanism of cancellation of anomalies in superstring models.

The study of quantum properties of dual realizations of one and the same representation of supersymmetry is of special interest.

The theory of $\mathcal{N} = 1$ supersymmetric massive tensor multiplet as a dual version of a massive vector multiplet is well-known (see, for example, [39]). In $\mathcal{N} = 2$ supersymmetry, the antisymmetric tensor strength is contained within a tensor multiplet that is described by analytic harmonic $\mathcal{N} = 2$ superfield G^{++} subject to certain constraints [51]. The action for a massless tensor multiplet contains only G^{++}; however, if the antisymmetric tensor has a certain mass, gauge invariance requires including the interaction of the corresponding Stueckelberg fields with the vector multiplet [67].

In the present section, we will consider quantum properties of $\mathcal{N} = 2$ massive Yang—Mills theory using the Stueckelberg superfield. This model is a direct $\mathcal{N} = 2$ supersymmetrization of a nonsupersymmetric ($\mathcal{N} = 0$) massive Yang—Mills theory in the Kunimasa—Goto formalism [171]. Certain aspects of this problem have already been discussed in [173]. It was demonstrated there that the theory is finite in the second order in dimensionless Yang—Mills coupling constant g^2; the mass term is not renormalized, but the theory is nonrenormalizable in the sector that contains dimensionful coupling constant $\frac{m^2}{g^2}$. This allowed the authors of [173] to conclude that the theory is finite in the vector multiplet sector to all orders of loop expansion. Note, however, that the action of $\mathcal{N} = 2$ Yang—Mills theory within the harmonic superspace formalism [56] is written as an infinite sum over all powers of the vector multiplet potential V^{++} at the classical level. In addition, the sigma-model Lagrangian of the Stueckelberg superfield has a highly nonlinear form. Therefore, we cannot reduce the gauge-covariant analysis of quantum properties of the theory (even at the one-loop level) to the analysis of only the simplest diagrams, since this leads to gauge-noninvariant counterterms. All one-loop diagrams corresponding to the effective action with all external legs should be summed. If the conventional noncovariant diagram technique is used, this task appears extremely complicated (if not impossible).

In the present section, we use the formulation of $\mathcal{N} = 2$ supersymmetric theory and the corresponding Stueckelberg formalism in harmonic superspace [173] and apply the background field method [41] that allows one to sum all diagrams with an increasing number of insertions to construct the effective action. The obtained results are conceptually similar to the results presented in [174, 175], where the problem of construction of invariant counterterms for the nonsupersymmetric massive Yang—Mills theory off the mass shell was solved. The use of the invariant perturbation theory, which was developed for the models of principal chiral fields in [176—178], is an important mean that allows one to preserve gauge invaraince at all calculation stages.

The section is organized as follows. The formulation of $\mathcal{N} = 2$ supersymmetric massive Yang—Mills field theory in harmonic superspace with the Stueckelberg superfield taken into account is given in Subsection 5.2. Eliminating the nonphysical degrees of freedom, we obtain an explicitly gauge-invariant nonlocal

[31] Actually, the primary motivation behind incorporating the additional degrees of freedom into a linear multiplet is the renormalizability requirement. This requirement is not satisfied in the “minimal” case when these degrees of freedom are described by the nonlinear sigma model.

[32] The quantum equivalence of various dual formulations is a more delicate problem and requires separate analysis for each specific theory (see, for example, [168, 169]).

expression for the mass term in the Lagrangian. It is reasonable to expect that this form of the mass term will shed light on the dual coupling to the theory of $\mathcal{N} = 2$ massive tensor multiplet [67]. The procedure of construction of effective action based on the $\mathcal{N} = 2$ supersymmetric background field method is discussed in Subsection 5.3. The specific features of application of the background field method to the considered theory are also noted there. Subsection 5.4 is focused on the calculation of one-loop divergences of effective action. Gauge-invariant and manifestly $\mathcal{N} = 2$ supersymmetric counterterms depending on the Stueckelberg superfield are also presented there. The derivation of the component structure of the bosonic sector of one such counterterm is discussed in Subsection 5.5. The results are summarized in Subsection 5.6.

5.2. $\mathcal{N} = 2$ *Supersymmetric Massive Yang–Mills Theory in Harmonic Superspace*

It was already noted that the formulation of $\mathcal{N} = 2$ supersymmetric theories in terms of unconstrained superfields defined on the analytic subspace of harmonic superspace [56] turned out to be extremely useful in studying the quantum effects in such theories (see, for example, [41, 52]).

Recall that the $\mathcal{N} = 2$ vector multiplet with a finite number of physical and auxiliary fields and with an infinite set of gauge degrees of freedom is described by a real analytic superfield $V^{++} = V_a^{++}T_a$ that takes the values in the Lie algebra of the gauge group. Hypermultiplets ω and q^+, which are transformed by a certain representation R of the gauge group and have infinite sets of auxiliary fields off the mass shell, are described by analytic superfields: real superfield $\omega(\zeta)$ and complex superfield $q^+(\zeta)$ with its conjugates $\tilde{q}^+(\zeta)$ (see [56] for the definition of generalized conjugation as a combination of complex conjugation and antipodal mapping on a 2-sphere). Scalar component fields of the ω-hypermultiplet ($\omega(x_A)$ and $\omega^{(ij)}(x_A)$), which are transformed as an isoscalar and an isotriplet under the $SU(2)$ group of internal automorphisms of the supersymmetry, and a doublet of Weyl fermions ψ^i_α, $\bar{\psi}_{i\dot{\alpha}}$ emerge as the leading components of expansion of $\omega(\zeta)$ in powers of Grassmannian coordinates $\theta^+, \bar{\theta}^+$ and harmonics $u_i^\pm$. Other $\mathcal{N} = 2$ matter multiplets with a finite number of auxiliary fields also exist. They are described by analytic superfields that are subject to properly chosen harmonic constraints.

Gauge $\mathcal{N} = 2$ potential V^{++} satisfies the reality relation $\widetilde{V^{++}} = V^{++}$ and is transformed under gauge transformations as $\delta V^{++} = -\mathcal{D}^{++}\lambda$, where λ is an arbitrary real analytic superparameter and $\mathcal{D}^{++} = D^{++} + iV^{++} = e^{ib(z,u)}D^{++}e^{-ib(z,u)}$ is the covariant harmonic derivative in analytic basis.[33] Superfield $b(z,u)$ is the so-called gauge bridge. The gauge freedom associated with superparameter λ allows one to eliminate an infinite number of gauge degrees of freedom in V^{++} and impose the Wess–Zumino gauge in which this analytic superfield has a finite number of physical and auxiliary fields forming a vector supermultiplet off the mass shell. This multiplet contains vector field $A_m(x)$, complex scalar field $M(x) + iN(x)$, Majorana spinor isodoublet $\lambda^i_\alpha(x)$, $\bar{\lambda}^i_{\dot{\alpha}}(x)$, and a triplet of scalar auxiliary fields $F^{ij}(x)$. It was demonstrated in [56] that all geometrical objects of the theory (such as various connections and strengths) are expressed in terms of a single unconstrained potential $V^{++}(\zeta, u)$.

