# On Quasiprobability Distributions of Composite Quantum Systems


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# Elementary vs. Composite

**Question:**

What is an elementary and what is a composite ?

**Standpoint:**

"It is the theory finally which decides what can be observed as an elementary and what as a composite one".

**Composite system postulate in QM:**

The state space of a composite physical system is the tensor product of the state spaces of the individual component physical systems.

# Elementary Quantum Systems

Irreducibility condition:

From the standpoint of symmetry an **elementary system** means its states change under the irreducible transformation of a certain physical symmetry

E.P. Wigner (1939)

Decomposition into elementary systems:

"Every system, even one consisting of an arbitrary number of particles, can be decomposed into elementary systems. **The usefulness of the decomposition into elementary** systems depends of how often one has deal with linear combinations containing several elementary systems. We consider a particle "elementary" if it does not appear to be useful to attribute structure to it"

T.T.Newton and E.P. Wigner (1949)

# Quasiprobability representation of Quantum Mechanics

## Quantum world in classical wordings

"How far the [quantum] phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms"

N. Bohr (1949)

Quantum-classical correspondence:

The Stratonovich-Weyl (SW) correspondence between formulations of quantum mechanics in the Hilbert space and in the phase-space is based on the set of physically comprehended postulates.

My task:

Generalizing of Stratonovich-Weyl correspondence under the condition of a priori knowledge about the composite character of a quantum system.

# The statistical model of QM

Expectation value of a Hermitian operator $\hat{A}$ acting on the Hilbert space $\mathcal{H}$ in a state $\varrho$:

$$\mathbb{E}(\hat{A}) = \mathrm{Tr}(\hat{A}\varrho)$$

Statistical averages of a function $A(q,p)$ defined over the phase space $\Omega$ on via the probability distribution function (PDF) $f(q,p)$:

$$\mathbb{E}(A) = \int \mathrm{d}\Omega\, A(q,p)\, f(q,p)\,, \quad \text{with} \quad \int \mathrm{d}\Omega\, f(q,p) = 1\,.$$

Weyl correspondence: Mapping of operators $\hat{A}$ on $\mathcal{H}$ to functions $A(q,p)$ on a phase space $\Omega$.

# Probability space

$(\Omega, \mathcal{F}, \mathbb{P})$ - normalised measure space. $\Omega$-sample space, $2n-$dimensional symplectic manifold, a space of events $\mathcal{F} \subset \Omega$ is represented by elements of $\sigma-$algebra and the probability measure $\mathbb{P} : \mathcal{F} \to [0,1]$. The measure $\mathbb{P}(A)$ of event $A \in \mathcal{F}$ is specifically given by the Lebesgue integral:

$$\mathbb{P}(A) = \int_A \mu(dz)\rho(z)\,.$$

with a non-negative $\rho(z)-$ probability distribution function.

The Kolmogorov's axioms:

$A_I$. Non-negativity $\quad \mathbb{P}(A) \geq 0\,.$

$A_{II}$. Norm $\quad \mathbb{P}(\Omega) = 1\,.$

$A_{III}$. $\sigma-$additivity for pairwise disjoint sets, $\mathbb{P}(\cup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i)\,.$

# Quasiprobability distributions

What is PDF in QM ?

Quasiprobability distribution

- Wigner function (WF) and all its relatives;
    - WF over non-compact phase space /continuous;
    - WF of a finite level quantum system over compact phase space;
    - WF of finite-dimensional system with discrete phase-space;

# Wigner function for a system with continuous variables

The Wigner function for a pure quantum state $\varrho = |\psi\rangle\langle\psi|$ residing in a 2-dimensional phase space with coordinates $(x, p)$ reads,

$$W_{|\psi\rangle}(x,p) = \frac{1}{\pi\hbar}\int \mathrm{d}y \langle x+y|\varrho|x-y\rangle e^{-2\imath py/\hbar}\,.$$

Assuming that states are normalised, i.e., $\int \mathrm{d}y |\langle y|\psi\rangle|^2 = 1$ we conclude that WF of quantum system with continuous variables is bounded,

$$-\frac{1}{\pi\hbar} \leq W_{|\psi\rangle}(x,p) \leq \frac{1}{\pi\hbar}\,.$$

# Wigner function of N-level quantum system

WF associated to a quantum system in $\mathbb{C}_N$ is determined by two elements:

- The density matrix $\varrho \in \mathfrak{P}_N$
- The SW kernel $\Delta(\Omega_N) \in \mathfrak{P}^*_N$ ,

$$W_\varrho(\Omega_N) = \mathrm{Tr}\left(\varrho\Delta(\Omega_N)\right) ,$$

Quantum state $\varrho$ from the semi-positive cone of $N \times N$ :

$$\mathfrak{P}_N = \{X \in M_N(\mathbb{C}) \,|\, X = X^\dagger \,, \mathrm{Tr}X = 1 \,, X \geq 0 \,.\}$$

# Stratonovich-Weyl axioms

The WF of a state $\varrho$ is linear functional given by the kernel $\Delta(\Omega)$ defined over a phase space $\Omega$ subject to the physically motivated conditions:

I. Reconstruction; $\rho$ can be reconstructed from the WF as

$$\rho = \int_\Omega \mathrm{d}\Omega_N \, \Delta(\Omega) W_\rho(\Omega) \, .$$

II. Hermicity; $$\Delta(\Omega) = \Delta(\Omega)^\dagger$$

III. Finite Norm; The state norm is given by the integral of WF

$$\mathrm{tr}\rho = \int_\Omega \mathrm{d}\Omega \, W_\rho(\Omega), \qquad \int_\Omega \mathrm{d}\Omega \, \Delta(\Omega) = 1$$

IV. Covariance: Under the unitary transformations $\rho' = U(\alpha)\rho(\Omega)U^\dagger(\alpha)$

$$\Delta(\Omega') = U(\alpha)\Delta(\Omega)U^\dagger(\alpha)$$

# Master equations for SW kernel

SW kernel $\Delta(\Omega_N)$ of $N-$level quantum system belongs to the subspace of dual space $\mathfrak{P}^*_N$ defined via the following master equations:

$$\mathfrak{P}^*_N = \{X \in M_N(\mathbb{C}) \,|\, X = X^\dagger \,, \mathrm{Tr}X = 1 \,, \mathrm{Tr}X^2 = N \,.\}$$

# Symplectic manifold of $N-$ level quantum system

The quasiprobability distribution associated to $\mathcal{H}_N$ is defined over symplectic space $\Omega_N$ identified with the orbits of adjoint action of $SU(N)$ group,

$$\Omega_N\Big|_{\mathcal{H}_N} \to \frac{U(N)}{\mathrm{Iso}(\Delta)}\,.$$

where $\mathrm{Iso}(\Delta) \subset U(N)$ is an isotropy group of SW kernel.

$$\Delta(\Omega_N) = U(\Omega_N)\mathrm{diag}(\pi_1, \pi_2, \dots, \pi_N)U(\Omega_N)^\dagger\,,$$

At the same time the moduli space of SW kernels is given by the master equations on the corresponding orbit space $\mathcal{O}[\mathfrak{P}^*_N/U(N)]$, determined by the "master equations"

$$\mathcal{O}[\mathfrak{P}^*_N/U(N)]: \qquad \sum_{i=1}^{N}\pi_i = 1 \qquad \sum_{i=1}^{N}\pi_i^2 = N$$

# QM of composite systems

If $n_A$-dimensional system $A$ and $n_B$-dimensional system $B$ are joint together the Hilbert space of the resulting composite system is a subspace of the tensor product of the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ of subsystems:

$$\mathcal{H}_{AB} \subset \mathcal{H}_A \otimes \mathcal{H}_B \,.$$

The state is given by the density matrix $\varrho_{AB}$ acting on $\mathcal{H}_{AB}$, while an information on each subsystem is encoded in the density matrices $\varrho_A$ and $\varrho_B$ which are determined from $\varrho_{AB}$ using the partial trace operation:

$$\varrho_A = \mathrm{tr}_B \varrho_{AB} \,, \quad \varrho_B = \mathrm{tr}_A \varrho_{AB} \,.$$

The partial trace operation is equivalent to the invariant integration over the unitary groups of subsystems,

$$\int_{U_B} \mathrm{d}\mu \, (I_A \otimes U_B) \, \varrho \, (I_A \otimes U_B^\dagger) = \varrho_A \otimes I_B \,.$$

# Quasiprobability distributions of composite systems

Towards composition of quasiprobability distribution

1. Kolmogorov's $\sigma-$additivity axiom ;
2. Conditional probabilities ;
3. Compositions of SW kernels ;

A conventional assumption for SW kernels of a binary composite system:

$$\Delta_{AB} = \Delta_A \otimes \Delta_B$$

# Composite system postulate for SW kernel

The fifth SW postulate applicable to the case of composite systems.

