

One-loop divergences in $6D$, $\mathcal{N} = (1, 0)$ SYM theory

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ABSTRACT: We consider, in the harmonic superspace approach, the six-dimensional $\mathcal{N} = (1, 0)$ supersymmetric Yang-Mills gauge multiplet minimally coupled to a hypermultiplet in an arbitrary representation of the gauge group. Using the superfield proper-time and background-field techniques, we compute the divergent part of the one-loop effective action depending on both the gauge multiplet and the hypermultiplet. We demonstrate that in the particular case of $\mathcal{N} = (1, 1)$ SYM theory, which corresponds to the hypermultiplet in the adjoint representation, all one-loop divergencies vanish, so that $\mathcal{N} = (1, 1)$ SYM theory is one-loop finite *off shell*.

KEYWORDS: Supersymmetric gauge theory, Extended Supersymmetry, Superspaces

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1 Introduction

The ultraviolet behavior of extended supersymmetric Yang-Mills (SYM) theories in higher dimensions ($D \geq 5$) represents an exciting subject with the long history [1–5]. In this work we focus on the $6D$ SYM theory coupled to hypermultiplets. We formulate the theory in $6D$, $\mathcal{N} = (1, 0)$ harmonic superspace [6–12] and develop the corresponding background superfield method. As the basic topic, we expose the ultraviolet properties of the one-loop effective action for this theory in the general case when the hypermultiplet lies in an arbitrary representation of the gauge group. In the particular case of the adjoint representation, the considered $\mathcal{N} = (1, 0)$ SYM - hypermultiplet system amounts to $6D$, $\mathcal{N} = (1, 1)$ SYM theory formulated in terms of $\mathcal{N} = (1, 0)$ harmonic superfields.

In a recent work [13] we have calculated the divergent part of the one-loop effective action for the abelian $6D$, $\mathcal{N} = (1, 0)$ gauge theory, in which the vector (gauge) multiplet interacts with a hypermultiplet. The basic tools were the background superfield method and proper time technique appropriately adapted to $6D$, $\mathcal{N} = (1, 0)$ harmonic superspace. By explicit calculations we confirmed the general structure of one-loop counterterms which was analyzed earlier in refs. [1, 14] on the pure symmetry grounds. In the present paper we generalize this study to the non-abelian case. We consider the $6D$, $\mathcal{N} = (1, 0)$ model in which the SYM multiplet interacts with the hypermultiplet in an arbitrary representation of gauge group, the adjoint and fundamental representations being particular cases. We extend the background superfield method to this general case of $6D$, $\mathcal{N} = (1, 0)$ SYM theory with the hypermultiplet matter. In many aspects, it is similar to the well-developed background superfield method for $4D$, $\mathcal{N} = 2$ SYM theory with hypermultiplets [15–17]. Using the $6D$, $\mathcal{N} = (1, 0)$ harmonic background superfield method constructed and the proper time technique, we calculate the divergent part of the one-loop effective action in the considered $6D$, $\mathcal{N} = (1, 0)$ model. It should be emphasized that we take into account the full set of contributions depending on both the background gauge multiplet and the hypermultiplet. To the best of our knowledge, the explicit calculation of the hypermultiplet-dependent divergent contributions to effective action of $6D$ SYM theories has never been accomplished earlier, and it is the pivotal point of our consideration.

It is well known that both $6D, \mathcal{N} = (1, 0)$ and $6D, \mathcal{N} = (1, 1)$ SYM theories at one loop are on-shell finite [1, 14]. For the $6D, \mathcal{N} = (1, 0)$ theory without hypermultiplets this result is easily recovered from the quantum calculations. The main result of the present work is the explicit proof of the absence of one-loop logarithmic divergencies in $6D, \mathcal{N} = (1, 1)$ SYM theory *off shell*. We demonstrate this by calculating the divergent part of the one-loop effective action in $\mathcal{N} = (1, 1)$ SYM theory formulated in terms of $\mathcal{N} = (1, 0)$ harmonic gauge and hypermultiplet superfields, both in the adjoint representation of the gauge group [14]. We start with a general $6D, \mathcal{N} = (1, 0)$ SYM - hypermultiplet action and find the one-loop contributions to the divergent part of the effective action. We demonstrate that the numerical factors depending on the gauge group and on the representation of the hypermultiplet vanish in the case when the hypermultiplet is in the adjoint representation of the gauge group. Hence, for the $\mathcal{N} = (1, 1)$ SYM theory we establish the absence of logarithmic divergencies in the one-loop effective action. The similar phenomenon takes place in $\mathcal{N} = 4$ SYM theory in four dimensions formulated in terms of $\mathcal{N} = 2$ superfields (see, e.g., [16, 17]).

It should be pointed out that in a certain sense the *off-shell* absence of the one-loop divergencies in that part of the total $\mathcal{N} = (1, 1)$ SYM effective action which depends only on gauge background superfields is an expected result. It is dictated by the formal structure of this one-loop effective action, in which the contributions from the ghost superfields are canceled by the corresponding contribution from quantum hypermultiplet in the adjoint representation. Once again, this happens in the full analogy with $4D, \mathcal{N} = 4$ case [15]. However, taking the background hypermultiplet parts of the one-loop effective action into account entails a few technical problems. The basic one is that, after making the background-quantum splitting, we encounter the mixed terms involving the quantum gauge superfields along with the hypermultiplet ones. In order to diagonalize the action, we are led to make a non-local shift of hypermultiplet variables [18–20] which induces an additional background hypermultiplet dependence in the one-loop effective action caused by the contributions from the quantum gauge multiplet.

The paper is organized as follows. In section 2 we briefly outline the gauge theory in $6D, \mathcal{N} = (1, 0)$ harmonic superspace and fix our $6D$ notations and conventions. Section 3 presents the harmonic superspace background superfield method for $\mathcal{N} = (1, 0)$ SYM theory. In section 4 we perform the direct calculations of the one-loop divergences in the model under consideration. In section 5 we summarize the results and discuss the problems for further study.

2 Gauge theory in $6D, \mathcal{N} = (1, 0)$ harmonic superspace

Our consideration in this section (including notations, conventions and terminology) will closely follow ref. [14].