Without elaborating on the construction of the $\mathcal{N} = 2$ gauge theory (see previous sections for details), we recall that the non-Abelian gauge vector supermultiplet action, which is important for further analysis, takes the following form in terms of V^{++}:

$$S = \frac{1}{2g^2}\mathrm{tr}\sum_{n=2}^{\infty}\frac{(-i)^n}{n}\int d^{12}z du_1\ldots du_n \times \frac{V^{++}(z,u_1)\ldots V^{++}(z,u_n)}{(u_1^+u_2^+)\ldots(u_n^+u_1^+)} = -\frac{1}{2g^2}\mathrm{tr}\int d^8z W^2. \tag{5.1}$$

Harmonic distributions $\frac{1}{u_1^+u_2^+}$ (in other words, Green's functions on sphere $G^{(-1,-1)}(u_1,u_2)$ that are defined by equation $\partial^{++}G^{(-1,-1)}(u_1,u_2) = \delta^{(1,-1)}(u_1,u_2)$) were used here. The rules of differentiation and integration with respect to harmonics were developed in pioneering paper [56].

The harmonics-independent chiral superstrength $W = -\frac{1}{4}(\bar{D}^+)^2V^{--}$ is defined in terms of nonanalytic superfield

$$V^{--}(z,u) = \int du' \frac{e^{ib(z,u)}e^{-ib(z,u')}V^{++}(z,u')}{(u^+u'^+)^2},$$

satisfying the zero-curvature equation

$$D^{++}V^{--} - D^{--}V^{++} + i[V^{++}, V^{--}] = 0. \tag{5.2}$$

[33] $D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}} - 2i\theta^+\sigma^m\bar{\theta}^+\frac{\partial}{\partial x_A^m} + \theta^{+\alpha}\frac{\partial}{\partial\theta^{-\alpha}} + \bar{\theta}^{+\dot{\alpha}}\frac{\partial}{\partial\bar{\theta}^{-\dot{\alpha}}}$.

In λ-basis, action (5.1) is invariant under gauge transformations

$$
\begin{gathered}
{}^{g}V^{++} = e^{i\lambda}(V^{++} - iD^{++})e^{-i\lambda}, \quad V^{++} = V_a^{++}T_a, \\
\lambda = \lambda(\zeta, u) = \bar{\lambda}(\zeta, u), \quad \lambda = \lambda_a T_a.
\end{gathered} \tag{5.3}
$$

Here T_a are generators of the gauge group,

$$[T_a, T_b] = if_{abc}T_c, \quad \mathrm{tr}(T_aT_b) = \delta_{ab}.$$

Transformation law (5.3) is the finite form of the infinitesimal gauge transformation defined above.

Let us construct now a gauge-invariant expression for the term in the superfield action that is responsible for the gauge field mass. We use a generalization of the known Kunimasa–Goto formalism [171, 166] that was developed for the gauge-invariant description of the mass of gauge fields in the conventional Yang–Mills theory. In $\mathcal{N} = 2$ superfield gauge theory, this formalism requires the introduction of an additional Goldstone ω-hypermultiplet in the adjoint representation of the gauge group. The corresponding massive term in the action is then written as

$$S_m = -\frac{m^2}{2g^2}\mathrm{tr}\int d\zeta^{(-4)}du[\Omega^{-1}(V^{++} - iD^{++})\Omega]^2, \tag{5.4}$$

where $\Omega = \Omega(\omega) = e^{-i\omega}$. The mass term in this formulation[34] is manifestly invariant under simultaneous transformations (5.3) and transformations

$${}^{g}\Omega = e^{i\lambda}\Omega, \tag{5.5}$$

where ${}^{g}\Omega = g\Omega$, and g is the same gauge group element. Note that mass term (5.4) is also invariant under the right global transformations g_R: $(V^{++})^{g_R} = V^{++}$, $\Omega^{g_R} = \Omega g_R$. Since the action of $\mathcal{N} = 2$ supersymmetric massless Yang–Mills theory (5.1) is gauge-invariant, the substitution $V^{++} \to {}^{\Omega}V^{++}$ does not change the structure of action (5.1).

It is instructive to examine another (explicitly gauge-invariant, but nonlocal) representation of the mass term through the superfield strength W. Let us consider action

$$
\begin{aligned}
S[V^{++}, \omega] = &-\frac{1}{2g^2}\mathrm{tr}\int d^4\theta W^2 - \frac{m^2}{2g^2} \\
&\times \mathrm{tr}\int d\zeta^{(-4)}du\mathcal{V}^{++}\mathcal{V}^{++},
\end{aligned} \tag{5.6}
$$

where

$$
\begin{aligned}
\mathcal{V}^{++} &= V^{++} - L^{++}, \quad L^{++} = i(D^{++}\Omega)\Omega^{-1} \\
&= \int_0^1 d\tau e^{-i\tau\omega}D^{++}\omega e^{i\tau\omega} = \int_0^1 d\tau D^{++}\omega_a R_{ab}(\tau\omega)T_b
\end{aligned} \tag{5.7}
$$

[35]

and write down the equation of motion that follows from it:

$$\frac{1}{4}(D^+)^2W + m^2\mathcal{V}^{++} = 0. \tag{5.8}$$

Since W is independent of harmonics, the self-consistency condition for this equation is the equation of motion for the ω-multiplet:

$$D^{++}\mathcal{V}^{++} = 0. \tag{5.9}$$

It can be seen from definition (5.7) that the component content of gauge-covariant $({}^{g}\mathcal{V}^{++} = g\mathcal{V}^{++}g^{-1})$ potential $\mathcal{V}^{++}$ is defined by complex nonpolynomial combinations formed from the physical components of the vector multiplet in the Wess–Zumino gauge and the ω-multiplet components. However, on the mass shell under condition (5.9), the component expansion can be parameterized as follows:

$$
\begin{aligned}
\mathcal{V}^{++}(\zeta, u) &= f^{++}(x_A) + (\theta^+)^2\bar{\varphi}(x_A) + (\bar{\theta}^+)^2\varphi(x_A) \\
&+ 2i(\theta^+\sigma^m\bar{\theta}^+)(A_m(x_A) + \partial_m f^{+-}(x_A)) \\
&+ 2[\theta^{+\alpha}\psi^i_\alpha - \bar{\theta}^+_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}i}]u^+_i + 2i[(\bar{\theta}^+)^2\theta^{+\alpha}\partial_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}i} \\
&+ (\theta^+)^2\bar{\theta}^+_{\dot{\alpha}}\partial^{\dot{\alpha}\alpha}\psi^i_\alpha]u^-_i - (\theta^+)^2(\bar{\theta}^+)^2\Box f^{--}(x_A).
\end{aligned} \tag{5.10}
$$

The mass term for the vector component of superfield $\mathcal{V}^{++}$ then takes the form similar to the mass term in the Stueckelberg formalism for the nonsupersymmetric massive Yang–Mills field theory [175]:

$$S_m = -\frac{m^2}{2g^2}\mathrm{tr}\int d^4x du(A_m - L_m)^2,$$

where Cartan form L_m on the group is defined in terms of isoscalar ω, and isotriplet $\omega^{(ij)}$ physical components of the ω-supermultiplet. It is technically complicated enough to derive an explicit component expression for S_m, since this requires integrating over harmonics in each term of an infinite series in the expansion of Cartan form $L_m = e^{-if^{ij}u^+_iu^-_j}\partial_m e^{if^{ij}u^+_iu^-_j}$, where $f^{ij} = \omega\varepsilon^{ij} + \omega^{ij}$. However, the similarity with the corresponding nonsupersymmetric theory is evident.

