Net Baryon Number Probability Distribution and the Quark Chemical Potential

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03.03.2023

Outline

1. Hadron Resinance Gas (HRG) model
   - average abundances of particular resonances
   - partial probabilities of a particular net baryon charge
2. μ_B as an observable in grand canonical formalism
3. Partial probabilities on a lattice
4. Canonical versus grand canonical
5. A comment on B, Q, S

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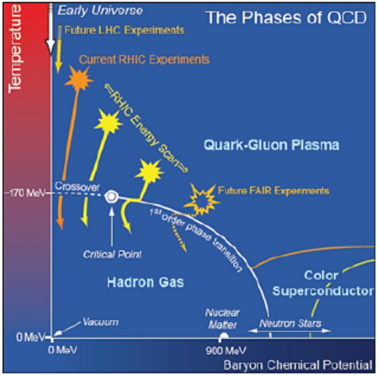
Hypothetical QCD phase diagram



Fireball evolution at various collision energies:

$$T_{ini}, \mu_B^{(ini)} \longrightarrow T_F, \mu_B^{(F)} \;,$$

ini =initial, F =freezeout.

How can one measure these parameters?

Strictly speaking,

$$\text{evolution:} \qquad T_{ini}, \mu_B^{(ini)}, \mu_Q^{(ini)}, \mu_S^{(ini)} \longrightarrow T_F, \mu_B^{(F)}, \mu_Q^{(F)}, \mu_S^{(F)}$$

Statistical hadronization approach

Hadron Resonance Gas model: thermal production of noninteracting stable particles and resonances with subsequent decay of the resonsnces

Particle-number density of ith resonance

$$n_i(T,\vec{\mu}) = \frac{V m_i^2 g_i}{2\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \lambda_i^n K_2\left(\frac{n\, m_i}{T}\right)$$

where

$$\lambda_i = \exp\left(\frac{\mu_B B_i + \mu_Q Q_i + \mu_S S_i}{T}\right)$$

Measurements by ALICE, RHIC

Statistical hadronization approach (HRG) gives:

- Chemical potential is measured by fitting
  average values of particle and nuclei abundances
  by the prediction of the HRG
- In 2017 ALICE attained exellent agreement
  with data @$\sqrt{s_{NN}} = 2.76$ GeV
  over the range of abundances from 10^{-9} to 1 when

$$T_F = 156.5(1.5)\mathrm{MeV}, \mu_B = 0.7(3.8)\mathrm{MeV}, V = 5280(410)\mathrm{fm}^3$$

- Poisson distribution of each particle (as well as antiparticle) species
- Skellam distribution of each net-charge density

HRG prediction for higher-order cumulants

$$\langle B^r \rangle = \sum_{n=0}^{\infty} \mathcal{S}(n) n^r$$

was compared with

- experimental data
- results of lattice simulations

Let $\mathcal{P}(n)$ be the probabiliy that
the net baryon charge of the fireball equals n.

The HRG model predicts:

$$\mathcal{P}(n) = \mathcal{S}(n) = \left(\frac{b}{\bar{b}}\right)^{n/2} I_n(2\sqrt{b\bar{b}}) \exp(-(b+\bar{b}))$$

The probabilities $\mathcal{P}_n$

should be evaluated on the lattice and measured in experiments

Grand canonical partition function

$$Z_{GC}(\theta, T, V) = \sum_j \langle j| \exp\left(\frac{-\hat{H} + \mu \hat{B}}{T} \right) |j\rangle$$

can be expanded in Laurent series in fugacity $\xi = e^{\theta}$ ($\theta = \mu/T = \theta_R + \imath\theta_I$):

$$Z_{GC}(\theta, T, V) \; = \; \sum_{n=-\infty}^{\infty} Z_C(n, T, V)\xi^n,$$

The inverse traqnsform has the form

$$Z_C\,(n, T, V) = \int_{-\pi}^{\pi} \frac{d\theta_I}{2\pi} e^{-in\theta_I} Z_{GC}(\theta, T, V)\Big|_{\theta_R=0} \; .$$

Phenomenological issues

$\mathcal{P}_n = \frac{Z_C(n,T,V)\xi^n}{Z_{GC}(\theta,T,V)}$ - is the probability that

the baryon charge at the given T and μ_B equals n.

