# Few-nucleon systems in the Bethe-Salpeter approach

S. Bondarenko
*BLTP JINR, Dubna*

Dedicated to memory of Professor Valery Burov

## Why a relativistic approach?

- Elastic electron-deuteron scattering experiments
  "Large Momentum Transfer Measurements of the Deuteron Elastic Structure Function A($Q^2$) at Jefferson Laboratory"
  JLab Hall A Collaboration, Phys.Rev.Lett.82:1374-1378,1999
  $Q^2$=0.7-6.0 $(\text{GeV/c})^2$

  Lorentz transformation: $\eta_{LOR} = -Q^2/4M_d^2 \sim 0.43$, $\sqrt{1+\eta_{LOR}} \sim 1.19$
- Exclusive disintegration of the deuteron experiments
  JLab Hall C Deuteron Electro-Disintegration at Very High Missing Momenta (E12-10-003) proposal
  https://www.jlab.org/exp_prog/proposals/10/PR12-10-003.pdf:
  "We propose to measure the D(e,e'p)n cross section at $Q^2 = 4.25$ $(\text{GeV/c})^2$ and xbj = 1.35 for missing momenta ranging from pm = 0.5 GeV/c to pm = 1.0 GeV/c expanding the range of missing momenta explored in the Hall A experiment (E01-020)"

  Lorentz transformation: $\eta_{LOR} = -Q^2/4s_{np} \sim 0.30$, $\sqrt{1+\eta_{LOR}} \sim 1.14$

## Bethe-Salpeter equation for the nucleon-nucleon $T$ matrix

$$T(p',p;P) = V(p',p;P) + \frac{i}{4\pi^3}\int d^4k\, V(p',k;P)\, S_2(k;P)\, T(k,p;P)$$

$p'$, $p$ - the relative four-momenta
$P$ - the total four-momentum

$V(p',p;P)$ - the interaction kernel

$S_2^{-1}(k;P) = \left(\frac{1}{2}\, P\cdot\gamma + k\cdot\gamma - m\right)^{(1)}\left(\frac{1}{2}\, P\cdot\gamma - k\cdot\gamma - m\right)^{(2)}$
free two-particle Green function

## Separable kernels of the NN interaction

The separable kernels of the nucleon-nucleon interaction are widely used in the calculations. The separable kernel as a *nonlocal* covariant interaction representing complex nature of the space-time continuum.

Separable ansatz for the kernel

$$V(p';p;s) = \sum_{m,n=1}^{N} \lambda_{mn}(s) g_m(p') g_n(p)$$

Solution for the $T$ matrix

$$T(p';p;s) = \sum_{m,n=1}^{N} \tau_{mn}(s)\, g_m(p')\, g_n(p)$$

where

$$\Big[\tau_{mn}(s)\Big]^{-1} = \Big[\lambda_{mn}(s)\Big]^{-1} + h_{mn}(s),$$

$$h_{mn}(s) = -\frac{i}{(2\pi)^4} \int d^4k\, g_m(k) g_n(k) S_2(k;P)$$

$g$ - the model functions, $\lambda(s)$ - a matrix of model parameters.

## What is a separable kernel?

The integral equations in the nuclear physics (Lippmann-Schwinger, Bethe-Salpeter) are similar to the Fredholm (first or second) type of equations.
The separable kernel of the integral equation is the degenerated kernel.
Fredholm integral equation of the second type:

$$\phi(x) = f(x) + \lambda \int dy\ K(x, y)\phi(y)$$

Degenerated kernel of the equation:

$$K(x, y) = \sum_i a_i(x) b_i(y)$$

Solution of the equation:

$$\phi(x) = f(x) + \lambda \sum_i c_i a_i(x)$$

Constants $c_i$ can be found by solving the system of linear equations

$$c_i - \lambda \sum_j k_{ij} c_j = f_i$$

Matrix $k_{ij}$ and $f_i$ are:

$$k_{ij} = \int dy\ b_i(y) a_j(y), \qquad f_i = \int dy\ f(y) b_i(y)$$

## Separable NN kernels for BS equation

- NN scattering with spinor nucleon propagators

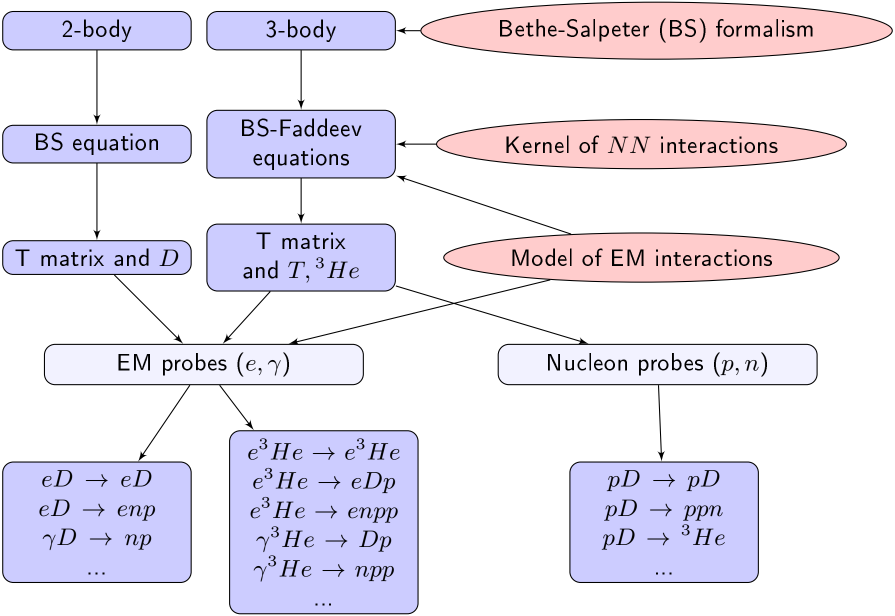
  *G. Rupp and J. A. Tjon* "Relativistic contributions to the deuteron electromagnetic form factors" Phys. Rev. C41. 472 (1990)
  Nonrelativistic Graz-II $\rightarrow$ relativistic Graz-II (only ${}^3S_1 -^3 D_1$ partial-wave states)

- NN scattering with scalar nucleon propagators

  *K. Schwarz, J. Haidenbauer, J. Frohlich* "A Separable Approximation of the NN Paris Potential in the Framework of the Bethe-Salpeter Equation" Phys.Rev. C33 456-466 (1986)
  partial-wave states with $J = 0, 1$ for Paris meson-exchange potentials

  *G. Rupp, J.A. Tjon* "Bethe-Salpeter calculation of three-nucleon with multirank observables separable interactions" Phys.Rev. C45 2133 (1991)
  ${}^1S_0$ and ${}^3S_1 -^3 D_1$ partial-wave states for Paris and Bonn meson-exchange potentials