Equation (5.8) can be regarded as a constraint for the ω-multiplet that can be resolved perturbatively in the general case of a non-Abelian theory. The solution in the Abelian case is particularly simple:

$$\omega(\zeta_1, u_1) = \int d\zeta_2^{(-4)}du_2 G_0^{(0,0)}(1|2)D_2^{++}V^{++}(2), \tag{5.11}$$

[34] It is useful to keep in mind the other form of the same expression (5.3):

$$S_m = -\frac{m^2}{2g^2}\mathrm{tr}\int d\zeta^{(-4)}du\left[V^{++} - i\Omega^{-1}(D^{++}\Omega + i[V^{++}, \Omega])\right]^2.$$

The specific structure of the gauge sigma model is seen clearer in this formulation.

[35] The definition of isotopic matrix $R_{ab}(\omega)$ is given below (see (5.16)).

where

$$G_0^{(0,0)}(1|2) = -\frac{1}{\Box}(D_1^+)^4(D_2^+)^4\delta^{12}(1|2)\frac{u_1^- u_2^-}{(u_1^+ u_2^+)^3} \quad (5.12)$$

is the Green's function of the omega multiplet [56]. Acting by operator D^{++} on both sides of (5.11), we obtain the following relation:

$$D_1^{++}\omega(\zeta_1, u_1) = -\int d\zeta_2^{(-4)} du_2 \frac{1}{\Box}(D_1^+)^4(D_2^+)^4$$
$$\times \delta^{12}(1|2)V^{++}(2)\left\{\frac{1}{(u_1^+ u_2^+)^2} + \frac{1}{2}(D_2^{--})^2\delta^{(2,-2)}(u_2, u_1)\right\}$$
$$= \int d\zeta_2^{(-4)} du_2 \{\Pi_T^{(2,2)}(1|2) + \delta_A^{(2,2)}(1|2)\} V^{++}(2),$$

where Π_T is an analytic distribution with the properties of a projection operator [56]. Eliminating the gauge degrees of freedom ω from (5.6), after some chain of transformations of the mass term the latter can be rewritten as follows:

$$S_m = -\frac{m^2}{2g^2}\int d\zeta_1^{(-4)} du_1 d\zeta_2^{(-4)} du_2 V^{++}(1)$$
$$\times \Pi_T^{(2,2)}(1|2)V^{++}(2) = \frac{m^2}{2g^2}\int d^8 z_c W\frac{1}{\Box}W. \quad (5.13)$$

As a result, the gauge-invariant form of the mass term is expressed completely through strengths W. It is evident that the ω-multiplet can also be eliminated perturbatively in the non-Abelian case within a polynomial expansion in powers of V^{++}. Of course, this expansion is a nonlocal expression, but it can be localized by introducing the proper tensor multiplets.[36] We thus obtain an interesting sequence of classical dualities for the model containing Stueckelberg superfields ω. The same occurs when calculating the $\langle A^2 \rangle$ condensates in the Yang–Mills theory. There the nonlocal gauge-invariant functional related to A^2 contains information on the topological vacuum structure of the theory with a nonvanishing vacuum expectation value of this operator (see [179] and references therein for a detailed description of the infrared dynamics of Yang–Mills theory).

Our next goal is to determine the effective action and to analyze the structure of one-loop divergences in the theory with action (5.6).

5.3. Background Field Method

The gauge-invariant loop expansion of the effective action in supersymmetric theories is carried out within the superfield background field method (see, for example, [37, 39] for $\mathcal{N} = 1$ theories and [41, 52] for $\mathcal{N} = 2$ theories). In the background field formalism, we should consecutively perform the background–quantum splitting of all fields, fix the gauge degrees of freedom of quantum fields applying the Faddeev–Popov procedure, and expand the generating functional of vertex functions in powers of quantum fields. The contribution to the effective action in a given loop order is then produced only by a finite number of terms in such an expansion.

Let us consider the theory of superfields V^{++}, ω with action (5.6). In the $\mathcal{N} = 2$ sector of vector multiplet V^{++}, we perform splitting $V^{++} \to V^{++} + gv^{++}$ and repeat all the steps corresponding to the massless theory [52]. In the ω-supermultiplet sector with a nonlinear chiral Lagrangian, we must construct the expansion by making use of the perturbation theory in terms of parameterization-independent invariant quantities [178].

The basic principle of background–quantum splitting of fields taking the values in the Lie group is the nonlinear rule for group addition of group elements $\Omega(\omega)$ and $\Omega(\chi)$ [178]:

$$\Omega(\omega \oplus \chi) = \Omega(\omega)\Omega\left(\frac{m}{g}\chi\right). \quad (5.14)$$

Under this addition rule, expression $\Omega(\omega \oplus \chi)$ is an element of the same space as $\Omega(\omega)$ and has the same group transformation law.

We define the splitting of superfield ω into background superfield ω and quantum superfield χ in accordance with (5.14). It is easy to demonstrate that the background ω-fields are transformed as in (5.5), while quantum fields χ are not transformed at all. With this splitting of fields into background and quantum ones, both the full Lagrangian and all terms of its Taylor expansion in quantum fields are invariant with respect to both local and global transformation groups.

[36] Let us write the action for the massive tensor multiplet in harmonic superspace [67] in the form

$$S = \frac{1}{2}\int d\zeta^{-4}(G^{++})^2 + \frac{1}{2}m\left(\int d^8 z W\psi + c.c.\right) + \frac{1}{2}\int d^8 z W^2,$$

where $G^{++}(z, u)$ is the real analytic superfield subject to the $D^{++}G^{++} = 0$ constraint. This constraint may be resolved in terms of the harmonic-independent unconstrained chiral prepotential $\psi(z)$ and its conjugate as $G^{++}(z, u) = \frac{1}{8}(D^+)^2\psi(z) + \frac{1}{8}(\bar{D}^+)^2\bar{\psi}(z)$. Superfield G^{++} remains invariant under gauge transformations $\delta\psi = i\Lambda$, $\bar{D}^i_{\dot{\alpha}}\Lambda = 0$, $D^{\alpha i}D^j_{\alpha}\Lambda = \bar{D}^i_{\dot{\alpha}}\bar{D}^{j\dot{\alpha}}\bar{\Lambda}$. Let us choose the gauge-fixing function in the form $F^{++} = \frac{1}{8}(D^+)^2\psi(z) - \frac{1}{8}(\bar{D}^+)^2\bar{\psi}(z)$. Integrating in the generating functional over prepotentials $\psi, \bar{\psi}$, we then obtain a nonlocal mass term for vector multiplet (5.13):

$$Z = \int D\psi D\bar{\psi}(\mathrm{Det}\Box)$$
$$\times e^{\frac{i}{2}\int d\zeta^{-4}\{(G^{++})^2 + (F^{++})^2\} + \frac{i}{2}m\int d^8 z(W\psi + c.c.)} = e^{i\int d^8 z W\frac{m^2}{\Box}W}.$$

As a result, the obtained counterterms are automatically invariant under the classical gauge and global group transformations.

