On distance indicator of non-classicality of qudits

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Physical motivation

Classically, a particle in one dimension with its position q and momentum p is described by a phase space distribution $P_{Cl}(q,p)$. The average of a function of the position and momentum $A(q,p)$ can then be expressed as

$$\langle A\rangle_{Cl} = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp\, A(q,p)\, P_{Cl}(q,p)\,.$$

A **quantum mechanical** particle is described by a density matrix $\hat{\varrho}$ and the average of a function of the position and momentum operators $\hat{A}(\hat{q},\hat{p})$ is

$$\langle A\rangle_{QM} = \mathrm{tr}\left(\hat{A}\,\hat{\varrho}\right)\,.$$

A quantum mechanical average can be expressed using a quasiprobability distribution $P_{QM}(q,p)$ as

$$\langle A\rangle_{QM} = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp\, A(q,p)\, P_{QM}(q,p)\,.$$

Objective

Because of Heisenberg's uncertainty principle, the function $P_{QM}(q, p)$ has **negative values** for certain quantum states. Hence, it is not a true probability density and is referred to as a quasiprobability distribution.

Due to this negativity property, quasiprobability distributions may serve as a tool for understanding the interrelations between quantum and classical statistical descriptions.

Aim of the talk:

To consider the Wigner quasiprobability distribution $W(q, p)$ and, specifying the notion of "classical states" as the states whose Wigner function is non-negative everywhere in the phase space, to quantify a state classicality.

Wigner function

The Wigner quasiprobability distribution

$$W(\Omega_N) = \operatorname{tr}\left[\varrho\, \Delta(\Omega_N)\right]$$

is constructed from the **density matrix** (describing a quantum state)

$$\varrho \in \mathfrak{P}_N = \{X \in M_N(\mathbb{C}) \mid X = X^\dagger\,, \quad X \geq 0\,, \quad \operatorname{tr}(X) = 1\}$$

and the **Stratonovich-Weyl** self-dual **kernel**

$$\Delta(\Omega_N) \in \mathfrak{P}_N^* = \{X \in M_N(\mathbb{C}) \mid X = X^\dagger\,, \quad \operatorname{tr}(X) = 1\,, \quad \operatorname{tr}\left(X^2\right) = N\}\,,$$

defined over the symplectic manifold Ω_N .

Density matrix

A state of an N-**level quantum system** is given by the density matrix

$$\varrho = \frac{1}{N}\,\mathbb{I}_N + \sqrt{\frac{N-1}{2N}}\,(\boldsymbol{\alpha}, \boldsymbol{\lambda})\ ,$$

where $\boldsymbol{\alpha}$ is (N^2-1)-dimensional Bloch vector and $\boldsymbol{\lambda} = \{\lambda_1\,, \cdots\,, \lambda_{N^2-1}\}$ is $\mathfrak{su}(N)$ algebra orthonormal Hermitian basis.

The singular value decomposition of the density matrix reads:

$$\varrho = U\,\mathrm{diag}\,(r_1, \dots, r_N)\ U^\dagger\,, \qquad U \in SU(N)\,,$$

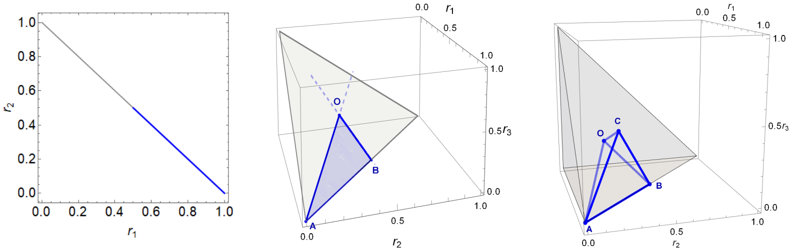
the spectrum $\{r_1, \dots, r_N\}$ of the density matrix forms $\underline{\Delta}_{N-1}$-simplex:

$$1 \geq r_1 \geq \cdots \geq r_N \geq 0\,, \quad \sum_{i=1}^{N} r_i = 1\,.$$

For $N=2$ (qubit) Δ_1 : $\{1 \geq r_1 \geq r_2 \geq 0\,, \quad \sum_{i=1}^{2} r_i = 1\}\,.$