The $6D, \mathcal{N} = (1, 0)$ gauge covariant derivatives in the “central basis” are defined by

$$\nabla_{\mathcal{M}} = D_{\mathcal{M}} + i\mathcal{A}_{\mathcal{M}}, \quad (2.1)$$

where $D_{\mathcal{M}} = (D_M, D_a^i)$ are the flat derivatives. Here $M = 0, \dots, 5$, is the $6D$ vector index and $a = 1, \dots, 4$, is the spinorial one. The superfield $\mathcal{A}_{\mathcal{M}}$ is the gauge superconnection. The covariant derivatives transform under the gauge group as

$$\nabla'_{\mathcal{M}} = e^{i\tau} \nabla_{\mathcal{M}} e^{-i\tau}, \quad \tau^+ = \tau. \quad (2.2)$$

The fundamental object of $6D, \mathcal{N} = (1, 0)$ SYM theory is revealed after extending the standard $6D, \mathcal{N} = (1, 0)$ superspace $z := (x^M, \theta_i^a)$ by $SU(2)$ harmonics u_i^\pm , $u^{+i}u_i^- = 1$, and singling out, in this extended harmonic $6D, \mathcal{N} = (1, 0)$ superspace (z, u) , an analytic subspace (ζ, u) containing four independent Grassmann coordinates along with the harmonics u_i^\pm . All geometric quantities of the theory are expressed in terms of the hermitian analytic gauge connection $V^{++}(\zeta, u) = \widetilde{V^{++}(\zeta, u)}$,

$$V^{++} = (V^{++})^A T^A, \quad (T^A)^+ = T^A, \quad (2.3)$$

where the generalized conjugation \sim is defined in [7] and T^A are the generators of the gauge group.

For simplicity, we will consider only simple gauge groups. In our notation the generators of the fundamental representation $T_f^A \equiv t^A$ are normalized by the condition $\text{tr}(t^A t^B) = \frac{1}{2} \delta^{AB}$. For an arbitrary representation R , which can be in general reducible,

$$[T^A, T^B] = if^{ABC} T^C, \quad \text{tr}(T^A T^B) = T(R) \delta^{AB}, \quad (T^A)_m^l (T^A)_l^n = C(R)_m^n. \quad (2.4)$$

If R is irreducible, we obtain:

$$C(R)_m^n = C_2(R) \delta_m^n, \quad C_2(R) = T(R) \frac{d_G}{d_R}, \quad (2.5)$$

where $C_2(R)$ is the second Casimir for the representation R , $d_G \equiv \delta_{AA}$ is the dimension of the gauge group, and $d_R \equiv \delta_m^m$ is the dimension of the irreducible representation R . In the case when R is a reducible representation, $R = \sum_i R_{(i)}$, we have (in the matrix notation)

$$T(R) = \sum_i T(R_{(i)}) d_{R_{(i)}}, \quad C(R) = \sum_i C_2(R_{(i)}) I_{(i)}, \quad d_{R_{(i)}} = \text{tr} I_{(i)}, \quad (2.6)$$

whence

$$T(R_{(i)}) = C_2(R_{(i)}) \frac{d_{R_{(i)}}}{d_G}.$$

For the adjoint representation the generators are written as $(T_{\text{Adj}}^C)_A^B = if^{ACB}$. Consequently,

$$T(\text{Adj}) = C_2, \quad C(\text{Adj})_m^n = C_2 \delta_m^n. \quad (2.7)$$

The connection V^{++} , (2.3), covariantizes the flat analyticity-preserving harmonic derivative D^{++} :

$$\begin{aligned} D^{++} &\Rightarrow \nabla^{++} = D^{++} + iV^{++}, \\ (V^{++})' &= -ie^{i\lambda^A T^A} D^{++} e^{-i\lambda^A T^A} + e^{i\lambda^A T^A} V^{++} e^{-i\lambda^A T^A}, \end{aligned} \quad (2.8)$$

where $\lambda^A(\zeta, u) = \widetilde{\lambda^A(\zeta, u)}$ is the real gauge group parameter in the “ λ -basis”. Another important object is the non-analytic harmonic connection $V^{--} = (V^{--})^A T^A$ covariantizing the flat derivative D^{--}

$$\begin{aligned} D^{--} &\Rightarrow \nabla^{--} = D^{--} + iV^{--}, \\ (V^{--})' &= -ie^{i\lambda^A T^A} D^{--} e^{-i\lambda^A T^A} + e^{i\lambda^A T^A} V^{--} e^{-i\lambda^A T^A}. \end{aligned} \quad (2.9)$$

It is not independent and is related to V^{++} by the harmonic flatness condition

$$[\nabla^{++}, \nabla^{--}] = D^0 \Leftrightarrow D^{++}V^{--} - D^{--}V^{++} + i[V^{++}, V^{--}] = 0, \quad (2.10)$$

where D^0 is the operator counting the harmonic U(1) charges of the involved superfields. The formal solution of (2.10) is

$$V^{--}(z, u) = \sum_{n=1}^{\infty} (-i)^{n+1} \int du_1 \dots du_n \frac{V^{++}(z, u_1) \dots V^{++}(z, u_n)}{(u^+ u_1^+)(u_1^+ u_2^+) \dots (u_n^+ u^+)}. \quad (2.11)$$

Using the zero curvature condition (2.10), one can derive a useful relation between arbitrary variations of harmonic connections [14]

$$\delta V^{--} = \frac{1}{2}(\nabla^{--})^2 \delta V^{++} - \frac{1}{2} \nabla^{++}(\nabla^{--} \delta V^{--}). \quad (2.12)$$

All the geometric quantities of the theory are expressed in terms of V^{--} . The covariant derivatives in the λ -frame can be written as

$$\nabla_a^+ = D_a^+, \quad \nabla_a^- = D_a^- + i\mathcal{A}_a^-, \quad \nabla_{ab} = \partial_{ab} + i\mathcal{A}_{ab}, \quad (2.13)$$

where superfield connections are determined as

$$\mathcal{A}_a^- = iD_a^+ V^{--}, \quad \mathcal{A}_{ab} = \frac{1}{2} D_a^+ D_b^+ V^{--}. \quad (2.14)$$

The covariant derivatives satisfy the algebra

$$\{\nabla_a^+, \nabla_b^-\} = 2i\nabla_{ab}, \quad [\nabla_c^\pm, \nabla_{ab}] = \frac{i}{2} \varepsilon_{abcd} W^{\pm d}, \quad [\nabla_M, \nabla_N] = iF_{MN}, \quad (2.15)$$

where $\nabla_{ab} = \frac{1}{2}(\gamma^M)_{ab} \nabla_M$ and $W^{a\pm}$ is the covariant superfield strength

$$W^{+a} = -\frac{1}{6} \varepsilon^{abcd} D_b^+ D_c^+ D_d^+ V^{--}, \quad W^{-a} = \nabla^{--} W^{+a}. \quad (2.16)$$

We also define the Grassmann-analytic superfield [14]

$$F^{++} \equiv \frac{1}{4} D_a^+ W^{+a} = (D^+)^4 V^{--}, \quad (2.17)$$

such that

$$D_a^+ W^{+b} = \delta_a^b F^{++}, \quad D_a^+ F^{++} = 0, \quad \nabla^{++} F^{++} = 0. \quad (2.18)$$

It will be used for constructing the background field formalism and counterterms in the next sections.