It is sufficient for one-loop calculations to know the expansion of the Lagrangian up to the terms of the second order in quantum fields v^{++}, χ:

$$\begin{aligned} S^{(2)} &= \frac{1}{2}\int d^{12}z du_1 du_2 \frac{v_a^{++}(1)v_a^{++}(2)}{(u_1^+u_2^+)^2} \\ &-\frac{1}{2}\int d\zeta^{(-4)}du\{m^2(v_a^{++})^2 - 2mD^{++}\chi_a R_{ab}v_b^{++} \\ &+ D^{++}\chi_a D^{++}\chi_a + f_{abc}D^{++}\chi_a\chi_b R_{cd}\mathcal{V}_d^{++}\}, \end{aligned} \tag{5.15}$$

where isotopic matrix $R_{ab}(\omega)$ is determined by $\Omega T_a \Omega^{-1} = R_{ab}T_b$. Just as in the nonsupersymmetric case [175], it has the following properties:

$$\begin{gathered} R_{ae}R_{be} = \delta_{ab}, \quad D^{++}R_{ab} = -R_{ae}f_{bec}L_c^{++}, \\ f_{abc} = R_{ad}R_{be}R_{cg}f_{deg}. \end{gathered} \tag{5.16}$$

We should also add the term fixing the gauge in the sector of the quantum vector superfield to (5.15). It is convenient to choose this term in the background gauge-invariant form

$$F^{(+4)} = \mathcal{D}^{++}v^{++}. \tag{5.17}$$

The action of Faddeev–Popov and Nielsen–Kallosh ghosts should also be added. We follow here the approach developed in [52]. According to the Faddeev–Popov procedure, in order to fix the gauge in functional integral $Z = N\int\mathcal{D}v^{++}e^{iS}$, we should introduce unity into it in the form $1 = \Delta_{FP}\delta(F^{(+4)} - f^{(+4)})$, where the Faddeev–Popov determinant is defined as $\Delta_{FP}[v^{++}, V^{++}] = \mathrm{Det}[\mathcal{D}^{++}(\mathcal{D}^{++} + iv^{++})]$. We then should insert unity in the form

$$\begin{aligned} 1 &= \Delta_{NK}\int\mathcal{D}f^{(+4)} \\ &\times \exp\left\{\frac{i}{2\alpha}\mathrm{tr}\int d^{12}z du_1 du_2 f_\tau^{(+4)}\frac{(u_1^-u_2^-)}{(u_1^+u_2^+)^3}f_\tau^{(+4)}\right\}, \end{aligned}$$

into the functional integral. Here α is the gauge parameter that is set as $\alpha = -1$ for convenience in further calculations, $\Delta_{NK}[V^{++}]$ is the Nielsen–Kallosh determinant, and $f_\tau^{(+4)} = e^{-ib}f^{(+4)}e^{ib}$ is the gauge-invariant function taking the values in the Lie algebra of the gauge group. Note that the Nielsen–Kallosh determinant depends on the background superfield, which means the presence of the third Nielsen–Kallosh ghost. The details of the procedure of calculation of this determinant, $\Delta_{NK}[V^{++}] = [\mathrm{Det}^{-1/2}(\mathcal{D}^{++})^2]\,\mathrm{Det}^{1/2}\widehat{\square}_{(4,0)}$, are given in [52]. Here $\widehat{\square}$ is the covariant-analytic d'Alembertian that transforms analytic superfields into analytic ones [52]:

$$\begin{aligned} \widehat{\square} &= -\frac{1}{2}(\mathcal{D}^+)^4(\mathcal{D}^{--})^2 = \frac{1}{2}\mathcal{D}^{\alpha\dot\alpha}\mathcal{D}_{\alpha\dot\alpha} + \frac{i}{2}(\mathcal{D}^{+\alpha}W)\mathcal{D}_\alpha^- \\ &+ \frac{i}{2}(\bar{\mathcal{D}}_{\dot\alpha}^+\bar{W})\bar{\mathcal{D}}^{-\dot\alpha} + \frac{1}{2}\{W, \bar{W}\} \\ &- \frac{i}{4}(\bar{\mathcal{D}}^+\bar{\mathcal{D}}^+\bar{W})\mathcal{D}^{--} + \frac{i}{8}[\mathcal{D}^+, \mathcal{D}^-]W. \end{aligned} \tag{5.18}$$

The final result for the Lagrangian that defines one-loop quantum corrections to the effective action in the vector multiplet sector is

$$S_2 + S_{GF} = -\frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}du\, v^{++}(\widehat{\square} + m^2)v^{++}. \tag{5.19}$$

The ghost action is written as

$$\begin{aligned} S_{ghost} &= \mathrm{tr}\int d\zeta^{(-4)}du\, b(\mathcal{D}^{++})^2c \\ &+ \frac{1}{2}\int d\zeta^{(-4)}du\,\phi(\mathcal{D}^{++})^2\phi + \mathrm{tr}\int d\zeta^{(-4)}du\,\rho^{(+4)}\widehat{\square}_{4,0}\sigma. \end{aligned} \tag{5.20}$$

Here b, c are anticommuting superfield Faddeev–Popov ghosts, $\rho^{(+4)}, \sigma$ are anticommuting superfield Nielsen–Kallosh ghosts, and ϕ are additional commuting ghosts taking values in the Lie algebra of the gauge group.

It is convenient to write the superfield action for quantum field χ and its interaction with quantum field v^{++} in the matrix form

$$\begin{aligned} S_{SYM}^{(2)} + S_m^{(2)} &= -\frac{1}{2}\int d\zeta^{(-4)}(v_a^{++}, \chi_a) \\ &\times \begin{pmatrix} \widehat{\square}_{ab} + m^2 - mR_{ba}D^{++} & mD^{++}R_{ab} \\ mD^{++}R_{ab} & -\nabla^{++}D^{++} \end{pmatrix}\begin{pmatrix} v_b^{++} \\ \chi_b \end{pmatrix}. \end{aligned} \tag{5.21}$$

Here D^{++} is the ordinary harmonic derivative, and

$$\nabla^{++}\chi_a = D^{++}\chi_a + f_{abc}\chi_b R_{cd}\mathcal{V}_d^{++} \tag{5.22}$$

is the "long" derivative in the λ-basis, where $\Omega\mathcal{V}_a^{++}T_a\Omega^{-1}$ plays a role of the gauge-invariant connection. Since this superfield is analytic, standard constraints $[\nabla^{++}, \nabla_{\alpha,\dot\alpha}^+] = 0$ take place. For iniformity, the notation $\nabla_{\alpha,\dot\alpha}^+ = D_{\alpha,\dot\alpha}^+$ is used hereafter. Other commutation relations, such as

$$\begin{gathered} [\nabla^{++}, \nabla^{--}] = D^0, \quad [\nabla^{\mp\mp}, \nabla_{\alpha,\dot\alpha}^{\pm}] = \nabla_{\alpha,\dot\alpha}^{\mp}, \\ \{\bar\nabla_{\dot\alpha}^+, \nabla_\alpha^-\} = -\{\nabla_\alpha^+, \bar\nabla_{\dot\alpha}^-\} = 2i\nabla_{\alpha\dot\alpha}, \\ \{\nabla_\alpha^+, \nabla_\beta^-\} = -2i\epsilon_{\alpha\beta}\bar{\mathcal{W}}, \quad \{\bar\nabla_{\dot\alpha}^+, \bar\nabla_{\dot\beta}^-\} = 2i\epsilon_{\dot\alpha\dot\beta}\mathcal{W}, \end{gathered} \tag{5.23}$$

replicate exactly the commutation relations for covariant derivatives $\mathcal{D}^{\pm}_{\alpha,\dot\alpha}$, $\mathcal{D}^{\pm\pm,0}$ [56] and determine $\nabla^{-}_{\alpha,\dot\alpha}$ and the chiral superfield of harmonically independent strengths $\mathcal{W}[\mathcal{V}^{++}] = -\frac{1}{4}(\nabla^{+})^2\mathcal{V}^{--}$ and $\overline{\mathcal{W}}[\mathcal{V}^{++}] = -\frac{1}{4}(\overline{\nabla}^{+})^2\mathcal{V}^{--}$ for gauge-invariant potential $\Omega\mathcal{V}^{++}\Omega^{-1}$.