C-parity conservation implies $Z_C(n,T,V) = Z_C(-n,T,V)$

$$\Longrightarrow \qquad \frac{\mathcal{P}_n}{\mathcal{P}_{-n}} = \xi^{2n} \qquad \Longrightarrow \qquad \mu_B = \frac{T}{2n} \ln\left(\frac{\mathcal{P}_n}{\mathcal{P}_{-n}}\right)$$

This formula gives:

- possible procedure of measurement of μ_B [A.Nakamura, K.Nagato 2013]
- criterion of thermodynamical equilibrium: μ_B measured for different n coincide

$$\begin{aligned} Z_{GC}(\theta, T, V) &= \sum_j \langle j | \exp\left(\frac{-\hat{H} + \mu \hat{B}}{T} \right) | j \rangle \\ &= \int \mathbf{D} U e^{-S_G} (\det \mathcal{D}(\mu_q))^{N_f} \end{aligned}$$

$$B(\theta) = \frac{1}{N_c} \frac{\partial \ \ln Z_{GC}}{\partial \theta} \qquad P(\theta, T) = -\frac{1}{V} \ln Z_{GC} \qquad \theta = \frac{\mu_q}{T}$$

Thus we find the net baryon number density B and $\Longrightarrow$ the grand canonical partition function

$$B(\theta) = \frac{1}{N_c} \frac{\partial (T \ln Z_{GC})}{\partial \mu_q} \Longrightarrow Z_{GC}(\theta_I)|_{\theta_R = 0} = \exp\left(N_c \int_0^{\theta_I} B(x)\, dx \right)$$

as well as the canonical partition functions

$$Z_C\,(n, T, V) = \int_0^{2\pi} \frac{d\theta_I}{2\pi} e^{-in\theta_I} Z_{GC}(\theta_I, T, V)\ ,$$

High-precision computations are needed

Instead of $Z_C(n,T,V)$ it is convenient to use $Z_n = \frac{Z_C(n,T,V)}{Z_C(0,T,V)}$, related to the partial probabilities

$$\mathcal{P}_n = \frac{Z_C(n,T,V)e^{n\theta}}{Z_{GC}(\theta,T,V)} = \frac{Z_n e^{n\theta}}{1+2\sum_{k=1}^{\infty} Z_k \cosh k\theta} \qquad (Z_n = Z_{-n})$$

$B(\theta_I)$ can be fitted by

$$B(\theta_I)\Big|_{\theta_R=0} \simeq \imath \sum_{n=1}^{N_{param}} a_n f_n(\theta)$$

Problem: given a_n, find Z_n

Roberge-Weiss approach:

Fock space includes only colorless states at all temperatures and chemical potentials.

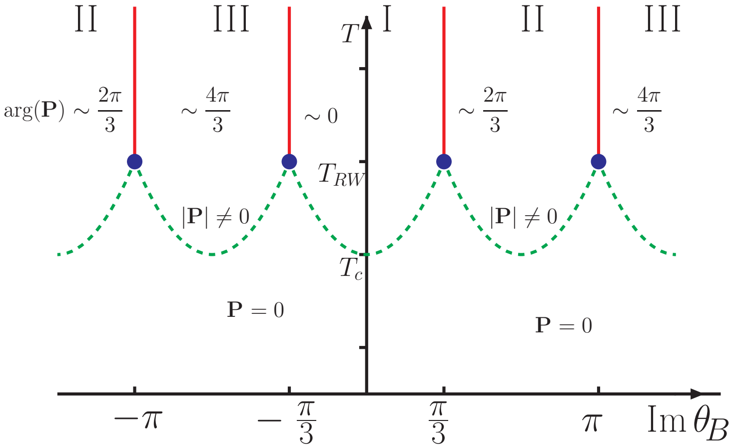
$$Z_{GC}(\theta_I) = Z_{GC}(\theta_I + 2\pi/N_c)$$