**V. Composite system postulate** *The partially reduced matrices $\Delta_A$ and $\Delta_B$ are SW kernels of subsystems $A$ and $B$ providing the SW mapping with the Wigner functions of the partially reduced states $\varrho_A$ and $\varrho_B$ respectively:*

$$
\begin{aligned}
W_{\varrho_A} &= \operatorname{Tr}\left(\varrho_A \Delta_{N_A}\right) , \\
W_{\varrho_B} &= \operatorname{Tr}\left(\varrho_B \Delta_{N_B}\right) .
\end{aligned}
$$

*This is equivalent to the following equations for SW kernel of joint system:*

$$
\mathrm{tr}_A\left(\mathrm{tr}_B \Delta(\Omega_N)\right)^2 = N_A\,, \qquad \mathrm{tr}_B\left(\mathrm{tr}_A \Delta(\Omega_N)\right)^2 = N_B\,.
$$

# Symplectic manifold of a composite system

If a priory it is known that a quantum system is a composite, the question of where the Weyl mapping is performed is raising up. Searching for the phase space of composite system $\Omega_{N_A \times N_B}$ we found on the correspondence:

$$\text{Global Unitary Symmetry} \quad \Longleftrightarrow \quad \text{Local Unitary Symmetry}$$

The symplectic manifold $\Omega_{N_A \times N_B}$ is defined as the $LU$ group orbits

$$\Omega_N \Big|_{\mathcal{H}_A \otimes \mathcal{H}_B} \to \frac{U(N_A) \times U(N_B)}{H_X} \tag{1}$$

of element $X$ from the moduli space $\mathcal{P}_{N_A \times N_B}$ of a composite SW kernel:

$$\mathcal{P}_{N_A \times N_B} = \left\{ X \in \mathfrak{P}^*_{N_A N_B} \,\Big|\, \mathrm{tr}_A \left(\mathrm{tr}_B X\right)^2 = N_A \,, \ \mathrm{tr}_B \left(\mathrm{tr}_A X\right)^2 = N_A \right\}$$

with a certain isotropy grou $H_X \subset U(N_A) \times U(N_B)$.

# The Wigner function of 2-qubits

Comparing 4-level system, the quatrit, and 2-qubit systems:

- Phase space – $\Omega_4$ vs. $\Omega_{2\times 2}$
- Moduli space – $\mathcal{P}_4$ vs. $\mathcal{P}_{2\times 2}$

# Quatrit

A generic **quatrit** $(N = 4)$ state is given by the density matrix

$$\varrho_{\mathrm{Quatrit}} = \frac{1}{4}\left(\mathbb{I}_4 + \sqrt{6}\sum_{\alpha=1}^{15}\xi_\alpha\lambda_\alpha\right) .$$

The **Stratonovich-Weyl kernel**

$$\Delta(\Omega_N|\boldsymbol{\nu}) = U(\Omega_N)\,\frac{1}{4}\left[I + \sqrt{30}\,(\mu_3\lambda_3 + \mu_8\lambda_8 + \mu_{15}\lambda_{15})\right]\,U(\Omega_N)^\dagger .$$

$$\mu_3^2 + \mu_8^2 + \mu_{15}^2 = 1$$

The **Wigner function** of a quatrit

$$W_{\boldsymbol{\xi}}^{(\boldsymbol{\nu})}(\Omega_4) = \frac{1}{4} + \frac{3\sqrt{5}}{4}\left[\mu_3(\boldsymbol{n}^{(3)},\boldsymbol{\xi}) + \mu_8(\boldsymbol{n}^{(8)},\boldsymbol{\xi}) + \mu_{15}(\boldsymbol{n}^{(15)},\boldsymbol{\xi})\right] ,$$

with

$$n_\alpha^{(s)} = \frac{1}{2}\mathrm{Tr}\left[U\lambda_s U^\dagger\lambda_\alpha\right] \qquad s = 3, 8, 15$$