## Reactions in the BS approach

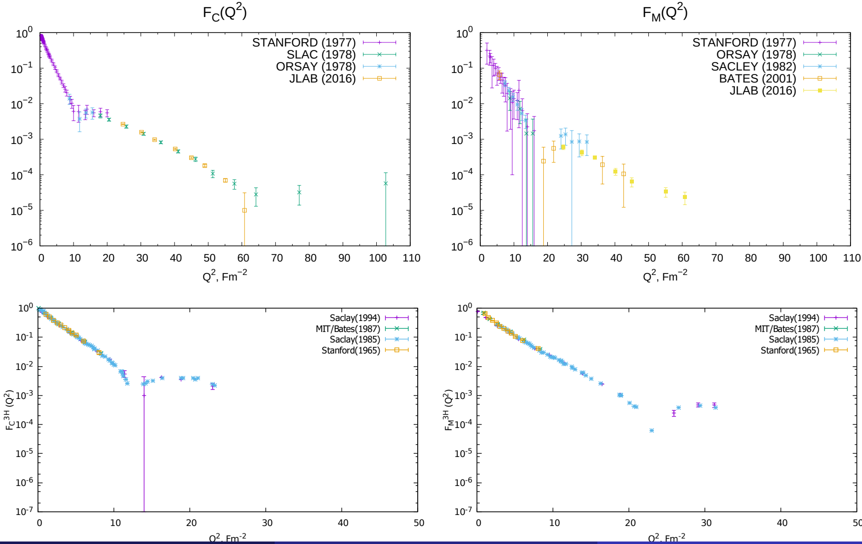


**Bethe-Salpeter approach with separable kernel:**

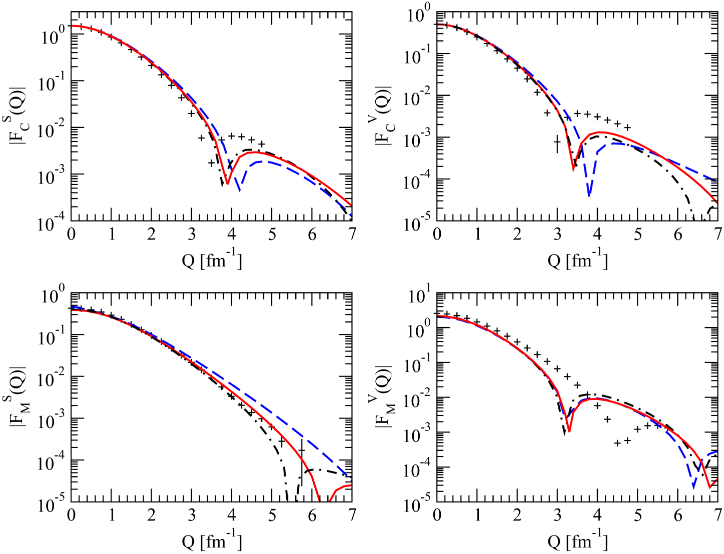
- $NN$-scattering, dispersion relations for separable $T$ matrix: S.Bondarenko, V.Burov, S.Dorkin
- elastic $eD$-scattering: S.Bondarenko, V.Burov, S.Dorkin, A.Bekzhanov, M.Beyer, H.Toki, A.Hosaka, N.Hamamoto, Y.Manabe
- deep-inelastic scattering: V.Burov, A.Molochkov, G.Smirnov
- deuteron photodisintegration at threshold: S.Bondarenko, V.Burov, K.Kazakov, D.Shulga
- partial-wave analisys of $NN$-scattering: S.Bondarenko, V.Burov, E.Rogochaya, P.Hwang
- exclusive deuteron electrodisintegration: S.Bondarenko, V.Burov, E.Rogochaya
- three-nucleon systems, elastic election-3N scattering: S.Bondarenko, V.Burov, S.Yurev

# Elastic $e +^3 He(^3H) \to e +^3 He(^3H)$ scattering

## Experimental data for $^3He$ (first line) and $^3H$ (second line)



## Relativistic covariant spectator theory (F. Gross, M. Pena *et al.*)



## The relativistic three-particle equation for $T$ matrix

is considered in the Faddeev form with the following assumptions:

- no three-particles interaction $V_{123} = \sum_{i \neq j} V_{ij}$
- two-particles interaction is separable
- nucleon propagators are chosen in a scalar form
- the only strong interactions are considered (not EM), so ${}^3He \equiv T$

## Bethe-Salpeter-Faddeev equation

$$\begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} - \begin{bmatrix} 0 & T_1G_1 & T_1G_1 \\ T_2G_2 & 0 & T_2G_2 \\ T_3G_3 & T_3G_3 & 0 \end{bmatrix} \begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix},$$

Three-particles $T$ matrix $T = \sum_i T^{(i)}$

Two-particle Green functions $G_i$ is the free two-particles ($j$ and $n$)

Two-particle $T_i$ matrix

G. Rupp, J. A. Tjon, Phys. Rev. C 37 (1988), 1729
G. Rupp, J. A. Tjon, Phys. Rev. C 45 (1992), 2133.

# Bethe-Salpeter-Faddeev equation

Introducing the equal-mass Jacobi momenta

$$p_i = \frac{1}{2}(k_j - k_n), \quad q_i = \frac{1}{3}K - k_i, \quad K = k_1 + k_2 + k_3.$$

Amplitude of three-particle state as a projection of $T$ matrix to the bound state:

$$\Psi^{(i)}(p_i, q_i; s) = \langle p_i, q_i | T^{(i)} | M_B \rangle,$$

with $\sqrt{s} = M_B = 3m_N - E_t$.