$\Downarrow$ $\Downarrow$

Quark number $\mathcal{Q}$ is a multiple of N_c

$$Z_C\,(n, T, V) = \int_0^{2\pi/N_c} \frac{d\theta_I}{2\pi} e^{-in\theta_I} Z_{GC}(\theta_I, T, V)\ ,$$

$\theta_B = N_c\theta_q = N_c\theta;\qquad \xi = e^{\theta N_c}$



II
III
I
II
III
T
$\arg(\mathbf{P}) \sim \frac{2\pi}{3}$
$\sim \frac{4\pi}{3}$
~ 0
$\sim \frac{2\pi}{3}$
$\sim \frac{4\pi}{3}$
T_{RW}
$|\mathbf{P}| \neq 0$
$|\mathbf{P}| \neq 0$
T_c
$\mathbf{P} = 0$
$\mathbf{P} = 0$
$-\pi$
$-\frac{\pi}{3}$
$\frac{\pi}{3}$
π
$\mathrm{Im}\,\theta_B$

Results of lattice simulations

$T > T_{RW}$:

$\mathrm{Im}B(\theta_I)$ is $\dfrac{2\pi}{3}$-periodic function
with discontinuities at $\theta_I = \frac{(2n+1)\pi}{3}$),
at $|\theta_I| < \dfrac{\pi}{3}$ is well fitted by the polynomial

$$\mathrm{Im}B(\theta_I) \simeq a_1\theta_I - a_3\theta_I^3$$

$T < T_c$:

$\mathrm{Im}B(\theta_I)$ is well fitted by the sine

$$\mathrm{Im}B(\theta_I) \simeq a\sin(3\theta_I)$$

$T > T_{RW}$:

$$Z_n \simeq \exp\left(-\frac{n^2}{2a_1 VT^3}\right), \qquad \text{when} \quad n \ll VT^3$$

$$Z_n \simeq \exp\left(-\frac{3}{4}\sqrt[3]{\frac{3}{a_1}}\left(\frac{n}{VT^3}\right)^{4/3}\right), \qquad \text{when} \quad n \gg VT^3$$

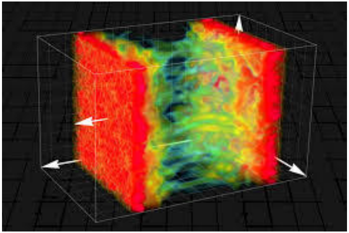
$T < T_c$

$Z_n \simeq e^{-a} I_n(a)$ [Bornyakov *et al.*, 1611.04229] $\Longrightarrow$ $a = 2\sqrt{b\bar{b}}$

$T_c < T < T_{RW}$ yet to be studied

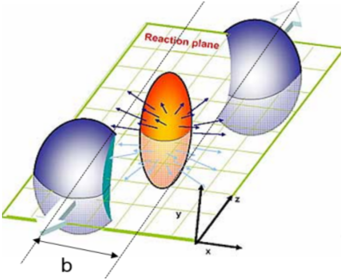
of phenomenological interest $\quad \dfrac{\mathcal{P}_{n-k}\mathcal{P}_{n+k}}{\mathcal{P}_n^2} = \dfrac{Z_{n-k}Z_{n+k}}{Z_n^2}$

Two scenarios of thermalization



1. Exchange of conserved charges (B, Q, S) proceeds during the fireball expansion.

Grand canonical approach works down to $T_{freezeout}$



2. The fireball after formation at an early stage is isolated from the remnants of colliding nuclei.

Evolution starts with the $Z_{GC}(T_i, V_i, \vec{\mu}_{ini})$ and proceeds with $Z_C(T, V, B_{ini}, Q_{ini}, S_{ini})$.