# The modul space of 4-level system

The **master equations**

$$\operatorname{Tr}(\Delta(\Omega)) = 1\,, \qquad \operatorname{Tr}\left(\Delta(\Omega)^2\right) = 4$$

determine 2-parametric family of kernels that differ by their spectrum $\operatorname{spec}(\Delta^{(4)}) = \{\pi_1, \pi_2, \pi_3, \pi_4\}$. Fixing the decreasing order of eigenvalues $\pi_1 \geq \pi_2 \geq \pi_3 \geq \pi_4$, the spectrum of kernels can be written as functions of two least eigenvalues:

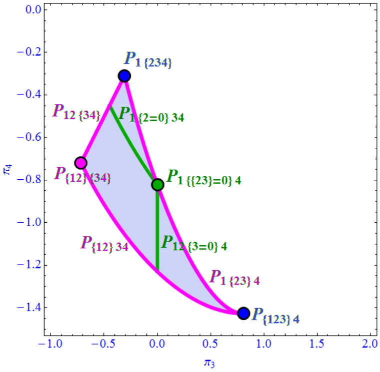
- **Generic** kernel:

$$\operatorname{spec}(\Delta^{(4)}) = \left\{ \frac{\gamma+\delta}{2}\,, \frac{\gamma-\delta}{2}\,, \pi_3\,, \pi_4 \right\}\,,$$

where

$$\gamma = 1 - \pi_3 - \pi_4\,, \quad \delta = \sqrt{8 - 2(\pi_3^2 + \pi_4^2) - \gamma^2}\,.$$

- **Degenerate** kernels:




- Triple degenerate

$$P^{(4)}_{\{123\}4} : \pi_1 = \pi_2 = \pi_3 \neq \pi_4\,,$$
$$P^{(4)}_{1\{234\}} : \pi_1 \neq \pi_2 = \pi_3 = \pi_4\,.$$

- Double degenerate

$$P_{\{12\}\{34\}} : \pi_1 = \pi_2 \neq \pi_3 = \pi_4\,,$$
$$P_{\{12\}34} : \pi_1 = \pi_2 \neq \pi_3 \neq \pi_4\,,$$
$$P_{1\{23\}4} : \pi_1 \neq \pi_2 = \pi_3 \neq \pi_4\,,$$
$$P_{12\{34\}} : \pi_1 \neq \pi_2 \neq \pi_3 = \pi_4\,.$$

- **Singular** kernels

$$P_{1\{2=0\}34} : \pi_1 \neq \pi_2 = 0 \neq \pi_3 \neq \pi_4\,,$$
$$P_{12\{3=0\}4} : \pi_1 \neq \pi_2 \neq \pi_3 = 0 \neq \pi_4\,,$$
$$P_{1\{23=0\}4} : \pi_1 \neq \pi_2 = \pi_3 = 0 \neq \pi_4$$

# 2-Qubit

### Composition and symmetry

As soon as the Hilbert space is assumed to be associated with the tensor product of 2-qubits the Local Unitary symmetry, i.e., covariance under the adjoint action of the subgroup $K = SU(2) \times SU(2) \subset SU(4)$ comes into play.

# Local Unitary group

Below I consider the adjoint action $K = SU(2) \times SU(2) \subset SU(4)$ induced by a certain embedding of the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ into $\mathfrak{su}(4)$via the compositions of two types of embedding:

$$\mathfrak{su}(2) \hookrightarrow \mathfrak{su}(4): \quad \begin{bmatrix} x & z \\ \bar{z} & -x \end{bmatrix} \hookrightarrow \left[\begin{array}{cc|cc} x & 0 & z & 0 \\ 0 & x & 0 & z \\ \hline \bar{z} & 0 & -x & 0 \\ 0 & \bar{z} & 0 & -x \end{array}\right]$$

and

$$\mathfrak{su}(2) \hookrightarrow \mathfrak{su}(4): \quad \begin{bmatrix} y & w \\ \bar{w} & -y \end{bmatrix} \hookrightarrow \left[\begin{array}{cc|cc} y & w & 0 & 0 \\ \bar{w} & -y & 0 & 0 \\ \hline 0 & 0 & y & w \\ 0 & 0 & \bar{w} & -y \end{array}\right]$$

The corresponding exponent mapping $\mathfrak{su}(4) \to SU(4)$ defines the embedding of the group $SU(2) \times SU(2)$ into $SU(4)$.