The harmonic covariant derivatives $\nabla^{\pm\pm} = D^{\pm\pm} + iV^{\pm\pm}$ act on the arbitrary analytic superfields \mathcal{F} in an arbitrary representation of the gauge group as

$$(\nabla^{\pm\pm}\mathcal{F})_m = \left(D^{\pm\pm}\delta_m^n + i(V^{\pm\pm})^C(T^C)_m^n\right)\mathcal{F}_n \equiv (\nabla^{\pm\pm})_m^n \mathcal{F}_n. \quad (2.19)$$

If \mathcal{F} belongs to the adjoint representation, then the above equation gives

$$(\nabla^{\pm\pm}\mathcal{F})^A = \left(D^{\pm\pm}\delta^{AB} - f^{ACB}(V^{\pm\pm})^C\right)\mathcal{F}^B \equiv (\nabla^{\pm\pm})^{AB}\mathcal{F}^B. \quad (2.20)$$

The superfield action of $6D$, $\mathcal{N} = (1, 0)$ SYM interacting with a hypermultiplet has the form

$$\begin{aligned} S_0[V^{++}, q^+] &= \frac{1}{f^2} \sum_{n=2}^{\infty} \frac{(-i)^n}{n} \text{tr} \int d^{14}z du_1 \dots du_n \frac{V^{++}(z, u_1) \dots V^{++}(z, u_n)}{(u_1^+ u_2^+) \dots (u_n^+ u_1^+)} \\ &\quad - \int d\zeta^{(-4)} du \tilde{q}^{+m} (\nabla^{++})_m^n q_n^+, \end{aligned} \quad (2.21)$$

where f is a dimensionful coupling constant ($[f] = -1$). In the SYM part of this action $V^{++} = V^{++A}t^A$ with t^A being generators of the fundamental representation, while in the hypermultiplet part of the action $(V^{++})_m^n = V^{++A}(T^A)_m^n$, where T^A are generators of the representation for the hypermultiplet. The action (2.21) is invariant under the gauge transformation (2.8) and

$$(q_m^+)' = (e^{i\lambda^A T^A})_m^n q_n^+. \quad (2.22)$$

Classical equations of motion following from the action (2.21) read

$$\frac{\delta S}{\delta(V^{++})^A} = 0 \Rightarrow \frac{1}{f^2} (F^{++})^A + i\tilde{q}^{+m} (T^A)_m^n q_n^+ = 0, \quad (2.23)$$

$$\frac{\delta S}{\delta \tilde{q}^{+m}} = 0 \Rightarrow (\nabla^{++})_m^n q_n^+ = 0. \quad (2.24)$$

The \sim -reality of eq. (2.23) (as well as of the action (2.21)) is guaranteed by the conjugation rules $\widetilde{\tilde{q}^+} = -q^+$, $\widetilde{F^{++}} = F^{++}$ [7].

3 Background field formalism for $\mathcal{N} = (1, 0)$ SYM theory

In the present paper we generalize the background field method developed in [13] for the abelian case to the non-abelian model (2.21). The construction of gauge invariant effective action in the model under consideration is very similar to that in $4D, \mathcal{N} = 2$ supersymmetric gauge theories [15, 21] (see also the reviews [16, 17]).¹

One splits the superfields V^{++}, q^+ into the sum of the “background” superfields V^{++}, Q^+ and the “quantum” ones v^{++}, q^+ ,

$$V^{++} \rightarrow V^{++} + f v^{++}, \quad q^+ \rightarrow Q^+ + q^+, \quad (3.1)$$

¹The background field method can be also constructed in the ordinary $\mathcal{N} = 2$ superspace [22, 23]. However, this approach encounters a problem of an infinite number of the Faddeev-Popov ghosts.

and then expand the action in a power series in quantum fields. As a result, we obtain the classical action as a functional of background superfields and quantum superfields. The original infinitesimal gauge transformations are realized in two different ways: as the *background* transformations:

$$\delta V^{++} = -\nabla^{++}\lambda, \quad \delta v^{++} = -i[v^{++}, \lambda], \quad (3.2)$$

and as the *quantum* transformations²

$$\delta V^{++} = 0, \quad \delta v^{++} = -\nabla^{++}\lambda - i[v^{++}, \lambda]. \quad (3.3)$$

To construct the gauge invariant effective action, we need to impose the gauge-fixing conditions only on quantum superfields. We introduce the gauge-fixing function in the full analogy with 4D case [15–17]

$$\mathcal{F}_\tau^{(+4)} = D^{++}v_\tau^{++} = e^{-ib}(\nabla^{++}v^{++})e^{ib} = e^{-ib}\mathcal{F}^{(+4)}e^{ib}, \quad (3.4)$$

where $b(z)$ is a background-dependent gauge bridge superfield and τ means τ -frame (see, e.g., [7]). We consider the non-abelian gauge theory, where the gauge-fixing function (3.4) is background-dependent. The gauge-fixing function transforms according to the law

$$\delta\mathcal{F}_\tau^{(+4)} = -e^{-ib}\{\nabla^{++}(\nabla^{++}\lambda + i[v^{++}, \lambda])\}e^{ib} \quad (3.5)$$

under the quantum transformations (3.3). Eq. (3.5) leads to the Faddeev-Popov determinant

$$\Delta_{FP}[v^{++}, V^{++}] = \text{Det}(\nabla^{++}(\nabla^{++} + iv^{++})).$$

Following the standard procedure, we can obtain a path-integral representation for $\Delta_{FP}[v^{++}, V^{++}]$ by introducing two real analytic fermionic ghosts \mathbf{b} and \mathbf{c} , both in the adjoint representation of the gauge group. The corresponding ghost action is

$$S_{FP}[\mathbf{b}, \mathbf{c}, v^{++}, V^{++}] = \text{tr} \int d\zeta^{(-4)} du \, \mathbf{b} \nabla^{++}(\nabla^{++}\mathbf{c} + i[v^{++}, \mathbf{c}]). \quad (3.6)$$

As a result, we arrive at the effective action $\Gamma[V^{++}, Q^+]$ in the form

$$e^{i\Gamma[V^{++}, Q^+]} = \int \mathcal{D}v^{++} \mathcal{D}q^+ \mathcal{D}\mathbf{b} \mathcal{D}\mathbf{c} \, \delta[\mathcal{F}^{(+4)} - f^{(+4)}] e^{i\{S_0[V^{++}+fv^{++}, Q^++q^+]+S_{FP}[\mathbf{b}, \mathbf{c}, v^{++}, V^{++}]\}}, \quad (3.7)$$

where $f^{(+4)}(\zeta, u)$ is an external Lie-algebra valued analytic superfield which is independent of V^{++} , and $\delta[\mathcal{F}^{(+4)} - f^{(+4)}]$ is the functional analytic delta-function. As the next step, we average the right-hand side in eq. (3.7) with the weight

$$\Delta[V^{++}] \exp \left\{ \frac{i}{2} \text{tr} \int d^{14}z du_1 du_2 f_\tau^{(+4)}(z, u_1) \frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3} f_\tau^{(+4)}(z, u_2) \right\}. \quad (3.8)$$

²We denote the parameters of these transformations by the same letter, hoping that this will not lead to confusion.