## Partial-wave three-nucleon functions for a one-rank separable kernel

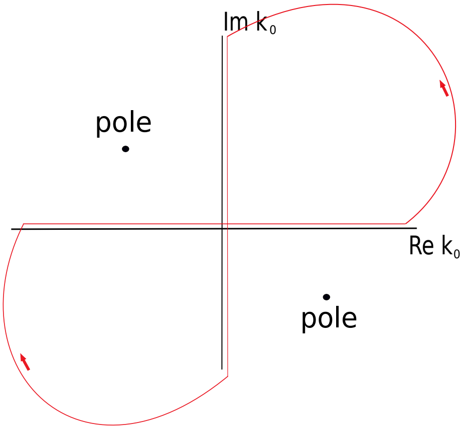
$$\Psi^{(a)}(p,q;s) = g^{(a)}(p)\tau^{(a)}[(\frac{2}{3}\sqrt{s}+q)^2]\,\Phi^{(a)}(q;s)$$

## System of the integral equations

$$\Phi^{(a)}(q;s) = \frac{i}{4\pi^3}\sum_{a'}\int d^4q'\,Z^{(aa')}(q;q';s)\frac{\tau^{(a')}[(\frac{2}{3}\sqrt{s}+q')^2]}{(\frac{1}{3}\sqrt{s}-q')^2-m_N^2+i\epsilon}\Phi^{(a')}(q';s)$$

with effective kernels of equation

$$Z^{(aa')}(q;q';s) = C_{(aa')}\frac{g^{(a)}(-q/2-q')g^{(a')}(q+q'/2)}{(\frac{1}{3}\sqrt{s}+q+q')^2-m_N^2+i\epsilon}$$


Im k0
pole
Re k0
pole

## Method of solution

- Wick-rotation procedure: $q_0 \to iq_4$
- The Gaussian quadrature with $N_1 \times N_2[q_4 \times |\mathbf{q}|]$ grid

$$q_4 = (1+x)/(1-x)$$

$$|\mathbf{q}| = (1+y)/(1-y)$$

- Iteration method to obtain the triton binding energy

$$\lim_{n\to\infty} \frac{\Phi_n(s)}{\Phi_{n-1}(s)}\Big|_{s=M_B^2} = 1$$

The convergence was investigated and $N_1 = 96$, $N_2 = 15$ was used in calculations

## Rank-one Yamagichi kernels

Two-nucleon low-energy parameters for $^1S_0^+$ channel, $^3S_1^+ -^3 D_1^+$ channels

| | Exp. | $^1S_0$ |
|---|---|---|
| $a_L$ (fm) | -23.748 | -23.753 |
| $r_L$ (fm) | 2.75 | 2.75 |

| | Exp. | $^3S_1 -^3 D_1$ $(p_d = 4\%)$ | $^3S_1 -^3 D_1$ $(p_d = 5\%)$ | $^3S_1 -^3 D_1$ $(p_d = 6\%)$ |
|---|---|---|---|---|
| $a_L$ (fm) | 5.424 | 5.454 | 5.454 | 5.453 |
| $r_L$ (fm) | 1.756 | 1.81 | 1.81 | 1.80 |

## Triton binding energy (MeV) with the one-rank Yamaguchi-type separable kernel

| $p_D$ | $^1S_0 - ^3S_1$ | $^3D_1$ | $^3P_0$ | $^1P_1$ | $^3P_1$ |
|---|---|---|---|---|---|
| 4 | 9.221 | 9.294 | 9.314 | 9.287 | 9.271 |
| 5 | 8.819 | 8.909 | 8.928 | 8.903 | 8.889 |
| 6 | 8.442 | 8.545 | 8.562 | 8.540 | 8.527 |
| | | Exp. | 8.48 | | |