# Double coset $SU(2) \times SU(2) \backslash SU(4)/T^3$ decomposition

**Proposition:** The $\mathfrak{su}(4)$ algebra admits decomposition into the direct sum:

$$\mathfrak{su}(4) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{a}' \oplus \mathfrak{k}' ,$$

where $\mathfrak{a}$ and $\mathfrak{a}'$ are Abelian subalgebras such that

$$[\mathfrak{a}' , \mathfrak{a}] \subset \mathfrak{k} ,$$

and $\quad \mathfrak{k} := \mathfrak{su}(2) \oplus \mathfrak{su}(2) , \quad \mathfrak{k}' := \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$, with the relations:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} , \qquad [\mathfrak{k}', \mathfrak{k}'] \subset \mathfrak{k}' , \qquad [\mathfrak{k}, \mathfrak{k}'] \subset \mathfrak{a} \oplus \mathfrak{a}' ,$$

The exponential map $\exp : \mathfrak{su}(4) \to SU(4)$ results in corresponding Cartan type coordinates description of the group in vicinity of the identity,

$$g := K \exp(\mathfrak{a}) \exp\left(\mathfrak{a}'\right) T_3 , \qquad K \in SU(2) \times SU(2) ,$$

where $T_3$ is the maximal torus in $SU(4)$ ,

## Fano form of 2-qubit SW kernel

2-qubit SW kernel in Fano basis:

$$\Delta(\Omega_4) = \frac{1}{4}\,\mathbb{I}_4 + \frac{\sqrt{30}}{4}\,\left[\boldsymbol{\zeta_A}\cdot\boldsymbol{\sigma_A} + \boldsymbol{\zeta_B}\cdot\boldsymbol{\sigma_B} + \frac{1}{\sqrt{2}}\mathcal{E}_{ij}\,\sigma_i\otimes\sigma_j\right]\,, \qquad (2)$$

where $\boldsymbol{\sigma_A} = \frac{1}{\sqrt{2}}(\sigma_{10},\sigma_{20},\sigma_{30})\,,\ \boldsymbol{\sigma_B} = \frac{1}{\sqrt{2}}(\sigma_{01},\sigma_{02},\sigma_{03})\,.$ The coefficients of expansion $\boldsymbol{\zeta_A}$ and $\boldsymbol{\zeta_B}$ are a real 3-vectors and $\mathcal{E}$ is a real $3\times 3$ matrix. According to the master equations for composite system the norm of these vectors and matrices is fixed:

$$\boldsymbol{\zeta_A}^2 = \frac{1}{5}\,, \qquad \boldsymbol{\zeta_B}^2 = \frac{1}{5}\,, \qquad \mathrm{tr}\left(\mathcal{E}\mathcal{E}^T\right) = \frac{3}{5}\,.$$

Hence, for 2-qubit all three primary second order $SU(2)\times SU(2)$ polynomial invariants of SW kernel are fixed, while the higher order invariants characterize all admissible types of SW kernels.

## Subsystems kernels

**Proposition:** From SVD of SW kernel with the Cartan type coordinates for $SU(4)$ factor

$$\Delta(\boldsymbol{z}) = U(\boldsymbol{z})\mathrm{diag}(\pi_1, \pi_2, \dots, \pi_N)U(\boldsymbol{z})^\dagger\,,$$

it follows that the reduced SW kernels of subsystems are:

$$\Delta_A = \frac{1}{2}\,U_A\left(\mathbb{I}_2 + \sqrt{15}\,(\zeta^A \cdot \boldsymbol{\sigma})\right)U_A^\dagger\,,$$

and

$$\Delta_B = \frac{1}{2}\,U_B\left(\mathbb{I}_2 + \sqrt{15}\,(\zeta^B \cdot \boldsymbol{\sigma})\right)U_B^\dagger\,,$$