Following the Faddeev-Popov method, the functional $\Delta[V^{++}]$ is determined from the equation

$$1 = \Delta[V^{++}] \int \mathcal{D}f^{(+4)} \exp \left\{ \frac{i}{2} \text{tr} \int d^{14}z du_1 du_2 f_{\tau}^{(+4)}(z, u_1) \frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3} f_{\tau}^{(+4)}(z, u_2) \right\}. \quad (3.9)$$

Passing in this expression to the analytic subspace, we obtain

$$\begin{aligned} \Delta^{-1}[V^{++}] &= \int \mathcal{D}f^{(+4)} \exp \left\{ \frac{i}{2} \text{tr} \int d\zeta_1^{(-4)} d\zeta_2^{(-4)} du_1 du_2 f^{(+4)}(\zeta_1, u_1) A(1, 2) f^{(+4)}(\zeta_2, u_2) \right\} \\ &= \text{Det}^{-1/2} A. \end{aligned} \quad (3.10)$$

Here, like in 4D case [15–17], we have introduced the special background-dependent operator A , which arose when we passed from (3.9) to (3.10). This operator depends on the background field through a background-dependent bridge $b(z)$ and has the form

$$A(1, 2) = \frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3} (D_1^+)^4 (D_2^+)^4 \left[(e^{ib_1} e^{-ib_2})_{\text{Adj}} \delta^{14}(z_1 - z_2) \right], \quad (3.11)$$

where

$$(e^{ib_1} e^{-ib_2})_{\text{Adj}} f^{(+4)}(\zeta_2, u_2) = e^{ib_1} e^{-ib_2} f^{(+4)}(\zeta_2, u_2) e^{ib_2} e^{-ib_1}. \quad (3.12)$$

We note that operator $A(1, 2)$ acts in the space of analytic superfields, which take values in the Lie algebra of the gauge group. Thus, we have derived the following formal expression for the functional $\Delta[V^{++}]$

$$\Delta[V^{++}] = \text{Det}^{1/2} A. \quad (3.13)$$

To calculate the functional determinant for the operator A , we do not need the explicit form for it. We represent the determinant for this operator through a functional integral over analytic superfields,

$$\text{Det}^{-1} A = \int \mathcal{D}\chi^{(+4)} \mathcal{D}\rho^{(+4)} \exp \left\{ i \text{tr} \int d\zeta_1^{(-4)} du_1 d\zeta_2^{(-4)} du_2 \chi^{(+4)}(1) A(1, 2) \rho^{(+4)}(2) \right\}, \quad (3.14)$$

and, as in 4D case, make use of the following substitution of the functional variables

$$\rho^{(+4)} = (\nabla^{++})^2 \sigma, \quad \text{Det} \left(\frac{\delta \rho^{(+4)}}{\delta \sigma} \right) = \text{Det}(\nabla^{++})^2. \quad (3.15)$$

Then we find (see a similar calculation in [15–17])

$$\text{Det}^{-1} A = \text{Det}(\nabla^{++})^2 \int \mathcal{D}\chi^{(+4)} \mathcal{D}\sigma \exp \left\{ i \text{tr} \int d\zeta^{(-4)} du \chi^{(+4)} \widehat{\square}_{\lambda} \sigma \right\}. \quad (3.16)$$

Here, the operator $\widehat{\square}_{\lambda}$ is the covariant d'Alembertian. Hereafter we use the formal definition for this covariant d'Alembertian $\widehat{\square}_{\lambda}$ in λ -frame

$$\widehat{\square}_{\lambda} = \frac{1}{2} (D^+)^4 (\nabla^{--})^2. \quad (3.17)$$

It is possible to present this operator as a sum of two terms,

$$\widehat{\square}_\lambda = \widehat{\square} + X, \quad (3.18)$$

where

$$\widehat{\square} = \eta^{MN} \nabla_M \nabla_N + W^{+a} \nabla_a^- + F^{++} \nabla^{--} - \frac{1}{2} (\nabla^{--} F^{++}), \quad (3.19)$$

$$\begin{aligned} X = & \left(W^{-a} - W^{+a} \nabla^{--} + 2i \nabla^{ab} \nabla_b^- \right) D_a^+ + \left(i \nabla^{ab} \nabla^{--} - \frac{1}{4} \varepsilon^{abcd} \nabla_c^- \nabla_d^- \right) D_a^+ D_b^+ \\ & - \nabla^{--} \nabla_d^- (D^+)^{3d} + \frac{1}{2} (\nabla^{--})^2 (D^+)^4. \end{aligned} \quad (3.20)$$

In this equation we use the notation

$$(D^+)^{3d} \equiv -\frac{1}{6} \varepsilon^{dabc} D_a^+ D_b^+ D_c^+; \quad \nabla^{ab} \equiv \frac{1}{2} \varepsilon^{abcd} \nabla_{cd}. \quad (3.21)$$

The presentation (3.18) is convenient, because the operator X gives vanishing contribution acting on the analytic superfields. Therefore, when acting on the analytic superfields, the operator $\widehat{\square}_\lambda$ is reduced to the operator $\widehat{\square}$.

In every case we should determine the space of superfields on which the operator (3.17) acts, namely, the harmonic $U(1)$ charge of superfield and the representation of gauge group to which it belongs. Using eqs. (3.13)–(3.16), one obtains

$$\Delta[V^{++}] = \text{Det}^{-1/2} (\nabla^{++})^2 \text{Det}^{1/2} \widehat{\square}. \quad (3.22)$$

Finally, we can represent the functional determinant $\Delta[V^{++}]$ as the functional integral over bosonic real analytic superfield φ taking values in the Lie algebra of the gauge group,

$$\Delta[V^{++}] = \text{Det}^{1/2} \widehat{\square} \int \mathcal{D}\varphi \exp \{ i S_{NK}[\varphi, V^{++}] \}, \quad (3.23)$$

$$S_{NK} = \frac{1}{2} \text{tr} \int d\zeta^{(-4)} du \varphi (\nabla^{++})^2 \varphi. \quad (3.24)$$

Like in $4D$ case, φ is the Nielsen-Kallosh ghost. As a result, we see that the $6D$, $\mathcal{N} = (1, 0)$ SYM theory, in the close analogy with $4D$, $\mathcal{N} = 2$ SYM, in the background field approach is described by the three ghosts: two fermionic ghosts \mathbf{b} and \mathbf{c} together with the single bosonic ghost φ .

According to (3.4), the gauge-fixing part of the quantum field action has the form

$$S_{GF}[v^{++}, V^{++}] = -\frac{1}{2} \text{tr} \int d^{14}z du_1 du_2 \frac{v_\tau^{++}(1) v_\tau^{++}(2)}{(u_1^+ u_2^+)^2} + \frac{1}{4} \text{tr} \int d^{14}z du v_\tau^{++} (D^{--})^2 v_\tau^{++}. \quad (3.25)$$

The action (3.25) depends on the background field V^{++} through the background gauge bridge b , $v_\tau^{++} = e^{-ib} v^{++} e^{ib}$.