For $N=3$ (qutrit) Δ_2 : $\{1 \geq r_1 \geq r_2 \geq r_3 \geq 0\,, \quad \sum_{i=1}^{3} r_i = 1\}\,.$

For $N=4$ (quatrit) Δ_3 : $\{1 \geq r_1 \geq r_2 \geq r_3 \geq r_4 \geq 0\,, \quad \sum_{i=1}^{4} r_i = 1\}\,.$



$N=2$ $N=3$ $N=4$

State space $\mathfrak{P}_N$

Unitary $U(N)$ automorphism of the Hilbert space of an N-level quantum system induces the adjoint $SU(N)$-action on state space $\mathfrak{P}_N$:

$$g \cdot \varrho = g\, \varrho\, g^{\dagger}\,, \qquad g \in SU(N)\,,$$

which sets equivalence relations between elements of $\mathfrak{P}_N$ and gives rise to its decomposition over the strata:

$$\mathfrak{P}_{[H_\alpha]} := \{x \in \mathfrak{P}_N |\ H_x \text{ is conjugate to } H_\alpha\}\,, \ \mathfrak{P}_N = \bigcup_{\text{orbit types}} \mathfrak{P}_{[H_\alpha]}\,.$$

A subgroup $H_x \subset SU(N)$ is the isotropy group of a point $x \in \mathfrak{P}_N$,

$$H_x = \{g \in SU(N) \mid g \cdot x = x\}\,,$$

and points $x\,, y \in \mathfrak{P}_N$ are said to be of the same type if their stabilizers H_x and H_y are conjugate subgroups of $SU(N)$ group.

Stratonovich-Weyl kernel

The Stratonovich-Weyl kernel is the following:

$$\Delta(\Omega_N) = \frac{1}{N}\mathbb{I}_N + \sqrt{\frac{N^2-1}{2N}}\sum_{\lambda_s \in K} \mu_s \lambda_s \,,$$

$K \in \mathfrak{su}(N)$ is Cartan subalgebra, real coefficients $\sum_{s=2}^{N} \mu^2_{s^2-1} = 1\,.$

The SVD of the Stratonovich-Weyl kernel reads:

$$\Delta(\Omega_N) = V \operatorname{diag}(\pi_1, \dots, \pi_N)\, V^\dagger \,, \qquad V \in SU(N)\,.$$

Ordering of the spectrum $\{\pi_1, \dots, \pi_N\}$ of the SW kernel cuts out the moduli space of $\Delta(\Omega_N)$ in the form of a spherical polyhedron:

$$\pi_1 \geq \dots \geq \pi_N \,, \qquad \sum_{i=1}^{N} \pi_i = 1\,, \qquad \sum_{i=1}^{N} \pi_i^2 = N\,.$$

For $\kappa = \sqrt{\frac{N(N^2-1)}{2}}$ the SW kernel spectrum $\boldsymbol{\pi}$ may be presented as:

$$\pi_i = \frac{1}{N}\left(1 + \sqrt{2}\kappa \sum_{s=i+1}^{N} \frac{\mu_{s^2-1}}{\sqrt{s\,(s-1)}} - \kappa\sqrt{\frac{2\,(i-1)}{i}}\mu_{i^2-1}\right).$$

The conventional parameterization by $N-2$ spherical angles:

$$\begin{aligned}
\mu_3 &= \sin\psi_1 \cdots \sin\psi_{N-2}\,; \ldots\,; \\
\mu_{i^2-1} &= \sin\psi_1 \cdots \sin\psi_{N-i} \cos\psi_{N-i+1}\,; \ldots\,; \\
\mu_{N^2-1} &= \cos\psi_1\,, \qquad i = \overline{2, N}\,,
\end{aligned}$$

where for $\pi_1 \geq \cdots \geq \pi_N$ the constraints on μ_i are:

$$\mu_3 \geq 0\,, \qquad \mu_{(i+1)^2-1} \geq \sqrt{\frac{i-1}{i+1}}\,\mu_{i^2-1}\,, \quad i = \overline{2, N-1}\,.$$