- the main contribution is from $S$-states
- the $D$-state contribution is about $0.8 - 1.2$ % depending on $D$-wave (pseudo)probability in deuteron
- the $P$-state contributions are alternating and give about $-0.2\%$

## Multi-rank kernels

Two-nucleon low-energy parameters for ${}^1S_0^+$ channel, ${}^3S_1^+ - {}^3D_1^+$ channels
The relativistic generalization of the NR Graz-II and Paris separable kernel:

- Graz-II: ${}^1S_0^+$ – rank 2, ${}^3S_1^+ - {}^3D_1$ – rank 3
- Paris-1,2: ${}^1S_0^+$ – rank 3, ${}^3S_1^+ - {}^3D_1$ – rank 4

### Results for ${}^1S_0^+$ channel

| | Exp. | Graz-II | Paris-1 | Paris-2 |
|---|---|---|---|---|
| $a$ (fm) | -23.748 | -23.77 | -23.72 | -23.72 |
| $r_0$ (fm) | 2.75 | 2.683 | 2.810 | 2.817 |

### Results for ${}^3S_1^+ - {}^3D_1$ channels

| | Exp. | Graz-II | Graz-II | Graz-II | Paris-1 | Paris-2 |
|---|---|---|---|---|---|---|
| $p_d$ (%) | | 4 | 5 | 6 | 5.77 | 5.77 |
| $a$ (fm) | 5.424 | 5.419 | 5.420 | 5.421 | 5.426 | 5.413 |
| $r_0$ (fm) | 1.759 | 1.780 | 1.779 | 1.778 | 1.775 | 1.765 |
| $E_d$ (MeV) | 2.2246 | 2.2254 | 2.2254 | 2.2254 | 2.2246 | 2.2250 |

## Results for multi-rank separable kernel

Table: Triton binding energy (MeV) with Graz-II kernel

| Kernel | Nonrelativistic | | Relativistic | |
|---|---|---|---|---|
| | ${}^1S_0, {}^3S_1$ | ${}^1S_0, {}^3S_1, {}^3D_1$ | ${}^1S_0, {}^3S_1$ | ${}^1S_0, {}^3S_1, {}^3D_1$ |
| GRAZ-II $p_D = 4\ \%$ | 8.372 | 8.334 | 8.628 | 8.617 |
| GRAZ-II $p_D = 5\ \%$ | 7.964 | 7.934 | 8.223 | 8.217 |
| GRAZ-II $p_D = 6\ \%$ | 7.569 | 7.548 | 7.832 | 7.831 |

Table: Triton binding energy (MeV) with Paris-1(2) kernel

| Kernel | $E_t$ |
|---|---|
| Paris-1 | 7.535 |
| Paris-2 | 7.474 |

## Electromagnetic form factors of three-nucleon systems:

$$2F_{\mathrm{C}}(^3\mathrm{He}) = (2F_{\mathrm{C}}^p + F_{\mathrm{C}}^n)F_1 - \frac{2}{3}(F_{\mathrm{C}}^p - F_{\mathrm{C}}^n)F_2,$$

$$F_C(^3\mathrm{H}) = (2F_{\mathrm{C}}^n + F_{\mathrm{C}}^p)F_1 + \frac{2}{3}(F_{\mathrm{C}}^p - F_{\mathrm{C}}^n)F_2,$$

$$\mu(^3\mathrm{He})F_{\mathrm{M}}(^3\mathrm{He}) = \mu_n F_{\mathrm{M}}^n F_1 + \frac{2}{3}(\mu_n F_{\mathrm{M}}^n + \mu_p F_{\mathrm{M}}^p)F_2 + \frac{4}{3}(F_{\mathrm{M}}^p - F_{\mathrm{M}}^n)F_3,$$

$$\mu(^3\mathrm{H})F_{\mathrm{M}}(^3\mathrm{H}) = \mu_p F_{\mathrm{M}}^p F_1 + \frac{2}{3}(\mu_n F_{\mathrm{M}}^n + \mu_p F_{\mathrm{M}}^p)F_2 + \frac{4}{3}(F_{\mathrm{M}}^n - F_{\mathrm{M}}^p)F_3,$$

Electric and magnetic form factors of the proton and neutron $F_{\mathrm{C,M}}^{p,n}$.