Summarizing, one can write the final expression for the effective action (3.7) as follows

$$e^{i\Gamma[V^{++}, Q^+]} = \text{Det}^{1/2} \widehat{\square} \int \mathcal{D}v^{++} \mathcal{D}q^+ \mathcal{D}\mathbf{b} \mathcal{D}\mathbf{c} \mathcal{D}\varphi e^{iS_{quant}[v^{++}, q^+, \mathbf{b}, \mathbf{c}, \varphi, V^{++}, Q^+]}. \quad (3.26)$$

Here, the quantum action S_{quant} has the structure

$$S_{quant} = S_0[V^{++} + f v^{++}, Q^+ + q^+] + S_{GF}[v^{++}, V^{++}] \\ + S_{FP}[\mathbf{b}, \mathbf{c}, v^{++}, V^{++}] + S_{NK}[\varphi, V^{++}]. \quad (3.27)$$

In the one-loop approximation, the first quantum correction to the classical action, $\Gamma^{(1)}[V^{++}, Q^+]$, is given by the following path integral [15, 21]:

$$e^{i\Gamma^{(1)}[V^{++}, Q^+]} = \text{Det}^{1/2} \widehat{\square} \int \mathcal{D}v^{++} \mathcal{D}q^+ \mathcal{D}\mathbf{b} \mathcal{D}\mathbf{c} \mathcal{D}\varphi \, e^{iS_2[v^{++}, q^+, \mathbf{b}, \mathbf{c}, \varphi, V^{++}, Q^+]}. \quad (3.28)$$

In this expression, the full quadratic action S_2 is the sum of three terms. These are the classical action (2.21) in which the background-quantum splitting was performed, the gauge-fixing action (3.25) and the actions for the ghost superfields (3.6) and (3.23):

$$S_2 = \frac{1}{2} \int d\zeta^{(-4)} du \, v^{++A} \widehat{\square}^{AB} v^{++B} + \int d\zeta^{(-4)} du \, \mathbf{b}^A (\nabla^{++})^2 AB \mathbf{c}^B \\ + \frac{1}{2} \int d\zeta^{(-4)} du \, \varphi^A (\nabla^{++})^2 AB \varphi^B - \int d\zeta^{(-4)} du \, \tilde{q}^{+m} (\nabla^{++})_m{}^n q_n^+ \\ - \int d\zeta^{(-4)} du \, \left\{ \tilde{Q}^{+m} i f(v^{++})^C (T^C)_m{}^n q_n^+ + \tilde{q}^{+m} i f(v^{++})^C (T^C)_m{}^n Q_n^+ \right\}. \quad (3.29)$$

Hereafter, we write all the group indices explicitly. The operator $\widehat{\square}$ (3.17) transforms the analytic superfields v^{++} into analytic superfields and, according to (2.20), has the following structure

$$\widehat{\square}^{AB} = \frac{1}{2} (D^+)^4 \left\{ (D^{--})^2 \delta^{AB} - 2 f^{ACB} (V^{--})^C D^{--} - f^{ACB} (D^{--} V^{--})^C \right. \\ \left. + f^{ACE} f^{EDB} (V^{--})^C (V^{--})^D \right\}. \quad (3.30)$$

The Green function, associated with (3.30), i.e. $G_{(2,2)}^{AB}(z_1, u_1 | z_2, u_2) = i \langle 0 | T(v_1^{++})^A (v_2^{++})^B | 0 \rangle$, is given by the expression which is similar to that of the $4D, \mathcal{N} = 2$ case [7]

$$G_{\tau(2,2)}^{AB}(z_1, u_1 | z_2, u_2) = - \left(\widehat{\square}_1^{-1} \right)^{AB} (\nabla_1^+)^4 \delta^{14}(z_1 - z_2) \delta^{(-2,2)}(u_1, u_2). \quad (3.31)$$

The action S_2 (3.29) contains terms with a mixture of quantum superfields v^{++} and q^+ . For further use, we diagonalize this quadratic form by means of the special substitution of the quantum hypermultiplet variables³ in the path integral (3.28), such that it removes the mixed terms,

$$q_n^+(1) = h_n^+(1) - f \int d\zeta_2^{(-4)} du_2 \, G_{(1,1)}(1|2)_n{}^p i v^{++C}(2) (T^C)_p{}^l Q_l^+(2), \quad (3.32)$$

with h_n^+ being a set of new independent quantum superfields. It is evident that the Jacobian of the variable change (3.32) is unity. Here $G_{\tau(1,1)}(\zeta_1, u_1 | \zeta_2, u_2)_m{}^n =$

³A similar substitution was used in [18, 21] and [19] for computing one- and two-loop effective actions in supersymmetric theories, and in [20] for non-local redefinition of fields in non-supersymmetric QED.

$i\langle 0 | T q_m^+(\zeta_1, u_1) \tilde{q}^{+n}(\zeta_2, u_2) | 0 \rangle$ is the superfield hypermultiplet Green function in the τ -frame. This Green function is analytic with respect to both its arguments and it satisfies the equation

$$(\nabla_1^{++})_m{}^p G_{\lambda(1,1)}(1|2)_p{}^n = \delta_m^n \delta_A^{(3,1)}(1|2). \quad (3.33)$$

In τ -frame the Green function can be written in the form

$$G_{\tau(1,1)}(1|2)_m{}^n = \left(\widehat{\square}_1^{-1} \right)_m{}^n (\nabla_1^+)^4 (\nabla_2^+)^4 \frac{\delta^{14}(z_1 - z_2)}{(u_1^+ u_2^+)^3}. \quad (3.34)$$

Here $\delta_A^{(3,1)}(1|2)$ is the covariantly-analytic delta-function and $(\widehat{\square})_m{}^n$ is the covariantly-analytic d'Alembertian (3.17) [25] which acts on analytic superfields q_m^+ , in accordance with (2.19), as follows

$$\begin{aligned} \widehat{\square}_m{}^n = \frac{1}{2} (D^+)^4 \Big\{ & (D^{--})^2 \delta_m^n + 2i (V^{--})^C (T^C)_m{}^n D^{--} + i (D^{--} V^{--})^C (T^C)_m{}^n \\ & - (V^{--})^C (V^{--})^D (T^C T^D)_m{}^n \Big\}. \end{aligned} \quad (3.35)$$

Note that the covariant d'Alembertian transforms the analytic superfields into analytic superfields.