Moduli space

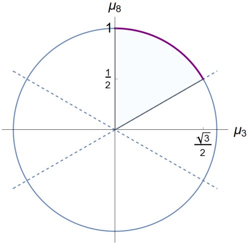
For $\boldsymbol{N=2}$: $\pi_1 \geq \pi_2\,,\ \sum_{i=1}^{2} \pi_i = 1\,,\ \sum_{i=1}^{2} \pi_i = 2\,,$ so:

$$\pi_1 = (1+\sqrt{3})/2\,, \quad \pi_2 = (1-\sqrt{3})/2\,.$$

For $\boldsymbol{N=3}$: $\pi_1 \geq \pi_2 \geq \pi_3\,,\ \sum_{i=1}^{3} \pi_i = 1\,,\ \sum_{i=1}^{3} \pi_i = 3,$ so:

$$\pi_2 = (1-\pi_1+\sqrt{5+2\pi_1-3\pi_1^2})/2\,, \quad 1 \leq \pi_1 \leq 5/3\,,$$

or, equivalently, for $\mu_3 = \sin\zeta\,,\ \mu_8 = \cos\zeta$: $\quad 0 \leq \zeta \leq \pi/3\,.$

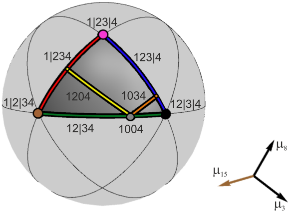


For $\boldsymbol{N = 4}$: $\pi_1 \geq \pi_2 \geq \pi_3 \geq \pi_4\,,\ \sum_{i=1}^{4} \pi_i = 1\,,\ \sum_{i=1}^{4} \pi_i = 4,$ so for

$$\mu_3 = \sin\psi_1 \sin\psi_2\,, \quad \mu_8 = \sin\psi_1 \cos\psi_2\,, \quad \mu_{15} = \cos\psi_1\,,$$

where $\mu_3 \geq 0\,,\ \mu_8 \geq \frac{\mu_3}{\sqrt{3}}\,,\ \mu_{15} \geq \frac{\mu_8}{\sqrt{2}}\,,$ the moduli space reads:

$$\left[\begin{array}{l} \begin{cases} \psi_2 \in \left(0, \frac{\pi}{3}\right]\,, \\ 0 < \psi_1 \leq arccot\left(\cos\psi_2/\sqrt{2}\right)\,; \end{cases} \\ \\ \begin{cases} \psi_2 = 0\,, \\ 0 < \psi_1 \leq arccot\left(1/\sqrt{2}\right)\,; \end{cases} \\ \\ \psi_1 = 0\,. \end{array}\right.$$



Quatrit moduli space as the Möbius spherical triangle $(2,3,3)$ on a unit sphere.

Wigner function positivity

A family of the Wigner functions:

$$W(\Omega_N) = \frac{1}{N}\left(1 + \frac{N^2-1}{\sqrt{N+1}}(\boldsymbol{n}, \boldsymbol{\alpha})\right) ,$$

vectors $\boldsymbol{n} = \mu_3 \boldsymbol{n}^{(3)} + \ldots + \mu_{N^2-1}\boldsymbol{n}^{(N^2-1)}$, $\boldsymbol{n}_\mu^{(s^2-1)} = \frac{1}{2}\operatorname{tr}\left(U\lambda_{s^2-1}U^\dagger\lambda_\mu\right)$.

For $\boldsymbol{r} \in \underline{\Delta}_N\,, \boldsymbol{\pi} \in \mathbf{spec}\,(\Delta(\Omega_N))$, the lower bound of Wigner function

$$W_N^{(-)} = \sum_{i=1}^{N} \pi_i r_{N-i+1} \equiv (\boldsymbol{r}^\uparrow \cdot \boldsymbol{\pi}^\downarrow) = r_1\pi_N + \ldots + r_N\pi_1$$

determines the WF positivity region.

At that: $\quad W_N^{(-)} \leq W(\Omega_N) \leq W_N^{(+)}\,, \qquad W_N^{(+)} = \sum_{i=1}^N \pi_i r_i\,.$

Classical states

The "classical states" form the subset $\mathfrak{P}_N^{(+)} \subset \mathfrak{P}_N$ of states whose Wigner function is non-negative everywhere over the phase space:

$$\mathfrak{P}_N^{(+)} = \{ \varrho \in \mathfrak{P}_N \mid W_\varrho(z) \geq 0 , \quad \forall z \in \Omega_N \} .$$

The "classical states on a fixed stratum" $\mathfrak{P}_{H_\alpha}$ are defined as:

$$\mathfrak{P}_{H_\alpha}^{(+)} = \mathfrak{P}_N^{(+)} \cap \mathfrak{P}_{H_\alpha} .$$

The unitary orbit space $\mathcal{O}[\mathfrak{P}_N]$ is the quotient space under the equivalence relation imposed by the adjoint $SU(N)$-action on the state space $\mathfrak{P}_N$ with quotient mapping $\pi : \mathfrak{P}_N \longrightarrow \mathcal{O}[\mathfrak{P}_N] = \mathfrak{P}_N/SU(N)$.