5.4. One-Loop Divergences

Let us proceed to the analysis of one-loop divergences in the considered theory. The effective action is the sum of contributions from action (5.20) of quantum superfields of ghosts and action (5.21) of quantum superfields v^{++}, χ:

$$\Gamma^{(1)}[V^{++},\omega] = \Gamma^{(1)}_1[V^{++}] + \Gamma^{(1)}_2[V^{++},\mathcal{V}^{++}], \quad (5.24)$$

where $\Gamma^{(1)}_1[V^{++}]$ is the ghost contribution to the effective action and $\Gamma^{(1)}_2[V^{++},\mathcal{V}^{++}]$ is the contribution of superfields v^{++} and χ. Here $\mathcal{V}^{++}$ is defined by (5.7). Note that the complete dependence of effective action on Stueckelberg superfield ω is contained within superfield $\mathcal{V}^{++}$. Actions (5.20) and (5.21) determine completely the structure of the perturbative expansion for the calculation of one-loop effective action of the massive $\mathcal{N} = 2$ Yang–Mills theory in a manifestly supersymmetric and gauge-invariant form. We are interested only in the structure of divergences of the considered theory and will use dimensional regularization (see [39] for details of application of dimensional regularization in superfield theories) and the minimal subtraction scheme in its analysis.

The ghost contribution to the one-loop effective action depends only on potential V^{++} and coincides exactly with the corresponding contribution in the standard massless $\mathcal{N} = 2$ Yang–Mills theory:

$$\begin{aligned} i\Gamma^{(1)}_1[V^{++}] &= \mathrm{Tr}\ln(\mathcal{D}^{++})^2 \\ &- \frac{1}{2}\mathrm{Tr}\ln(\mathcal{D}^{++})^2 + \frac{1}{2}\mathrm{Tr}\ln\widehat{\Box}_{(4,0)}. \end{aligned} \quad (5.25)$$

Therefore, we can directly use the results presented in [41, 52] (without modifying them in any way) to derive the divergent part of effective action:

$$\Gamma^{(1)}_{1,div}[V^{++}] = -\frac{C_2}{32\pi^2\varepsilon}\mathrm{tr}\int d^8z W^2. \quad (5.26)$$

Here C_2 is the quadratic Casimir operator of the gauge group and ε is the dimensional regularization parameter.

New divergences are associated with one-loop corrections to the effective action that are induced by quantum fields v^{++}, χ within the loop and by their mixing. In order to simplify the calculation of the functional determinant of the matrix operator in action (5.21), one reduces the matrix to a diagonal form:

$$\begin{pmatrix} 1 & mRD^{++}\frac{1}{\nabla^{++}D^{++}} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \widehat{\Box} + m^2 - m^2RD^{++}\frac{1}{\nabla^{++}D^{++}}D^{++}R & 0 \\ 0 & -\nabla^{++}D^{++} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{\nabla^{++}D^{++}}mD^{++}R & 1 \end{pmatrix}. \quad (5.27)$$

All contributions to the effective action are then defined by the diagonal elements of the inner matrix:

$$\begin{pmatrix} \widehat{\Box} + m^2\Pi^T & 0 \\ 0 & -\nabla^{++}D^{++} \end{pmatrix}, \quad (5.28)$$

where Π^T is defined through a comparison with (5.27). It is well known that operator $\widehat{\Box} + m^2\Pi^T$, just as $\widehat{\Box}_{(4,0)}$ does not contribute to the holomorphic part of effective action. All possible contributions to the one-loop counterterm are then attributed to the known ghost contribution (5.26) and contribution

$$\Gamma^{(1)}_2[0,\mathcal{V}^{++}] = \Gamma^{(1)}[\mathcal{V}^{++}] = \frac{i}{2}\mathrm{Tr}\ln(\nabla^{++}D^{++}) \quad (5.29)$$

of quantum superfields χ, coming from the lower block of matrix (5.28). With the aim of applying the known $\Gamma^{(1)}[\mathcal{V}^{++}]$ calculation methods, we reduce the differential operator present in (5.29) to the following form:

$$(\nabla^{++})^2 + U^{(+4)}, \quad (5.30)$$

where

$$U^{(+4)}_{ab} = \frac{1}{2}f_{abc}D^{++}\mathcal{V}^{++}_c + \frac{1}{4}f_{ace}f_{bde}\mathcal{V}^{++}_c\mathcal{V}^{++}_d. \quad (5.31)$$

It is evident that the exact Green's function for operator $(\nabla^{++})^2$ coincides with Eq. (5.12) after the replacement $\mathcal{D}^{++} \to \nabla^{++}$, $W \to \mathcal{W}$. The Green's function for the ω-multiplet in external field $U^{(+4)}$ is determined by the following equation:

$$[(\nabla^{++}_1)^2 + U^{(+4)}_1]G^{(0,0)}_U(1|2) = \delta^{(4,0)}_A(1|2). \quad (5.32)$$

Let us determine the analytic superfield kernel[37] that takes all external field effects into account:

$$Q^{(4,0)}(1|2) = \delta_A^{(4,0)}(1|2) + U_1^{(+4)}G^{(0,0)}(1|2). \quad (5.33)$$

The effective action is then given by

$$\begin{aligned}\Gamma[\mathcal{V}^{++}] &= \frac{i}{2}\mathrm{Tr}\,\ln[(\nabla^{++})^2 + U^{(+4)}] \\ &= \frac{i}{2}\mathrm{Tr}\,\ln(\nabla^{++})^2 + \frac{i}{2}\mathrm{Tr}\,\ln Q^{(4,0)}.\end{aligned} \quad (5.34)$$

The first ($U^{(+4)}$-independent) contribution to $\Gamma[\mathcal{V}^{++}]$ divergences coincides exactly with the one-loop hypermultiplet contribution in external field $\mathcal{V}^{++}$; i.e., it is defined by Eq. (5.26) in which the sign is changed and superfield strength $W[V^{++}]$ is replaced with strength $\mathcal{W}[\mathcal{V}^{++}] = -\frac{1}{4}(\bar{\nabla}^+)^2\mathcal{V}^{--}$ calculated using potential (5.7):

$$_1\Gamma^{(1)}_{div}[\mathcal{V}^{++}] = \frac{C_2}{32\pi^2\varepsilon}\mathrm{tr}\int d^8z\mathcal{W}^2. \quad (5.35)$$

In terms of diagrams, the expansion of the second term in (5.34) in a power series in interactions of fields inside the loop with external insertions $U^{(+4)}$ and the propagator in external field $\mathcal{V}^{++}$ is written as

$$\Gamma[U^{(+4)}] = \sum_{n=1}^{\infty}\Gamma_n[U^{(+4)}], \quad (5.36)$$

where n-th term corresponds to a supergraph with n external $U^{(+4)}$ lines. Thus, (5.34) sums the contributions with an arbitrary number of external lines $\mathcal{V}^{++}$.