Which temperature should be used: T_{ini} or $T_{\text{freezeout}}$

when using the formula

$$\mu_B = \frac{T}{2n} \ln\left(\frac{\mathcal{P}_n}{\mathcal{P}_{-n}}\right) \ ?$$

- $T = T_{freezeout}$ if scenario 1) takes place
- $T = T_{ini}$ in the case of scenario 2)

Discussed in the literature:
"corrections due to conservation of the baryon number"

Let $\hat{H}, \hat{Q}_1, ..., \hat{Q}_s$ is a complete set of commuting observables. Then

$$Z_{GC}(\mu_1, ..., \mu_s, T, V) = \sum_n \left\langle n \right| \exp\left(\frac{-1}{T} \left(\hat{H} - \mu_1 \hat{Q}_1 - ... - \mu_s \hat{Q}_s \right) \right) \left| n \right\rangle$$

$\mu_1, \; ..., \; \mu_s$ are the respective chemical potentials.
The grand canonical partition function can be expanded in a series in fugacities $\xi_k = \exp\left(\frac{\mu_k}{T}\right)$:

$$Z_{GC} = \sum_{n_1, ..., n_s} Z_C(n_1, ..., n_s; T, V) \; \xi_1^{n_1} ... \; \xi_s^{n_s}$$

Example:

$$\hat{H}_{QCD} \to \hat{H} - \vec{\mu}\overrightarrow{Q} = \hat{H}_{QCD} - \mu_u \hat{N}_u - \mu_d \hat{N}_d - \mu_s \hat{N}_s$$

$$\vec{\mu}\vec{Q} = \mu_u \hat{N}_u + \mu_d \hat{N}_d + \mu_s \hat{N}_s = \mu_B \hat{B} + \mu_Q \hat{Q} + \mu_S \hat{S}$$

$$\begin{cases} \hat{B} = & \frac{1}{3}\hat{N}_u + \frac{1}{3}\hat{N}_d + \frac{1}{3}\hat{N}_s \\ \hat{Q} = & \frac{2}{3}\hat{N}_u - \frac{1}{3}\hat{N}_d - \frac{1}{3}\hat{N}_s \\ \hat{S} = & -\hat{N}_s \end{cases} \Longrightarrow \begin{cases} \mu_B = & \mu_u + 2\mu_d \\ \mu_Q = & \mu_u - \mu_d \\ \mu_S = & \mu_d - \mu_s \end{cases}$$

$$\vec{\mu}\vec{Q} = \mu_B \hat{B} + \mu_{I_3} \hat{I}_3 + \mu_Y \hat{Y}$$

$$\begin{cases} \hat{B} = & \frac{1}{3}\hat{N}_u + \frac{1}{3}\hat{N}_d + \frac{1}{3}\hat{N}_s \\ \hat{I}_3 = & \frac{1}{2}\hat{N}_u - \frac{1}{2}\hat{N}_d \\ \hat{Y} = & \frac{1}{3}\hat{N}_u + \frac{1}{3}\hat{N}_d - \frac{2}{3}\hat{N}_s \end{cases} \Longrightarrow \begin{cases} \mu_B = & \mu_u + \mu_d + \mu_s \\ \mu_{I_3} = & \mu_u - \mu_d \\ \mu_Y = & \frac{1}{2}\mu_u + \frac{1}{2}\mu_d - \mu_s \end{cases}$$

$$\mu_Y \sim \ln\left(\frac{\langle\pi^+\rangle}{\langle\pi^-\rangle}\ \frac{\langle K^-\rangle^2}{\langle K^+\rangle^2}\right)$$

An outline for future work

Two of the following charges

$$\hat{I}_3 = \hat{N}_u - \hat{N}_d, \qquad \hat{U} = \hat{N}_d - \hat{N}_s, \qquad \hat{V} = \hat{N}_u - \hat{N}_s$$

can be used in addition to $\hat{B}$. When

$$\begin{aligned} \mu_{I_3} \neq 0, &\quad \mu_s = \mu_u + \mu_d = 0 \quad \text{OR} \\ \mu_U \neq 0, &\quad \mu_u = \mu_s + \mu_d = 0 \quad \text{OR} \\ \mu_V \neq 0, &\quad \mu_d = \mu_u + \mu_s = 0 \end{aligned}$$

lattice simulations are possible both at real and imaginary part of the respective chemical potential.

Conclusions:

- Partial probabilities $\mathcal{P}_n$ of particular values of the net baryon number are evaluated on a lattice at $T > T_{RW}$ and at $T < T_c$
- At $T < T_c$ the results agree well with the HRG predictions
- The complete set of $\mathcal{P}_n$ involves much greater information than the average values of particular (anti)particles or cumulants of conserved charges
- A similar study should be performed for isospin and hypercharge.