The subset $\mathcal{O}[\mathfrak{P}_N^{(+)}] = \pi[\mathfrak{P}_N^{(+)}] = \{\pi(x) \mid x \in \mathfrak{P}_N^{(+)} \}$ represents the image of $\mathfrak{P}_N^{(+)}$ under the quotient mapping π.

Non-classicality characteristics of states

Non-classicality measures based on the violation of the Wigner function semi-positivity can be divided into different types:

1. (Global indicator of classicality) as the **relative volume** of a subspace $\mathfrak{P}_N^{(+)} \subset \mathfrak{P}_N$ of the state space $\mathfrak{P}_N$, consisting of states whose Wigner functions are **positive**:

$$\mathcal{Q}_N = \frac{\mathrm{Volume(Classical\ States)}}{\mathrm{Volume(All\ States)}}\,,$$

   where the Riemannian volume is calculated with respect to the measure dictated by the probability distribution function of an ensemble.

2. (Kenfack-Życzkowski indicator) based on the **volume** of a phase space region where the Wigner function is **negative**:

$$\delta_N = \int_{\Omega_N} \mathrm{d}\Omega_N \big| W(\Omega_N) \big| - 1\,.$$

Distance indicator of non-classicality of qudits

3. (Distance indicator of non-classicality) based on a **distance** D of a state ϱ from the "classical states" $\mathfrak{P}_N^{(+)}$:

$$d(\varrho;\mathfrak{P}_N^{(+)}) = \inf_{x\in\mathfrak{P}_N^{(+)}} D(\varrho, x)\,,$$

where states with positive Wigner functions are taken as the reference "classical states".

The distance on $\mathfrak{P}_N$ is assumed to be related to the Frobenious norm: $D(\varrho_1,\varrho_2) = ||\varrho_1 - \varrho_2||_2$, and so

$$d(\varrho;\mathfrak{P}_N^{(+)}) = \inf_{x\in\mathfrak{P}_N^{(+)}} \sqrt{\operatorname{Tr}\left(\varrho - x\right)^2} = \sqrt{\inf_{x_{diag}\in\mathcal{O}[\mathfrak{P}_N^{(+)}]} \sum_{i=1}^{N} (r_i - x_i)^2}\,.$$

Qubit state

The state of a qubit is given by the density matrix

$$\varrho_2 = \frac{1}{2}\left(\mathbb{I}_2 + \boldsymbol{\alpha}\cdot\boldsymbol{\sigma}\right) = U\,\mathrm{diag}(r_1, r_2)\,U^\dagger = U\,\frac{1}{2}(\mathbb{I}_2 + r\sigma_3)\,U^\dagger\,,$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ is a Bloch vector, $r = |\boldsymbol{\alpha}|$, and $\boldsymbol{\sigma}$ is the basis of $\mathfrak{su}(2)$ algebra – the standart Pauli matrices.

Qubit SW kernel: $\Delta(\Omega_2) = V\,\mathrm{diag}\left(\pi_1, \pi_2\right)\,V^\dagger\,.$

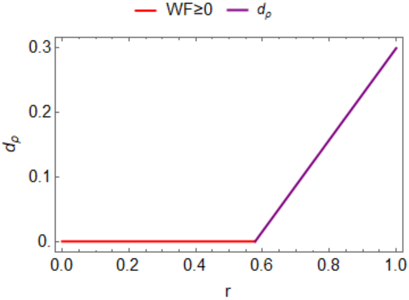
Qubit **Wigner function lower bound**: $W_2^{(-)} = r_1\pi_2 + r_2\pi_1\,.$

Qubit non-classicality distance

Qubit Wigner function: $W(\Omega_2) = \frac{1}{2}\left(1 + \sqrt{3}\,(\boldsymbol{n}, \boldsymbol{\alpha})\right).$