The first term in the expansion of $\Gamma[\mathcal{V}^{++}]$ (5.29) in powers of $U^{(+4)}$ equals zero, since it contains the harmonic product that vanishes in the coincidence limit: $u_1^-u_2^-\big|_{u_1=u_2} = 0$.The effective action in the second order is given by

$$\begin{aligned}{}_2\Gamma^{(1)}[\mathcal{V}^{++}] &= -\frac{i}{4}\mathrm{tr}\int d\zeta_1^{(-4)}d\zeta_2^{(-4)}du_1du_2U^{(+4)}(1) \\ &\times U^{(+4)}(2)\frac{1}{\widehat{\Box}_1}(\nabla_1^+)^4(\nabla_2^+)^4\delta^{12}(1|2)\frac{u_1^-u_2^-}{(u_1^+u_2^+)^3} \\ &\times\frac{1}{\widehat{\Box}_2}(\nabla_2^+)^4(\nabla_1^+)^4\delta^{12}(2|1)\frac{u_2^-u_1^-}{(u_2^+u_1^+)^3}.\end{aligned}$$

[37] According to the rule $G^{(0,0)}(1|2) = \int d\zeta_3^{(-4)}G_U^{(0,0)}(1|3)Q^{(4,0)}(3|2)$, where $G^{(0,0)}(1|2)$ is the Green's function in external field $\mathcal{V}^{++}$ satisfying the equation $(\nabla_1^{++})^2G^{(0,0)}(1|2) = \delta^{(4,0)}(1|2)$.

Restoring the full Grassmannian integration measure, we remove one delta function and use the equality $(D_1^+)^4(\bar{D}_1^+)^4\delta^8(\theta-\theta')\big|_{\theta=\theta'} = (u_1^+u_2^+)^4$. Following standard transformations, the divergent contribution takes the form

$$\begin{aligned}{}_2\Gamma^{(1)}_{div}[\mathcal{V}^{++}] &= \frac{1}{(8\pi)^2\varepsilon}\mathrm{tr}\int d^{12}zdu_1du_2 \\ &\times U^{(+4)}(z,u_1)U^{(+4)}(z,u_2)\frac{(u_1^-u_2^-)^2}{(u_1^+u_2^+)^2}.\end{aligned} \quad (5.37)$$

Subsequent terms in the expansion of (5.29) produce finite contributions to the effective action.

Expression (5.37) is the main result obtained in the present section. It is a new superfield counterterm, which depends on background superfield ω, in the $\mathcal{N} = 2$ supersymmetric massive Yang–Mills theory in the Stueckelberg formalism. This functional does not contain harmonic singularities on the mass shell. This follows from the fact that a nonzero contribution to the integral over odd coordinates must contain the maximum power of Grassmannian coordinates in each multiplicand; however, $(\theta_1^+)^2(\bar{\theta}_1^+)^2(\theta_2^+)^2(\bar{\theta}_2^+)^2 = (u_1^+u_2^+)^4(\theta)^4(\bar{\theta})^4$.

In addition to many other terms, functional (5.37) written in its component form contains nonstandard contact four-vector interactions and the terms needed for their supersymmetrization. For instance, these interactions for gauge group $SU(2)$ take on the form $a_m^ia_n^ia_m^ja_n^j + (a_m^ia_m^i)^2$, where $a_m = A_m - L_m$ is the vector component of the $SU(2)$ superfield $\mathcal{V}^{++}$ (5.7). The corresponding interactions for gauge group $SU(3)$ are given by $\frac{5}{6}(a_m^aa_m^a)^2 + a_m^aa_n^aa_m^ba_n^b - d_{abe}d_{ecd}a_m^aa_n^ba_m^ca_n^d$. It is this counterterm that emerges as an obstacle to renormalizability in the conventional nonsupersymmetric massive Yang–Mills theory [174, 175]. Such deviations from the Standard Model have intriguing phenomenological consequences, for example, for the processes of W^+W^-,WZ production (see [180] and references therein). Note that the mass term in the $\mathcal{N} = 2$ supersymmetric massive Yang–Mills field theory, in contrast to the nonsupersymmetric case [175], is not renormalized.

Counterterms (5.26), (5.35), and (5.37)[38] are manifestly gauge-invariant, and the latter two terms do not reproduce the form of the initial Lagrangian. Therefore, as well as the nonsupersymmetric massive Yang–Mills theory, the massive $\mathcal{N} = 2$ Yang–Mills theory

[38] The counterterms in the minimal subtraction scheme differ from divergences (5.26), (5.35), and (5.37) only in their sign.

can be regarded only as a low-energy effective theory. In other words, action (5.6) is not the most general $\mathcal{N} = 2$ supersymmetric functional compatible with the local left and global right gauge symmetries of the theory. In order to make the theory renormalizable (see, for example, [181]), one should include new vertices induced by functionals (5.35) and (5.37) into the next order of the expansion of effective action in derivatives.

5.5. On the Component Structure of $\mathcal{N} = 2$ Superfield Functional (5.35)

The study of the specifics of the component expansion of $\mathcal{N} = 2$ superfield strength $\mathcal{W}$ for gauge-invariant potential $\Omega\mathcal{V}^{++}\Omega^{-1}$, which contains an additional degree of freedom (associated with Stueckelberg superfield ω) in the vector multiplet, is a separate interesting problem. It is evident that this gauge-invariant superfield differs from the covariant chiral superfield strength constructed in terms of potential $\mathcal{V}^{++}$ only in matrix Ω dressing that makes no contribution under the trace.

The $\mathcal{N} = 2$ superfield formalism is the best suited for description of the supermultiplets off the mass shell and their manifestly $\mathcal{N} = 2$ supersymmetric interactions. The transition to component fields requires eliminating an infinite number of auxiliary fields, which is a rather involved technical problem. While the determination of the component structure of counterterm (5.26) presents no problems and its component form coincides with the classical action of $\mathcal{N} = 2$ Yang–Mills theory [56], the component form of expressions (5.35) and (5.37) warrants special investigation, since potential (5.7) is transformed homogeneously, ${}^{g}\mathcal{V}^{++} = e^{i\lambda}\mathcal{V}^{++}e^{-i\lambda}$ (i.e., in contrast to the law of transformation of V^{++}, its transformation law does not contain terms with derivative $D^{++}\lambda$). This is the reason why the Wess–Zumino gauge cannot be chosen for $\mathcal{V}^{++}$, and potential $\mathcal{V}^{++}$ contains nonremovable longitudinal degrees of freedom in the vector field sector. Thus, the problem of finding the component form of superfield functional (5.35) amounts to performing the component expansion of $\mathcal{W}$ with no gauge imposed on $\mathcal{V}^{++}$. No general solution to this problem is found in literature. Certain aspects of the component structure of the massive vector multiplet without gauge fixing were studied in [184] in the Abelian and non-Abelian (in the first order in the coupling constant expansion) cases.

In the present section, we describe a procedure for the determination of the component form of superfield functional (5.35) in the bosonic sector. The component content of superfield $\mathcal{V}^{++}$ needed to write (5.35) and (5.37) in terms of physical fields actually coincides with the component structure of superfield V^{++} in the Wess–Zumino gauge, but each component is endowed with an infinite tower of interactions with the longitudinal degrees of freedom emerging from the component fields of the ω-hypermultiplet.