After performing the shift (3.32), the quadratic part of the action S_2 (3.29) splits into few terms, each being bilinear in quantum superfields:

$$\begin{aligned} S_2 = & \frac{1}{2} \int d\zeta_1^{(-4)} d\zeta_2^{(-4)} v_1^{++A} \left\{ \widehat{\square}^{AB} \delta_A^{(3,1)}(1|2) - 2f^2 \tilde{Q}_1^{+m} (T^A G_{(1,1)} T^B)_m{}^n Q_{n2}^+ \right\} v_2^{++B} \\ & + \int d\zeta^{(-4)} du \mathbf{b}^A (\nabla^{++})^2{}^{AB} \mathbf{c}^B + \frac{1}{2} \int d\zeta^{(-4)} du \varphi^A (\nabla^{++})^2{}^{AB} \varphi^B \\ & - \int d\zeta^{(-4)} du \tilde{h}^{+m} (\nabla^{++})_m{}^n h_n^+. \end{aligned} \quad (3.36)$$

Starting from the action (3.36) one can construct the one-loop quantum correction $\Gamma^{(1)}[V^{++}, Q^+]$ to the classical action (2.21), which has the following formal expression

$$\begin{aligned} \Gamma[V^{++}, Q] = & \frac{i}{2} \text{Tr} \ln \left\{ \widehat{\square}^{AB} - 2f^2 \tilde{Q}^{+m} (T^A G_{(1,1)} T^B)_m{}^n Q_n^+ \right\} - \frac{i}{2} \text{Tr} \ln \widehat{\square} \\ & - i \text{Tr} \ln (\nabla^{++})_{\text{Adj}}^2 + \frac{i}{2} \text{Tr} \ln (\nabla^{++})_{\text{Adj}}^2 + i \text{Tr} \ln \nabla_{\text{R}}^{++}, \end{aligned} \quad (3.37)$$

where subscripts Adj and R mean that the corresponding operators are taken in the adjoint representation and that of the hypermultiplet.

The expression (3.37) is the starting point for studying the one-loop effective action in the model (2.21). In the next section we will calculate the divergent part of (3.37). The whole dependence on the background hypermultiplet is contained in the first term of the first line of eq. (3.37).

We also note that the possible structure of the one-loop divergences in the model under consideration was discussed in [14] and [13].

4 Divergent part of the one-loop effective action

The $(F^{++})^2$ part of the effective action depends only on the background vector multiplet V^{++} and is defined by the last three terms in eq. (3.37). More precisely,

$$\begin{aligned}\Gamma_{F^2}^{(1)}[V^{++}] &= -i\text{Tr} \ln(\nabla^{++})_{\text{Adj}}^2 + \frac{i}{2}\text{Tr} \ln(\nabla^{++})_{\text{Adj}}^2 + i\text{Tr} \ln \nabla_{\text{R}}^{++} \\ &= -i\text{Tr} \ln \nabla_{\text{Adj}}^{++} + i\text{Tr} \ln \nabla_{\text{R}}^{++}.\end{aligned}\quad (4.1)$$

Let us vary the expression (4.1) with respect to the background gauge multiplet $(V^{++})^A$, keeping in mind the explicit expressions for the covariant harmonic derivatives (2.20) and (2.19),

$$\delta\Gamma_{F^2}^{(1)}[V^{++}] = i\text{Tr} f^{ACB} \delta(V^{++})^C G_{(1,1)}^{BA} - \text{Tr} (T^C)_m{}^n \delta(V^{++})^C (G_{(1,1)})_n{}^m. \quad (4.2)$$

Here $(G_{(1,1)})_n{}^m$ is the superfield Green function (3.34) for operator $(\nabla^{++})_n{}^m$ (2.19) acting on the superfields in the representation R of gauge group to which the hypermultiplet belongs. Also we denoted $G_{(1,1)}^{BA}$ the Green function for the operator $(\nabla^{++})^{BA}$ (2.20), which acts on superfields in adjoint representation. The Green function $G_{(1,1)}^{BA}$ has the structure similar to (3.34), but it is constructed in terms of the covariant d'Alembertian (3.30), (3.18)–(3.20).

The calculation of (4.1) was discussed in details in recent works [13, 24, 26]. It is similar for abelian and non-abelian cases. Our aim is to calculate the divergent part of the effective action (4.1). In the proper-time regularization scheme [25, 26], the divergences are associated with the pole terms of the form $\frac{1}{\varepsilon}$, $\varepsilon \rightarrow 0$, with $d = 6 - \varepsilon$. Taking into account the expression for the Green functions (3.34), we obtain

$$\begin{aligned}\delta\Gamma_{F^2}^{(1)}[V^{++}] &= i \int d\zeta_1^{(-4)} du_1 \delta(V_1^{++})^C \left\{ f^{ACB} G_{(1,1)}^{BA}(1|2) + i(T^C)_m{}^n G_{(1,1)}(1|2)_n{}^m \right\} \Big|_{\text{div}}^{2=1} \\ &= -i \int d\zeta_1^{(-4)} du_1 \delta(V_1^{++})^C \int_0^\infty d(is) (is\mu^2)^{\frac{\varepsilon}{2}} \\ &\quad \times \left\{ f^{ACB} (e^{is\widehat{\square}_1})^{BA} + i(T^C)_m{}^n (e^{is\widehat{\square}_1})_n{}^m \right\} (\nabla_1^+)^4 (\nabla_2^+)^4 \frac{\delta^{14}(z_1 - z_2)}{(u_1^+ u_2^+)^3} \Big|_{\text{div}}^{2=1}.\end{aligned}\quad (4.3)$$

Here s is the proper-time parameter and μ is an arbitrary regularization parameter of mass dimension. Like in the four- and five-dimensional cases [27–31], one makes use of the identity

$$(\nabla_1^+)^4 (\nabla_2^+)^4 \frac{\delta^{14}(z_1 - z_2)}{(u_1^+ u_2^+)^3} = (\nabla_1^+)^4 \left\{ (u_1^+ u_2^+) (\nabla_1^-)^4 - (u_1^- u_2^+) \Omega_1^{--} + \widehat{\square}_1 \frac{(u_1^- u_2^+)^2}{(u_1^+ u_2^+)} \right\} \delta^{14}(z_1 - z_2), \quad (4.4)$$

where the operator $\widehat{\square}$ is given by eq. (3.19), and we have introduced the notation

$$\Omega^{--} = i\nabla^{ab} \nabla_a^- \nabla_b^- - W^{-a} \nabla_a^- + \frac{1}{4} (\nabla_a^- W^{-a}). \quad (4.5)$$

To find a part of eq. (4.3) corresponding to the first term in eq. (4.4), we use the identity

$$e^{is\widehat{\square}_1} (u_1^+ u_2^+) = e^{is\widehat{\square}_1} (u_1^+ u_2^+) e^{-is\widehat{\square}_1} e^{is\widehat{\square}_1} \quad (4.6)$$

and the well-known equation

$$e^A B e^{-A} = B + \frac{1}{1!}[A, B] + \frac{1}{2!}[A, [A, B]] + \dots \quad (4.7)$$