## Impulse approximation:

$$F_i(Q) = \int d^4p \int d^4q \;\; G_1'(\hat{k}_1')\, G_1(\hat{k}_1)\, G_2(\hat{k}_2)\, G_3(\hat{k}_3)\, f_i(p, q, q'; P, P')$$

Nucleon propagators:

$$G_i(\hat{k}_1) = \left[\hat{k}_i^2 - m_N^2 + i\epsilon\right]^{-1},$$

$$G_1'(q_0', q') = \left[(\frac{1}{3}\sqrt{s} - q_0')^2 - \mathbf{q}'^2 - m_N^2 + i\epsilon\right]^{-1},$$

Three-nucleon vertex functions:

$$f_1 = \sum_{i=1}^{3} \Psi_i^*(p, q; P)\, \Psi_i(p, q'; P')$$

$$f_2 = -3\Psi_1^*(p, q; P)\, \Psi_2(p, q'; P')$$

$$f_3 = \Psi_3^*(p, q; P)\, \Psi_3(p, q'; P')$$

Functions $\Psi_i$ are the definite combinations of the partial state functions.

## The Breit reference system

$$Q = (0, \mathbf{Q}), \qquad P = (E_B, -\frac{\mathbf{Q}}{2}), \qquad P' = (E_B, \frac{\mathbf{Q}}{2}), \tag{1}$$

with $s = M_{3N}^2$, $\eta = \mathbf{Q}^2/4s$, $E_B = \sqrt{\mathbf{Q}^2/4 + s} = \sqrt{s}\sqrt{1+\eta}$.

$$\begin{array}{lll} P = \mathrm{L} P_{c.m.}, & p = \mathrm{L} p_{c.m.}, & q = \mathrm{L} q_{c.m.} \\ P' = \mathrm{L}^{-1} P'_{c.m.}, & p' = \mathrm{L}^{-1} p'_{c.m.}, & q' = \mathrm{L}^{-1} q'_{c.m.} \end{array}$$

The explicit form of the transformation $\mathrm{L}$ can be obtained by using (1). Let us assume the boost of the system to be along the $Z$ axis:

$$\mathrm{L} = \begin{pmatrix} \sqrt{1+\eta} & 0 & 0 & -\sqrt{\eta} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sqrt{\eta} & 0 & 0 & \sqrt{1+\eta} \end{pmatrix}. \tag{2}$$

**Relation of the arguments of initial and final $3N$ functions:**

$$
\begin{aligned}
q'_0 &= (1+2\eta)\,q_0 - 2\sqrt{\eta}\sqrt{1+\eta}\,q_z + \frac{2}{3}\sqrt{\eta}\,Q, \\
q'_x &= q_x \qquad q'_y = q_y \\
q'_z &= (1+2\eta)\,q_z - 2\sqrt{\eta}\sqrt{1+\eta}\,q_0 - \frac{2}{3}\sqrt{1+\eta}\,Q,
\end{aligned}
\tag{3}
$$

here $q_z = q\cos\theta_{qQ}$ is the projection of momentum $\mathbf{q}$ onto the $Z$ axis

## Static approximation (SA) [G. Rupp, J. A. Tjon]:

$$q_0' = q_0, \qquad \mathbf{q}' = \mathbf{q} - \frac{2}{3}\mathbf{Q}$$

Propagator and final function:

$$G_1'(q_0', q') \to \left[ (\frac{1}{3}\sqrt{s} - q_0)^2 - \mathbf{q}^2 - \frac{2}{3}\mathbf{q}\cdot\mathbf{Q} - \frac{4}{9}\mathbf{Q}^2 - m_N^2 + i\epsilon \right]^{-1}$$

$$\Psi_i(p_0, p, q_0', q') \to \Psi_i(p_0, p, q_0, |\mathbf{q} - \frac{2}{3}\mathbf{Q}|)$$

with $\mathbf{q}\cdot\mathbf{Q} = qQ\cos\theta_{qQ}$.
The poles of $G_1'$ on $q_0$ do not cross the imaginary $q_0$ axis and always stay in the second and fourth quadrants. In this case, the Wick rotation procedure $q_0 \to iq_4$ can be applied.