A convenient way to determine the component content of the superfield strength for nonstandard theories is based on solving the harmonic zero-curvature equation [56] for nonanalytic potential $\mathcal{V}^{--}$:

$$D^{++}\mathcal{V}^{--} - D^{--}\mathcal{V}^{++} + i[\mathcal{V}^{++}, \mathcal{V}^{--}] = 0. \qquad (5.38)$$

Since $\mathcal{V}^{++} = V^{++} - L^{++}$ and superfield V^{++} undergoes the standard gauge transformations, we can impose the Wess–Zumino gauge on V^{++}. As a result, gauge-covariant analytic potential (5.7) takes the form (5.10):

$$\begin{aligned}\mathcal{V}^{++}(\zeta) &= f^{++}(x_A) + (\theta^+)^2\bar{\varphi}(x_A) + (\bar{\theta}^+)^2\varphi(x_A)\\ &+ 2i(\theta^+\sigma^m\bar{\theta}^+)(A_m(x_A) + \partial_m f^{+-}(x_A))\\ &- (\theta^+)^2(\bar{\theta}^+)^2\Box f^{--}(x_A) + \mathit{fermions},\end{aligned} \qquad (5.39)$$

where $\zeta = (x_A^m, \theta^{+\alpha}, \bar{\theta}^{+\dot{\alpha}}, u_i^{\pm})$ and the notation for the component form of the potential in the Wess–Zumino gauge is retained. However, one should bear in mind that each component V_{WZ}^{++} should be endowed with an infinite series in powers of interaction with the components of the ω-hypermultiplet $\omega(\zeta) = \omega(x_A) + \omega^{(ij)}(x_A)u_i^+u_j^- + \ldots$. Since superfield strength

$$\mathcal{W} = -\frac{1}{4}(\bar{D}^+)^2\mathcal{V}^{--} \qquad (5.40)$$

is a harmonic-independent $\mathcal{N} = 2$ chiral superfield, it is convenient to seek a solution of Eq. (5.38) in chirally analytic coordinates $Z_c = (z_c, \bar{\theta}^{\pm\dot{\alpha}})$, where

$$z_c = (x_L^m, \theta^{\pm\alpha}), \quad x_L^m = x_A^m + 2i\theta^-\sigma^m\bar{\theta}^+.$$

Each component of (5.39) is expanded in this basis as

$$\begin{aligned}f(x_A) &= f(x_L) - 2i\theta^-\sigma^m\bar{\theta}^+\partial_m f(x_L)\\ &+ (\theta^-)^2(\bar{\theta}^+)^2\Box f(x_L).\end{aligned}$$

Following [182], it is convenient to present the expansion of $\mathcal{V}^{--}$ in these coordinates as

$$\begin{aligned}\mathcal{V}^{--}(Z_c, u) &= v^{--}(x_L, \theta^{\pm}, u) + \bar{\theta}^+_{\dot{\alpha}}v^{(-3)\dot{\alpha}} + \bar{\theta}^+_{\dot{\alpha}}v^{-\dot{\alpha}}\\ &+ (\bar{\theta}^-)^2\mathcal{A} + (\bar{\theta}^+\bar{\theta}^-)\varphi^{--} + \bar{\theta}^{-\dot{\alpha}}\bar{\theta}^{+\dot{\beta}}\varphi^{--}_{\dot{\alpha}\dot{\beta}} + (\bar{\theta}^+)^2v^{(-4)}\\ &+ (\bar{\theta}^-)^2\bar{\theta}^+_{\dot{\alpha}}\tau^{-\dot{\alpha}} + (\bar{\theta}^+)^2\bar{\theta}^-_{\dot{\alpha}}\tau^{(-3)\dot{\alpha}} + (\bar{\theta}^+)^2(\bar{\theta}^-)^2\tau^{--}.\end{aligned}$$

Note that the expression for (5.40) incorporates not all the chiral superfields of this expansion: just $\mathcal{A}$, $\tau^{-\dot{\alpha}}$, and τ^{--} are included. However, it was demonstrated in [183] that only the chiral superfield

$$\mathcal{A} = \mathcal{A}_1 + (\theta^-)^2\mathcal{A}_4^{++} + (\theta^-\theta^+)\mathcal{A}_5 + \theta^{-\alpha}\theta^{+\beta}\mathcal{A}_{6\alpha\beta} + (\theta^+)^2\mathcal{A}_7^{--} + (\theta^-)^2(\theta^+)^2\mathcal{A}_{10} + fermions$$

actually specifies the component bosonic structure of superfield functional (5.35):

$$S_{bos} = \frac{1}{4}\mathrm{tr}\int d^4x_L du\Big(2\mathcal{A}_1\mathcal{A}_{10} + 2\mathcal{A}_4^{++}\mathcal{A}_7^{--} - \frac{1}{2}\mathcal{A}_5^2 - \frac{1}{4}\mathcal{A}_6^2\Big). \tag{5.41}$$

The equations that determine the components of superfield $\mathcal{A}$ emerge as coefficients in the expansion of superfield equation (5.38) in powers of $\theta^{\pm}$, $\bar{\theta}^{\pm}$:

$$\begin{gathered} d^{++}\mathcal{A}_1 = 0, \quad d^{++}\mathcal{A}_4^{++} = 0, \\ d^{++}\mathcal{A}_5 + 4\mathcal{A}_4^{++} = 0, \quad d^{++}\mathcal{A}_{6\alpha\beta} = 0, \\ d^{++}\mathcal{A}_7^{--} + 2\mathcal{A}_5 + [\bar{\phi}, \mathcal{A}_1] = 0, \\ d^{++}\mathcal{A}_{10} + [\bar{\phi}, \mathcal{A}_4^{++}] = 0, \end{gathered} \tag{5.42}$$

where

$$d^{++} = \partial^{++} + i[f^{++}, \ldots].$$

The general solution of this set of harmonic equations can be written in terms of Green's functions for operator ∂^{++}. For example, the solution of the first equation in chain (5.42) has the form:

$$\begin{aligned} \mathcal{A}_1(u) &= \varphi - i\int du_1 \frac{u^+u_1^-}{u^+u_1^+}[f^{++}(u_1), \mathcal{A}_1(u_1)] \\ &= \sum_{n=0}^{\infty}(-i)^n\int du_1\ldots du_n\frac{u^+u_1^-}{u^+u_1^+}\ldots\frac{u_{n-1}^+u_n^-}{u_{n-1}^+u_n^+} \\ &\times [f^{++}(u_1), [\ldots, [f^{++}(u_n), \varphi]]] = e^{ib}\varphi, \end{aligned} \tag{5.43}$$

where φ is a particular solution of the homogeneous equation, and nonanalytic bridge e^{ib} for field $f^{++} = -ie^{ib}\partial^{++}e^{-ib}$ can be constructed iteratively as a Taylor series in powers of f^{++} (see [56]).

Thus each component of $\mathcal{A}$ is determined by an infinite series in powers of interaction of the standard components of superfield W with field f^{++}:

$$\begin{gathered} \mathcal{A}_4^{++} = e^{ib}\Box f^{++}, \quad \mathcal{A}_7^{--} = 4e^{ib}\Box f^{--}, \\ \mathcal{A}_6^{\alpha\beta} = e^{ib}F^{\alpha\beta}, \quad \mathcal{A}_5 = e^{ib}\Big(-4\Box f^{+-} + \frac{1}{2}[\varphi, \bar{\varphi}]\Big), \\ \mathcal{A}_{10} = e^{ib}\Big(-\Box\bar{\varphi} + \frac{1}{8}[\bar{\varphi}, [\bar{\varphi}, \varphi]]\Big). \end{gathered} \tag{5.44}$$

Here component fields f^{ij}, $\varphi, \bar{\varphi}$, $F^{\alpha\beta}$ are defined by expansion (5.39), and the action of matrix operator e^{ib} is given by (5.43).

Our analysis therefore proves that a formal solution for the components of superfield strength (5.40) in a nonpolynomial form exists, and action (5.41) contains, in addition to the standard action in the Wess–Zumino gauge, the modified interaction with the ω-multiplet components that contains the fourth powers of the space-time derivatives of component fields $\omega, \omega^{(ij)}$. Finding the more detailed component form for expression (5.41) is a very complicated problem, although the above procedure technically allows one to find the complete solution. Note also that divergences (5.26) and (5.35) are cancelled out in the vector multiplet sector (i.e., when the dependence on the ω-multiplet components is omitted).

5.6. Summary

Let us summarize the main results presented in this section.