This gives the following terms which are relevant for calculating the divergent part of the effective action:

$$\begin{aligned} e^{is\widehat{\square}_1}(u_1^+ u_2^+) e^{-is\widehat{\square}_1} \Big|_{\text{div}}^{2=1} = & -\frac{(is)^2}{2} \left(\nabla^M \nabla_M F^{++} + F^{++} (\nabla^{--} F^{++}) - \frac{1}{2} [\nabla^{--} F^{++}, F^{++}] \right. \\ & \left. + W^{+a} (\nabla_a^- F^{++}) \right) \\ & - \frac{2(is)^3}{3} \nabla^M \nabla^N F^{++} \partial_M \partial_N + \dots, \end{aligned} \quad (4.8)$$

where dots denote terms which do not contribute to the one-loop divergences. Adding the relevant terms coming from the expansion of the last factor in eq. (4.6) we obtain

$$\begin{aligned} e^{is\widehat{\square}_1}(u_1^+ u_2^+) \Big|_{\text{div}}^{2=1} = & -\frac{(is)^2}{2} \left(\nabla^M \nabla_M F^{++} - \frac{1}{2} [\nabla^{--} F^{++}, F^{++}] + W^{+a} (\nabla_a^- F^{++}) \right) \\ & - \frac{2(is)^3}{3} \nabla^M \nabla^N F^{++} \partial_M \partial_N + \dots \end{aligned} \quad (4.9)$$

In calculating a divergent part of eq. (4.3) corresponding to the second term of eq. (4.4) we can commute the exponent with $(u_1^- u_2^+)$. After this, it is necessary to expand $\exp(is\widehat{\square})$ in a series and keep only terms containing $(D^+)^4 (D^-)^4$. Then calculating the divergent part of the effective action according to the standard technique, after some (rather non-trivial) transformations we obtain the result proportional to

$$\nabla^M \nabla_M F^{++} + \{W^{+a}, \nabla_a^- F^{++}\} - \frac{3}{2} [\nabla^{--} F^{++}, F^{++}] = \widehat{\square} F^{++}. \quad (4.10)$$

The fact that the operator $\widehat{\square}$ appears in the final expression is a non-trivial test of the calculation. Actually, the final expression has the form

$$\delta\Gamma_{F^2}^{(1)}[V^{++}] = \frac{(C_2 - T(R))}{3(4\pi)^3 \varepsilon} \int d\zeta^{(-4)} du \delta V^{++A} \widehat{\square} F^{++A}. \quad (4.11)$$

This implies that following the same procedure as in our previous work [13], it is possible to find the action the variation of which coincides with (4.11). Up to an unessential additive constant,

$$\Gamma_{F^2}^{(1)} = \frac{C_2 - T(R)}{6(4\pi)^3 \varepsilon} \int d\zeta^{(-4)} du (F^{++A})^2 = \frac{C_2 - T(R)}{3(4\pi)^3 \varepsilon} \text{tr} \int d\zeta^{(-4)} du (F^{++})^2, \quad (4.12)$$

where in the last equation $F^{++} = F^{++A} t^A$, with t^A being the generators of the fundamental representation.

The hypermultiplet-dependent part $\widetilde{Q}^+ F^{++} Q^+$ of the one-loop counterterm comes out from the first term in (3.37). To calculate this contribution, one expands the logarithm in

the first term (3.37) up to the first order and computes the functional trace,

$$\begin{aligned} \frac{i}{2} \text{Tr} \ln \left\{ \widehat{\square}^{AB} - 2f^2 \widetilde{Q}^{+m} (T^A G_{(1,1)} T^B)_m{}^n Q_n^+ \right\} &= \frac{i}{2} \text{Tr} \ln \widehat{\square} \\ &+ \frac{i}{2} \text{Tr} \ln \left\{ \delta^{AB} - 2f^2 \left(\widehat{\square}^{-1} \right)^{AC} \widetilde{Q}^{+m} (T^C G_{(1,1)} T^B)_m{}^n Q_n^+ \right\}. \end{aligned} \quad (4.13)$$

We note that, like in $4D$, $\mathcal{N} = 2$ SYM theory, the term $\frac{i}{2} \text{Tr} \ln \widehat{\square}$ does not contribute to the divergent part.⁴ To see this, let us expose some details of the structure of Green function for vector multiplet (3.31). In the limit of coincident points we need to collect eight spinorial derivatives on delta-function $\sim (D^+)^4 (D^-)^4 \delta^8(\theta - \theta')$ in order to obtain a non-vanishing contribution. However, the Green function (3.31) manifestly contains only four derivatives $(D^+)^4$, while the other four spinor derivatives could be taken from the expansion of the inverse operator $\widehat{\square}$ in (3.31) up to the fourth order in D^- . However, from this expansion we will simultaneously gain the fourth power of the inverse flat d'Alembertian. Thus, we will be left with the operator $\sim \frac{(D^-)^4}{\square^4}$ which can contribute only to the finite part of effective action and so is of no interest for our consideration.

Now, let us consider the second term in (4.13). Following [13], we decompose the logarithm up to the first order and compute the functional trace

$$\begin{aligned} \Gamma_{QFQ}^{(1)} &= -if^2 \int d\zeta^{(-4)} du \widetilde{Q}^{+m} Q_n^+ \left(\widehat{\square}^{-1} \right)^{AB} (T^B G_{(1,1)} T^A)_m{}^n \Big|_{\text{div}}^{2=1} \\ &= -if^2 \int d\zeta^{(-4)} du \widetilde{Q}^{+m} Q_n^+ \\ &\quad \times \left(\widehat{\square}^{-1} \right)^{AB} \left(T^B \widehat{\square}^{-1} T^A \right)_m{}^n (u_1^+ u_2^+) \delta^6(x_1 - x_2) \Big|_{2=1}. \end{aligned} \quad (4.14)$$

Here we made use of the explicit expression for the Green function $(G_{(1,1)})_m{}^n$ (3.34) and once again applied the identity (4.4) for extracting the divergent contribution to effective action. Then we decompose the inverse covariant d'Alembertians (3.30) and (3.35) up to the second order and obtain

$$\begin{aligned} \Gamma_{QFQ}^{(1)} &= -if^2 \int d\zeta^{(-4)} du \widetilde{Q}^{+m} Q_n^+ \left(\frac{\delta^{AB}}{\square_1} + 2f^{ACB} (F^{++})^C \frac{D_1^{--}}{\square_1^2} \right) \\ &\quad \times (T^B)_m{}^p \left(\frac{\delta_p^l}{\square_1} - 2i(F^{++})^C (T^C)_p{}^l \frac{D_1^{--}}{\square_1^2} \right) (T^A)_l{}^n (u_1^+ u_2^+) \delta^6(x_1 - x_2) \Big|_{2=1} \\ &= 2if^2 \int d\zeta^{(-4)} du \widetilde{Q}^{+m} Q_n^+ (F^{++})^C \\ &\quad \times \left\{ f^{ACB} (T^B T^A)_m{}^n - i(T^A T^C T^A)_m{}^n \right\} \frac{1}{\square_1^3} \delta^6(x_1 - x_2) \Big|_{2=1}. \end{aligned} \quad (4.15)$$

Let us rewrite the expression within the brackets in the last line of eq. (4.15), using the commutation relation

$$T^C T^A = T^A T^C + if^{CAD} T^D. \quad (4.16)$$

⁴A similar analysis can be done for the contribution $\text{Tr} \ln \widehat{\square}$ in (3.37).