with 3-vectors whose length is fixed by the “subsystem master equations”

$$(\zeta^A \cdot \zeta^A) = (\zeta^B \cdot \zeta^B) = \frac{1}{5}$$

## The moduli space of 2-qubit

**Proposition:** The moduli parameters of 2 qubits are determined by the moduli parameters of SW kernel of 4-level system as whole $\mu_3, \mu_6, \mu_{15}$ and the lengths of 3-vectors

$$\zeta_i^A = \sum_{\alpha \in H} \mu_\alpha O_{\alpha i}\,, \qquad i = 1, 2, 3\,,$$

$$\zeta_i^B = \sum_{\alpha \in H} \mu_\alpha O_{\alpha i}\,, \qquad i = 4, 5, 6\,,$$

defined in terms of the component $\mathcal{A}$ the Cartan decomposition $K\mathcal{A}T_3$ :

$$\mathcal{A}\lambda_\alpha\mathcal{A}^\dagger = O_{\alpha\beta}\,\lambda_\beta\,, \qquad \mathcal{A} = \exp\{\mathfrak{a}\}\exp\{\mathfrak{a}'\}$$

# The moduli space of 2-qubit

The moduli space $\mathcal{P}_{2\times 2}$ of 2-qubits is given by the bundle of a unit 2-sphere, two ellipsoids $\mathrm{E_A}$, and $\mathrm{E_B}$ in the moduli space space $\mathcal{P}_4$ with coordinates $\boldsymbol{\mu} = \{\mu_3, \mu_6, \mu_{15}\}$:

$$\boldsymbol{\mu}\boldsymbol{\mu}^T = 1\,, \qquad \mathrm{E_A}: \quad \boldsymbol{\mu}\mathbb{A}\boldsymbol{\mu}^T = 1\,, \qquad \mathrm{E_B}: \quad \boldsymbol{\mu}\mathbb{B}\boldsymbol{\mu}^T = 1\,,$$

The $3\times 3$ matrices $\mathbb{A}$ and $\mathbb{B}$ are:

$$\mathbb{A}_{\alpha\beta} := \frac{4}{3}\sum_{i=1,2,3} O_{\alpha i} O^T_{i\beta}\,, \qquad \mathbb{B}_{\alpha\beta} := \frac{4}{3}\sum_{i=4,5,6} O_{\alpha i} O^T_{i\beta}\,.$$

Properties of $\mathcal{P}_{2\times 2}$ are encoded in pairwise characteristic polynomials:

$$f_{\mathrm{E_A}\cap\mathbb{S}_2} = \det(t\mathbb{I}_3 + \mathbb{A}), \quad f_{\mathrm{E_B}\cap\mathbb{S}_2} = \det(t\mathbb{I}_3 + \mathbb{B}), \quad f_{\mathrm{E_A}\cap\mathrm{E_B}} = \det(t\mathbb{A} + \mathbb{B})\,.$$

**Proposition:** The ellipsoids and the 2-sphere overlap iff the characteristic polynomials $f_{\mathrm{E_A}\cap\mathbb{S}_2}, f_{\mathrm{E_B}\cap\mathbb{S}_2}$ and $f_{\mathrm{E_A}\cap\mathrm{E_B}}$ have no positive roots.

# Summary: ELEMENTARY vs. COMPOSITE

| | ELEMENTARY | COMPOSITE |
|---|---|---|
| Symmetry | $G := SU(N)$ | $LU := SU(N_A) \times SU(N_B)$ |
| STATE | $\mathrm{Tr}\varrho = 1, \varrho \geq 0$ | $\varrho_A = \mathrm{Tr}_B\varrho, \quad \varrho_B = \mathrm{Tr}_A\varrho$ |
| DUAL SPACE | $\mathrm{Tr}\Delta = 1\,, \mathrm{Tr}\Delta^2 = N$ | $\mathrm{Tr}_A(\mathrm{Tr}_B\Delta)^2 = N_A\,, \mathrm{Tr}_B(\mathrm{Tr}_A\Delta)^2 = N_B$ |
| PHASE SPACE | $\Omega_N = G/\mathrm{Iso}_G(\Delta)$ | $\Omega_{N_A \times N_B} = LU/\mathrm{Iso}_{LU}(\Delta)$ |

# Other topics

Non-classicality indicators for composite vs. elementary systems;
Positivity of WF and separability ;
Interrelations between marginals of WF and SW kernels

Thank you for your attention!