Qubit non-classicality distance for Hilbert-Schmidt metric:

$$d_\varrho = \theta[r - \tfrac{1}{\sqrt{3}}]\left(\frac{r}{\sqrt{2}} - \frac{1}{\sqrt{6}}\right).$$



Qutrit state

A generic qutrit state is given by the density matrix

$$
\begin{aligned}
\varrho_3 \;=\; & \frac{1}{3}(\mathbb{I}_3 + \sqrt{3}\sum_{\nu=1}^{8}\alpha_\nu\lambda_\nu) = U\,\mathrm{diag}(r_1, r_2, r_3)\,U^\dagger = \\
& U\,\frac{1}{3}(\mathbb{I}_3 + \sqrt{3}\sum_{i=3,8}\xi_i\lambda_i)\,U^\dagger\,,
\end{aligned}
$$

where $\boldsymbol{\alpha}$ is an 8-dimensional Bloch vector, $\boldsymbol{\lambda} = \{\lambda_1\,, \cdots, \lambda_8\}$ is $\mathfrak{su}(3)$ algebra basis – the Gell-Mann matrices, and coefficients ξ_3, ξ_8 are invariants under the adjoint $SU(3)$ transformations of ϱ_3 .

Qutrit SW kernel: $\quad \Delta(\Omega_3) = V\,\mathrm{diag}\,(\pi_1, \pi_2, \pi_3)\;V^\dagger\,.$

Qutrit **Wigner function lower bound**: $W_3^{(-)} = r_1\pi_3 + r_2\pi_2 + r_3\pi_1\,.$

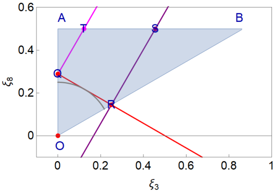
Qutrit non-classicality distance

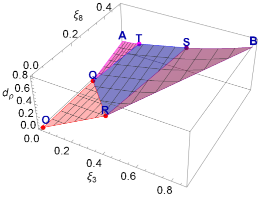
Qutrit Wigner function: $W(\Omega_3) = \frac{1}{3}\left(1 + 4\left(\boldsymbol{n}, \boldsymbol{\alpha}\right)\right).$

Qutrit non-classicality distance for Hilbert-Schmidt metric:

$$d_\varrho = \begin{cases} 0\,, & \text{if} \quad \xi_3, \xi_8 \in \triangle OQR\,, \\ \sqrt{\xi_3^2 + \left(\xi_8 - \frac{1}{4\cos\left(\zeta - \frac{\pi}{3}\right)}\right)^2}\,, & \text{if} \quad \xi_3, \xi_8 \in \triangle AQT\,, \\ \xi_3 \cos\left(\zeta + \frac{\pi}{6}\right) + \xi_8 \sin\left(\zeta + \frac{\pi}{6}\right) - \frac{1}{4}\,, & \text{if} \quad \xi_3, \xi_8 \in \square QRST\,, \\ \sqrt{\left(\xi_3 - \frac{\sqrt{3}}{8}\sec(\zeta)\right)^2 + \left(\xi_8 - \frac{\sec(\zeta)}{8}\right)^2}\,, & \text{if} \quad \xi_3, \xi_8 \in \triangle BRS\,. \end{cases}$$

Qutrit $\underline{\Delta}_2$-simplex with WF positivity boundary and non-classicality distance ($\zeta = \frac{\pi}{6}$):

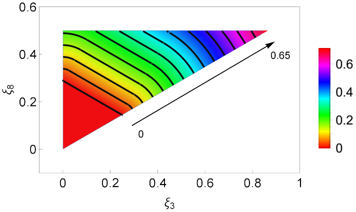




Results

The distance indicator of non-classicality $d(\varrho; \mathfrak{P}_N^{(+)})$ constructed out of the quasiprobability distributions was calculated for low-dimensional quantum systems.

One can also describe qutrit states that are equally distant from the classical states: $d(\varrho; \mathfrak{P}_3) = C$. Equal distant non-classical states comprise the line parallel to the separating one $(\boldsymbol{r}^{\uparrow} \cdot \boldsymbol{\pi}^{\downarrow}) = 0$.



Thank you!