Then we obtain for this expression

$$f^{ACB}T^BT^A - iT^AT^CT^A = 2f^{ACB}T^BT^A - iT^AT^AT^C. \quad (4.17)$$

Finally, we use eq. (2.4) and the identity

$$f^{ACB}T^BT^A = \frac{i}{2}f^{ACB}f^{BAD}T^D = \frac{i}{2}C_2T^C, \quad (4.18)$$

as well as the momentum representation of the space-time δ -function, and calculate the momentum integral in the ε -regularization scheme. This leads to

$$\frac{1}{\square^3}\delta^6(x_1 - x_2)\Big|_{2=1} = \frac{i}{(4\pi)^3}\frac{1}{\varepsilon}, \quad \varepsilon \rightarrow 0. \quad (4.19)$$

The result is

$$\Gamma_{QFQ}^{(1)}[V^{++}, Q^+] = -\frac{2if^2}{(4\pi)^3\varepsilon} \int d\zeta^{(-4)} du \tilde{Q}^{+m} (C_2\delta_m^l - C(R)_m^l) (F^{++})^A (T^A)_l^n Q_n^+. \quad (4.20)$$

Summing up the contributions (4.12) and (4.20), we finally obtain the total divergent contribution

$$\begin{aligned} \Gamma_{div}^{(1)}[V^{++}, Q^+] &= \frac{C_2 - T(R)}{3(4\pi)^3\varepsilon} \text{tr} \int d\zeta^{(-4)} du (F^{++})^2 \\ &\quad - \frac{2if^2}{(4\pi)^3\varepsilon} \int d\zeta^{(-4)} du \tilde{Q}^+ (C_2 - C(R)) F^{++} Q^+. \end{aligned} \quad (4.21)$$

We observe that the coefficients of the $(F^{++})^2$ and $\tilde{Q}^+ F^{++} Q^+$ terms in the divergent part of one-loop effective action are proportional to the differences between the second order Casimir operator for adjoint representation of gauge group and the operators $T(R)$ and $C(R)$ for the hypermultiplet representation R , respectively. Since $6D$, $\mathcal{N} = (1, 1)$ supersymmetric Yang-Mills theory involves only the hypermultiplet in the adjoint representation of gauge group, (4.21) vanishes for this case. Hence, the $6D$, $\mathcal{N} = (1, 1)$ SYM theory is one-loop finite, and there is no need to use the equations of motion (2.23), (2.24) to prove this property.

In general, for any other choice of the irreducible representation R , the expression (4.21) does not vanish even with taking into account the equations (2.23), (2.24), i.e. we meet the same situation as in the abelian case considered in [13], the theory is divergent already at the one-loop level.⁵ The case of pure $6D$, $\mathcal{N} = (1, 0)$ SYM theory corresponds to the evident choice $T(R) = 0$ and $C(R) = 0$ in (4.21), and the one-loop divergent part is vanishing *on shell*, where $F^{++} = 0$, in agreement with the old result of ref. [1].

⁵In principle, when the hypermultiplet is in some *reducible* representation of gauge group, we can pick up this representation in such a way that the coefficients before the corresponding divergent parts vanish. Such a theory will be also off-shell finite at one loop.

5 Summary and outlook

In the present paper we explicitly calculated the divergent part of the one-loop effective action in $6D$, $\mathcal{N} = (1, 0)$ SYM gauge theory coupled to the hypermultiplet in an arbitrary representation of the gauge group. The theory was formulated in the $6D$, $\mathcal{N} = (1, 0)$ harmonic superspace, which preserves the manifest $6D$, $\mathcal{N} = (1, 0)$ supersymmetry and provides a reliable ground for conducting the quantum field analysis.

We developed the background field quantization of the model under consideration. Although the $\mathcal{N} = (1, 0)$ SYM theories are in general anomalous⁶ (see, e.g., the papers [32, 33] and references therein), the one-loop divergences can be calculated in the manifestly gauge invariant and $\mathcal{N} = (1, 0)$ supersymmetric way. Anomalies are obtained by considering finite contributions and do not affect the one-loop divergences considered in this paper. Namely, we found one-loop divergences of the effective action both in the gauge multiplet sector and in the hypermultiplet sector for an arbitrary gauge group and an arbitrary hypermultiplet representation. The structure of the divergences in the gauge multiplet sector (with all hypermultiplet contributions being suppressed) completely matches with the results of the analysis in refs. [1, 2]. In particular, the divergences in this sector can be eliminated by a field redefinition. This implies that the theory is *on-shell* finite in the gauge multiplet sector. However, when the hypermultiplet sector is taken into account, the situation is drastically changed. The divergences cannot be eliminated by a field redefinition and the theory is divergent even *on-shell*.

However, there is a subclass of the general theory, which deserves a special consideration. It is the $\mathcal{N} = (1, 1)$ SYM theory which includes the interacting $\mathcal{N} = (1, 0)$ gauge multiplet and the $\mathcal{N} = (1, 0)$ hypermultiplet, both being in the same adjoint representation. The structure of the coefficients in various terms of the divergent part of the one-loop effective action (4.21) allows us to assert that the one-loop quantum effective action of the $\mathcal{N} = (1, 1)$ SYM theory does not contain the logarithmic divergences at all, even *off-shell*. Such a result is entirely unexpected.

We would like to emphasize that $6D$, $\mathcal{N} = (1, 1)$ SYM theory is in many aspects analogous to $4D$, $\mathcal{N} = 4$ SYM theory. The $4D$, $\mathcal{N} = 4$ SYM theory is formulated in $\mathcal{N} = 2$ harmonic superspace, the $6D$, $\mathcal{N} = (1, 1)$ SYM theory is formulated in $\mathcal{N} = (1, 0)$ harmonic superspace, both theories include vector multiplet and hypermultiplet in the same adjoint representation, both theories are described by the same set of harmonics. Both theories are off-shell finite at one loop. But $4D$, $\mathcal{N} = 4$ SYM theory is a completely finite field model. Taking into account these analogies and the results of this paper, we are led to assume that $6D$, $\mathcal{N} = (1, 1)$ SYM theory can be off-shell finite at higher loops as well.⁷ The first crucial test for such a conjecture would be the study of the structure of the two-loop divergences in $6D$, $\mathcal{N} = (1, 1)$ SYM theory. In the forthcoming paper, we plan to carry out an explicit calculation of the divergent part of the effective action of this theory in the two-loop approximation.

⁶The main object of our investigation, the $\mathcal{N} = (1, 1)$ SYM theory, is free from anomalies.

⁷It is possible that there are new non-renormalization theorems in $6D$, $\mathcal{N} = (1, 1)$ SYM theory (see the discussion of the non-renormalization theorem in $4D$, $\mathcal{N} = 2$ SYM theories in harmonic superspace approach in [34, 35]).

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