# Appendix

## The Stratonovich-Weyl kernel

$$\Delta(\Omega|\boldsymbol{\nu}) = \frac{1}{N} U(\Omega) \left[ I + \kappa \sum_{\lambda \in H} \mu_s(\boldsymbol{\nu}) \lambda_s \right] U(\Omega)^\dagger \,, \quad \kappa = \sqrt{N(N^2-1)/2}\,,$$

where

- $H$ is the **Cartan subalgebra** in $SU(N)$,
- parameter $\boldsymbol{\nu} = (\nu_1\,, \cdots\,, \nu_{N-2})$ labels members of the WF family,
- $\sum_{s=2}^{N} \mu^2_{s^2-1}(\boldsymbol{\nu}) = 1\,.$

## A density matrix of an $N$-dimensional quantum system

$$\varrho_\xi = \frac{1}{N} \left[ I + \sqrt{\frac{N\,(N-1)}{2}}\,(\boldsymbol{\xi}, \boldsymbol{\lambda}) \right] \,,$$

where $\boldsymbol{\xi}$– $(N^2-1)$-dimensional Bloch vector, $\boldsymbol{\lambda} = \{\lambda_1, \cdots, \lambda_{N^2-1}\}$ is $\mathfrak{su}(N)$ algebra basis.

A family of the Wigner functions

$$W^{(\boldsymbol{\nu})}_{\boldsymbol{\xi}}(\Omega_N) = \frac{1}{N}\left[1 + \frac{N^2-1}{\sqrt{N+1}}(\boldsymbol{n}, \boldsymbol{\xi})\right],$$

where

- $\boldsymbol{n} = \mu_3 \boldsymbol{n}^{(3)} + \cdots + \mu_{N^2-1}\boldsymbol{n}^{(N^2-1)}$,
- $\boldsymbol{n}^{(s^2-1)} = \frac{1}{2}\operatorname{tr}\left(U\lambda_{s^2-1}U^\dagger\lambda_\mu\right), \quad s = \overline{2, N}$.

The spectrum $\{\pi_1, \cdots, \pi_N\}$ of the Stratonovich-Weyl kernel:

$$\pi_i = \frac{1}{N}\left(1 + \sqrt{2}\kappa \sum_{s=i+1}^{N} \frac{\mu_{s^2-1}}{\sqrt{s\,(s-1)}} - \kappa\sqrt{\frac{2\,(i-1)}{i}}\mu_{i^2-1}\right).$$

# Decomposition in Fano basis of $\mathfrak{su}(4)$

Starting from the Fano basis of $\mathfrak{su}(4)$

$$\{\lambda_1, \lambda_2, \dots, \lambda_6\} = \frac{i}{2}\{\sigma_{10}, \sigma_{20}, \sigma_{30}, \sigma_{01}, \sigma_{02}, \sigma_{03}\}$$

and

$$\{\lambda_7, \lambda_8, \dots, \lambda_{15}\} = \frac{i}{2}\{\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{22}, \sigma_{23}, \sigma_{31}, \sigma_{32}, \sigma_{33}\}\,.$$

we decompose the algebra as follows,

$$\mathfrak{a} = \mathrm{span}\{\lambda_{11}\,, \lambda_9\,, \lambda_{13}\} \qquad \mathfrak{a}' = \mathrm{span}\{\lambda_4\,, \lambda_1\, \lambda_7\}$$

and

$$\mathfrak{k}' := \mathrm{span}\{\lambda_3\,, \lambda_6\, \lambda_{15}\}\,, \qquad \mathfrak{k} := \mathrm{span}\{-\lambda_{14}\,, \lambda_2\,, -\lambda_8\,; -\lambda_5\,, \lambda_{12}\,, -\lambda_{10}\}\,,$$