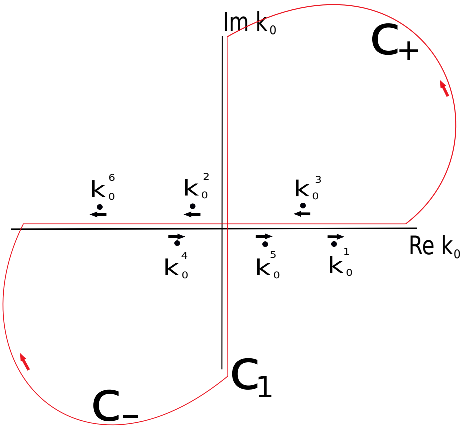
**Beyond the SA:**

1. Exact propagator

$$G_1' = \left[ q_0^2 + \frac{2}{3}\sqrt{s}(1+6\eta)q_0 + 4\sqrt{1+\eta}\sqrt{s}\sqrt{\eta}q_z - \frac{8}{3}\eta s + \frac{1}{9}s - \mathbf{q}^2 - m_N^2 + i\epsilon \right]$$

$$\Psi_i(p_0, p, q_0', q') \rightarrow \Psi_i(p_0, p, q_0, |\mathbf{q} - \frac{2}{3}\mathbf{Q}|).$$

For any $t = -Q^2 > -Q^2_{min} = 2/3\sqrt{s}(3m_N - \sqrt{s})$ the pole of $G_1'$ on $q_0$ crosses the imaginary $q_0$ axis and appears in the third quadrant.


Im $k_0$
$C_+$
$k_0^6$
$k_0^2$
$k_0^3$
Re $k_0$
$k_0^4$
$k_0^5$
$k_0^1$
$C_1$
$C_-$

## Beyond the SA:

2. Additional term from residue inside the countour of integration

Using the Cauchy theorem, one can transform the integrals over $p_0$, $q_0$ as follows:

$$\int d^4p \int d^4q \ldots f(p;q) = \tag{4}$$
$$- \int d^4p_E \int dq_E \, f(p_E; q_E)$$
$$+2\pi \operatorname*{Res}_{q_0=q_0^{(2)}} \int d^4p_E \int_{q_{min}}^{q_{max}} dq \int_{y_{min}}^{1} dy \ldots f(p_E, ; q_0^{(2)}, q, y),$$

where (...) means the two-fold integral $\int_0^\infty dp \int_{-1}^1 dx$ and

$$q_0^{(1,2)} = \frac{\sqrt{s}}{3}(1+6\eta) \pm \sqrt{4\eta(1+\eta)s - 4\sqrt{s}\sqrt{\eta}\sqrt{1+\eta}qy + \mathbf{q}^2 + m_N^2} \tag{5}$$

are the simple poles of the propagator $G_1'$.

## Beyond the SA:

3. Final function arguments transformation

Remembering that the BSF solutions are known for real values of $q_4$ only, the following assumption was made:

$$\Psi(p_0, p, q_0', q') \to g(p_0, p)\,\tau[(\frac{2}{3}\sqrt{s} + q_0^{(2)})^2 - \bar{\mathbf{q}}'^2]\,\Phi(0, \bar{q}'),$$

where value $\bar{q}'$ is obtained using (3) with $q_0 = q_0^{(2)}$.
The expansion of the function $\Phi(q_4', q')$ up to the first order of the parameter $\eta$:

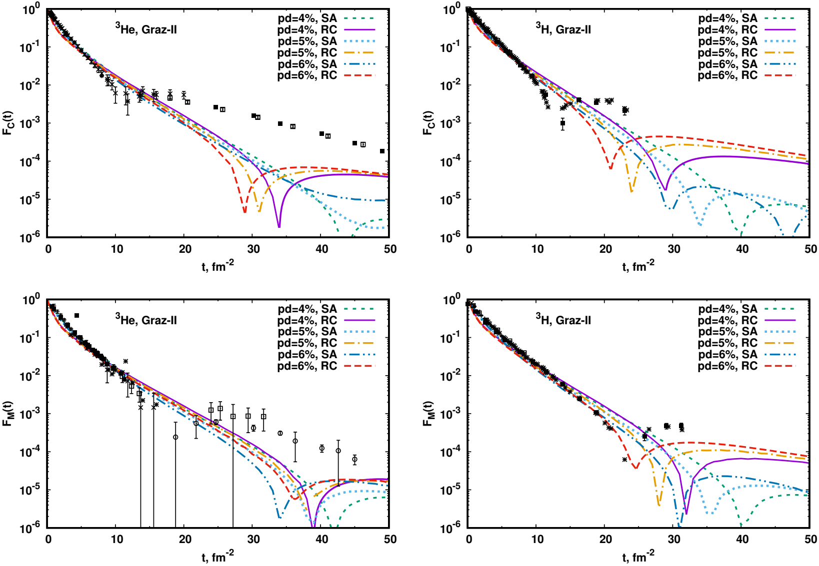
$$\Phi(iq_4', q') = \Phi(iq_4, |\mathbf{q} - \frac{2}{3}\mathbf{Q}|) + \left[C_{q_4}\frac{\partial}{\partial q_4}\Phi_j(iq_4, q)\right]_{q=|\mathbf{q}-\frac{2}{3}\mathbf{Q}|} + \left[C_q\frac{\partial}{\partial q}\Phi_j(iq_4, q)\right]_{q=|\mathbf{q}-\frac{2}{3}\mathbf{Q}|},$$

The function $\Phi'$ is determined by the integral

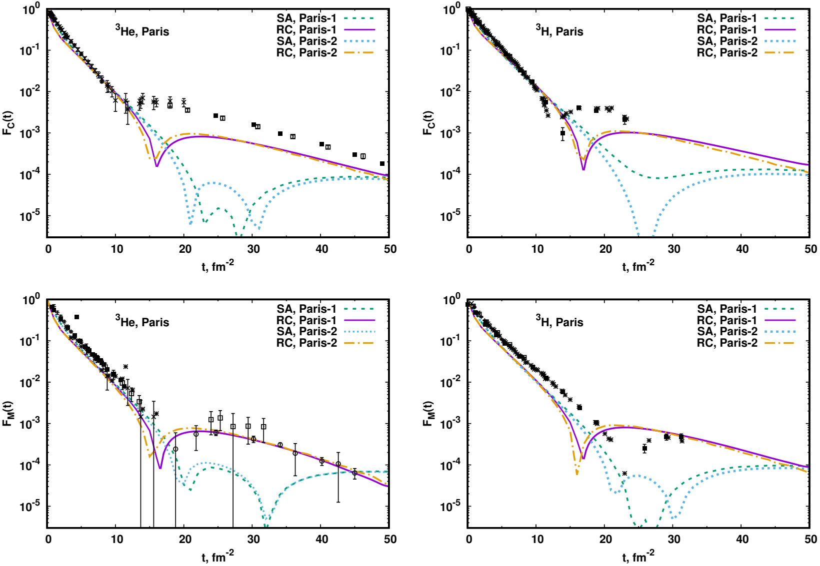
$$\Phi' = \int K'\Phi$$

where $K'$ is a derivative of the kernel of the integral equation.

# Graz-II relativistic kernel



# Paris relativistic kernel



## Summary

- the relativistic three-nucleon vertex functions were founs solving the BSF system of equations
- the charge and magnetic EM form factors of the 3N systems were calculated
- the static approximation and relativistic corrections were investigated
- the relativistic corrections were found to be significant in describing the experimental data