The $\mathcal{N} = 2$ supersymmetric massive Yang–Mills theory with its action depending on $\mathcal{N} = 2$ gauge superfield V^{++} and hypermultiplet Stueckelberg superfield ω was considered. Various dual-equivalent formulations of this theory, which differ in the way the gauge-invariant mass term is introduced into the superfield action, were proposed.

The background field method that allows one to obtain the loop expansion of effective action in a manifestly gauge-invariant and $\mathcal{N} = 2$ supersymmetric form was developed. It was demonstrated that the contribution of Stueckelberg superfield ω to the effective action can be given in terms of superfield $\mathcal{V}^{++}$ (5.7), which is a special gauge-covariant combination of background superfields V^{++} and ω.

The structure of one-loop divergences in the theory under consideration was investigated. Manifestly gauge-invariant and $\mathcal{N} = 2$ supersymmetric expressions for one-loop divergences (5.26), (5.35), and (5.37) were obtained. It is noteworthy that expression (5.37) is a new gauge-invariant and $\mathcal{N} = 2$ supersymmetric functional constructed from superfields V^{++} and ω. The emergence of this functional in one-loop divergences allows one to draw a conclusion about the (multiplicative) nonrenormalizability of the considered theory. This functional can be regarded as the $\mathcal{N} = 2$ supersymmetrization of the covariant counterterm in the nonsupersymmetric Yang–Mills field theory [174, 175]. However, in contrast to the nonsupersymmetric case, the mass term in the $\mathcal{N} = 2$ supersymmetric massive Yang–Mills theory is not renormalized. Thus, a complete analysis of the $\mathcal{N} = 2$ superfield structure of one-loop divergences in the considered theory was presented. The obtained results were verified in part by passing to the limiting case of no interaction of the vector multiplet with the Stueckelberg multiplet (in other words, for the zero mass limit). Only one divergence, which is stipulated by one-loop ghost contributions (5.26) and defines the known value [41] for the beta-function in the "pure" $\mathcal{N} = 2$ supersymmetric Yang–Mills theory, is left in this limit.

The component structure of a counterterm of form (5.35) in the bosonic sector was discussed. Since the gauge transformation for $\mathcal{V}^{++}$ (5.7) is homogeneous, it is not possible to set the Wess–Zumino gauge, which is often used to reduce the superfield description of the vector multiplet to its component description, for $\mathcal{V}^{++}$. Therefore, the problem of finding the component form of superfield functional (5.35) arises. A procedure to solve this problem in the bosonic sector was proposed (see (5.41) and (5.44)).

Let us discuss briefly the prospects for further investigation of $\mathcal{N} = 2$ supersymmetric massive Yang–Mills theory. As is well known, the $\mathcal{N} = 2$ supersymmetric massless Yang–Mills theory is finite beyond the one-loop approximation (see, for example, [52]). The problem of divergences of the considered massive theory in higher loops remains open. The determination of finite contributions to the one-loop effective action and the study of effective action in the presence of interaction between a massive gauge $\mathcal{N} = 2$ superfield and hypermultiplets are of some interest. In our opinion, the problem of quantum equivalence of the massive $\mathcal{N} = 2$ Yang–Mills theory within the Stueckelberg formalism and $\mathcal{N} = 2$ supersymmetric non-Abelian vector-tensor models, which were demonstrated to be dual at the classical level, is the most intriguing one.

6. CONCLUSIONS

In this paper, we have reviewed the methods and results concerning the structure of low-energy effective action in the four-dimensional quantum theory of gauge fields with $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetries. Our exposition is based on superfield description of the models under consideration. The general methods for constructing the superfield effective actions in various extended supersymmetric theories are formulated. The relation between the problems of effective action in the supersymmetric field theory and those of the low-energy limit in superstring theory is also discussed.

Let us briefly outline the basic research directions considered in the review, and summarize the results presented.

(1) The derivative expansion of the one-loop effective action of $\mathcal{N} = 4$ supersymmetric Yang–Mils theory, containing both the $\mathcal{N} = 2$ vector multiplet fields and the $\mathcal{N} = 2$ hypermultiplet fields, was developed. The formulation of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in terms of $\mathcal{N} = 1$ superfields was analyzed, and the gauge invariant one-loop effective action was obtained in the approximation of constant Abelian gauge field strengths and constant fields of hypermultiplets. The representation of the effective action in the form of expansion over supercovariant derivatives of the $\mathcal{N} = 2$ vector multiplet was found. In particular, the complete $\mathcal{N} = 4$ supersymmetric low-energy effective action, which was constructed earlier in [146], has been obtained in this way, and the next-to-leading corrections to this action were calculated. The self-consistent approach to finding the hypermultiplet-dependent correction terms, as well as the properly deformed hidden supersymmetry transformations which secure $\mathcal{N} = 4$ supersymmetry in the next-to-leading parts of effective action of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory, was developed.

(2) The structure of the hypermultiplet dependence of the low-energy effective action of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory formulated in the $\mathcal{N} = 2$ harmonic superspace was studied. Quantization of the model under consideration has been carried out and the superfield perturbation theory has been developed. An infinite series of covariant harmonic supergraphs with an arbitrary number of external hypermultiplet lines on the nontrivial $\mathcal{N} = 2$ vector multiplet background was summed up, and the general structure of the hypermultiplet dependence of effective action was determined. The method of operator symbols in the $\mathcal{N} = 2$ harmonic superspace was developed and then was used to calculate the one-loop effective action in the theory under consideration. The result was presented as an integral over analytic subspace of the harmonic superspace. It was demonstrated that each term of the expansion of effective action in spinor covariant derivatives can be equivalently represented as an integral over the full $\mathcal{N} = 2$ superspace. This provided a justification for the method used in another section, where the one-loop effective action in the hypermultiplet sector was formulated in terms of $\mathcal{N} = 1$ superfields with special gauge fixing and where it was taken for granted that the manifestly $\mathcal{N} = 2$ supersymmetric form of the effective action exists.

(3) The one-loop low-energy effective action of $\mathcal{N} = 2$ superconformal and UV_finite models formulated in the harmonic superspace was constructed. The effective action depending on the background

Abelian $\mathcal{N} = 2$ vector multiplet superfield and the hypermultiplet background superfield subjected to the special constraints defining the vacuum structure of these models, was studied. The universal expression for the effective action was found in the framework of the $\mathcal{N} = 2$ supersymmetric background field method under the assumption that the hypermultiplet satisfies the mass shell conditions. The special manifestly $\mathcal{N} = 2$ supersymmetric leading contribution, which is written as an integral over 3/4 of the full $\mathcal{N} = 2$ harmonic superspace, was constructed for the off-shell hypermultiplet. In the bosonic sector, this contribution contains terms with three space-time derivatives that are similar to Chern–Simons terms.

(4) The structure of effective action of $\mathcal{N} = 2$ supersymmetric massive Yang–Mills theory constructed with making use of the nonlinear sigma model for the Stueckelberg superfield, was analyzed. The background field method which allows one to obtain the loop expansion of effective action in the manifestly gauge-invariant and $\mathcal{N} = 2$ supersymmetric form, was developed. The structure of one-loop divergences in the theory under consideration was investigated. The component form of the counterterms contains the non-standard contact four-vector interactions together with the accompanying terms required by supersymmetry.

The methods, approaches, and results presented in this review were discussed in more detail in Subsections 2.10, 3.9, 4.6, and 5.6. We announce with deep sorrow that our friend and co-author Nicolay Pletnev passed away suddenly at the final stage of work on proofreading of this paper.

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