# The Wigner function of a single qubit

A generic **qubit** quantum state is parameterized in a standard way

$$\varrho_{qubit} = \frac{1}{2}\left(I + \boldsymbol{r}\cdot\boldsymbol{\sigma}\right)$$

by the Bloch vector $\boldsymbol{r} = (r\sin\psi\cos\phi\,,\ r\sin\psi\sin\phi\,,\ r\cos\psi)\,.$
The master equations determine the spectrum:

$$\operatorname{spec}\left(P^{(2)}\right) = \left\{\frac{1+\sqrt{3}}{2}\,,\ \frac{1-\sqrt{3}}{2}\right\}\,.$$

The **Wigner function** for a single qubit is

$$W_{\boldsymbol{r}}(\alpha,\beta) = \frac{1}{2} + \frac{\sqrt{3}}{2}\,(\boldsymbol{r},\boldsymbol{n})\ ,$$

where $\boldsymbol{n} = (-\cos\alpha\sin\beta\,,\ \sin\alpha\sin\beta\,,\ \cos\beta)$ is the unit 3-vector.

# Qutrit

A generic **qutrit** state is given by the density matrix

$$\varrho_{\mathrm{Qutrit}} = \frac{1}{3}\left(\mathbb{I}_3 + \sqrt{3}\sum_{\nu=1}^{8}\xi_\alpha\lambda_\alpha\right) .$$

The **Stratonovich-Weyl kernel**

$$\Delta(\Omega_3) = U(\Omega_3)\,\frac{1}{3}\left[I + 2\sqrt{3}\,(\mu_3\lambda_3 + \mu_8\lambda_8)\right]\; U(\Omega_3)^\dagger\,,$$

where the coefficients

$$\mu_3(\nu) = \frac{\sqrt{3}}{4}\sqrt{(1+\nu)(5-3\nu)}\,, \quad \mu_8(\nu) = \frac{1}{4}(1-3\nu)$$

are functions of the parameter $\nu = \frac{1}{3} - \frac{4}{3}\cos(\zeta)$ with $\zeta \in [0,\ \pi/3]$ being the moduli parameter of the unitary nonequivalent WF of a qutrit.

The **Wigner function** of a single qutrit

$$W^{(\nu)}_{\boldsymbol{\xi}}(\Omega_3) = \frac{1}{3} + \frac{4}{3}\left[\mu_3\left(\boldsymbol{n}^{(3)}, \boldsymbol{\xi}\right) + \mu_8\left(\boldsymbol{n}^{(8)}, \boldsymbol{\xi}\right)\right],$$

with two orthogonal unit 8-vectors

$$n^{(3)}_\nu = \frac{1}{2}\mathrm{tr}\left[U\lambda_3 U^\dagger \lambda_\nu\right], \qquad n^{(8)}_\nu = \frac{1}{2}\mathrm{tr}\left[U\lambda_8 U^\dagger \lambda_\nu\right].$$

The **master equations**

$$\mathrm{Tr}\left(\Delta(\Omega)\right) = 1, \qquad \mathrm{Tr}\left(\Delta(\Omega)^2\right) = 3$$

determine one-parametric family of kernels $\Delta^{(\nu)}$.

# One-parametric $\Delta^{(\nu)}$-family

- The spectrum of **generic** kernels:

$$\text{spec}\left(\Delta^{(\nu)}\right) = \left\{\frac{1-\nu+\delta}{2}, \frac{1-\nu-\delta}{2}, \nu\right\},$$

where $\quad \delta = \sqrt{(1+\nu)(5-3\nu)} \quad$ and $\quad \nu \in (-1, -\frac{1}{3})$.

- Two **degenerate** kernels:

$$\text{spec}\left(\Delta^{(-1)}\right) = \{1\,,1\,,-1\}\,, \quad \text{spec}\left(\Delta^{(-1/3)}\right) = \left\{\frac{5}{3}\,,-\frac{1}{3}\,,-\frac{1}{3}\right\}.$$

- The spectrum of **singular** kernel, $\det(\Delta) = 0$:

$$\text{spec}\left(\Delta^{(0)}\right) = \left\{\frac{1+\sqrt{5}}{2}\,,0\,,\frac{1-\sqrt{5}}{2}\right\}, \quad \text{Tr}\left((\Delta^{(0)})^m\right) = \mathcal{L}_m\,,$$

where the $m$-th **Lucas number** $\mathcal{L}_m = \phi^m + (-\phi)^{-m}$ and $\phi = \frac{1+\sqrt{5}}{2}$.